Limits

FORMAL APPROACH TO LIMIT

Before we proceed to formulate a definition of the limit of a function at a point on the basis of the discussion made in the previous article, we intend to discuss some basic concepts which will be used in further discussions.

NEIGHBOURHOOD (NBD) OF A POINT

Let a be a real number and let δ be a positive real number. Then the set of all real numbers lying between a-δ and a+δ is called the neighbourhood of a of radius 'δ' and is denoted by Nδ(a).

Thus Nδ(a) = (a-δ, a+δ) = {x. R|a-δ<x<a+δ}

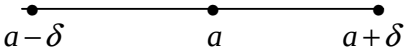
The set Nδ(a) – {a} is called deleted nbd of a of radiusδ. The set (a-δ, a) is called the left nbd of a and the set (a, a+δ) is known as the right nbd of a.

If δ is very small and x lies in the interval (a-δ, a), then x is said to approach to a from the left and we write $x \to \bar{a}$. If x. (a, a+δ), then x is said to approach to a from the right which is denoted by $x \to a^{+}$.

Consider the statement |x-a|<δ. We have

|x-a|<$\delta \Leftrightarrow$ -δ<x-a<$\delta \Leftrightarrow$ a-$\delta \Leftrightarrow$ x. Nδ(a).

Thus, |x – a|<δ means that x lies in the nbd of 'a' of radiusδ as shown in Fig.



Let f(x) be a function with domain D and let 'a ' be point such that every nbd of a contains infinitely many points of D.A real number *l* is limit of f(x) as x tends to a, if for every nbd of *l*, there exists a nbd of 'a ', such that images of all points in the deleted nbd of a are in the nbd of *l*.

In other words, a real number *l* is lmit of f(x) x tends to a, if for every. > 0 there exists $a\delta > 0$ such that 0 < |x – a| <$\delta \Rightarrow$ |f(x) – l | <$\varepsilon \Leftrightarrow$ x. (a - δ, a+δ), x$\neq$ a $\Rightarrow$ f(x). (l+ε)

If *l* is the limit of f(x) as x tends to a, then we write $\underset{x\to a}{Lim}\, f(x) = 1$

THE ALGEBRA OF LIMITS

Let f and g be two real functions with domain D. We define four new functions $f \pm g, fg, f/g$ on domain D by setting.

(f± g) (x) = f(x) ± g(x), (fg) (x) =f(x) g(x)

(f/g) (x) = f(x)/g(x), if g(x)$\neq$ 0 for any $x \in D$.

Following are some results concerning the limits of these functions.

Let $\lim_{x\to a}$ f(x) =*l* and $\lim_{x\to a} g(x) = m$. If l and m exist, then

(i) $\lim_{x\to a}(f \pm g)(x) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = l \pm m$

(ii) $\lim_{x\to a}(fg)(x) = \lim_{x\to a} f(x).\lim_{x\to a} g(x) = lm$

(iii) $\lim_{x\to a}\left(\frac{f}{g}\right)(x) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{l}{m}$, provided $m \neq 0$

(iv) $\lim_{x\to a} Kf(x) = K.\lim_{x\to a} f(x)$, where K is constant

(v) $\lim_{x\to a} | f(x) = \lim_{x\to a} f(x) | = | l |$

(vi) $\lim_{x\to a}\{f(x)\}g^{(x)} = l^{m}$

(vii) If f(x) ≤ g(x) for every x in the deleted nbd of a, then $\lim_{x\to a}$ f(x) ≤ $\lim_{x\to a}$ g(x

(viii) If f(x) ≤ g(x) ≤ h (x) for every x in the deleted nbd of a and, $\lim_{x\to a} f(x) = l = \lim_{x\to a} h(x)$ then $\lim_{x\to a} g(x) = l.$

(ix) $\lim_{x\to a} fog(x) = f\left(\lim_{x\to a} g(x)\right) = f(m)$

In particular

(i) $\lim_{x\to a} \log f(x) = \log\left(\lim_{x\to a} f(x)\right) = \log l$

(ii) $\lim_{x\to a} e^{f(x)} = e^{\lim_{x\to a} f(x)} = e^{1}.$

(iii) If $\lim_{x\to a} f(x) = +\infty$ *or* $-\infty$, *then* $\lim_{x\to a} \frac{1}{f(x)} = 0$

(iv) Evaluate $\lim_{x\to 0}\left\{\lim_{n\to\infty}\left(\frac{[1^2(\sin x)^x]+[2^2(\sin x)^x]+....+[n^2 9\sin x)^x]}{n^3}\right)\right\}$

(v) Evaluate $\lim_{n\to\infty}\prod_{r=3}^{n}\frac{r^3-8}{r^3+8}$ Here Π stands for the product.

(ix) $\lim_{x\to 1/\alpha}\frac{1-\cos(cx^2+bx+a)}{(1-\alpha x)^2}$

(x) If f(x+y)=f(x)+f(y) for all x, y $\in$ R and f(1) = 1, then evaluate

$$\lim_{x\to 0}\frac{2f(\tan x)-2f(\sin x)}{x^2 f(\sin x)}$$