Definite Integrals

THE NEWTON – LEIBNITZ FORMULA OR THE FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

If $\phi(x)$ is one the primitives or anti derivatives of a function f(x)defined on [a, b], then the definite integral of f(x) over [a, b] is given by $\phi(b) - \phi(a)$ and is denoted by $\int_a^b f(x)\,dx$.

Thus, if $\frac{d}{dx}(\phi(x)) = f(x)$, then $\int_a^b f(x)dx = \phi(b) - \phi(a)$

The numbers a and b are called the limits of integration 'a' is called the lower limit and 'b ' is the upper limit. The interval [a, b] is called the interval of integration.

INTEGRAL FUNCTION AND ITS PROPERTIES

PROPERTY I : The integral function of an integrable function is always continuous

PROPERTY II : If ϕ (x) is the integral function of a continuous function f(x) defined on [a, b], then $\phi(x)$ is differentiable on (a, b) and $\phi^1(x) = f(x)$ for all $x \in$ (a, b).

PROPERTY III : The integral function of an odd function is an even function

If f(x) is an odd function, then ϕ (x) = $\int_a^x f(t)\,dt$ is an even function

PROPERTY IV : If f(t) is an even function, then ϕ (x) = $\int_0^x f(t)\,dt$ an odd function

(1) Prove that $\int_0^{\pi/2} \frac{x \sin x \cos x}{(a^2\cos^2 x + b^2 \sin^2 x)}\,dx = \frac{\pi}{4ab^2(a+b)}$, where a, b > 0.

(2) Prove that $I = \int_0^{\pi} \frac{1}{a^2 - 2a\cos x + 1}\,dx = \begin{cases} \frac{\pi}{a^2-1}, & if \quad a > 1 \\ \frac{\pi}{1-a^2}, & if \quad 0 < a < 1 \end{cases}$

(3) Prove that $\int_0^{\pi} \frac{\sin^2 x}{a^2 - 2ab\cos x + b^2}\,dx = \begin{cases} \frac{\pi}{2a^2}, & if \quad a > b > 0 \\ \frac{\pi}{2b^2}, & if \quad b > a > 0 \end{cases}$

(4) Show that $\int_0^{\pi} \frac{1}{a - \cos x}\,dx = \frac{\pi}{\sqrt{a^2-1}}$, $a > 1$. Also, deduce that $\int_0^{\pi} \frac{1}{(2-\cos x)^2}\,dx = \frac{2\pi}{3\sqrt{3}}$

(5) If $\beta + 2\int_0^1 x^2 e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx$, then find the value of β.

(6) For x > 0, let f(x) = $\int_1^x \frac{\log_e t}{1+t}\,dt$. Find the function f(x) + f$\left(\frac{1}{x}\right)$ and show that $f(e) + f\left(\frac{1}{e}\right) = \frac{1}{2}$

(7) If in is positive integer, prove that

$\int_0^a e^{-x} x^n\,dx = n!\left[1 - e^{-a}\left\{1 + a + \frac{a^2}{2!} + ... + \frac{a^n}{n!}\right\}\right]$ Also, deduce the value of $\int_0^{\infty} e^{-x} x^n\,dx$

PROPERTIES OF DEFINITE INTEGRALS

(12) $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$ i.e., integration is independent of the change of variable.

(13) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where a < c < b.

(14) Property $\int_a^b f(x)dx = \int_a^b f(a+b-x)\,dx.$

(15) Property $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(x) \text{ is an even function} \end{cases}$

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(x) \text{ is an even function} \\ 0 \qquad\qquad , & \text{if } f(x) \text{ is an odd function} \end{cases}$$

(16) Prove that $\int_0^{\pi} \frac{x\sin 2x \sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi}\,dx = \frac{8}{\pi^2}$

(17) Evaluate $\int_{-\pi}^{\pi} \frac{2x\,(1+\sin x)}{1+\cos^2 x}\,dx$

(19) $\int_{-a}^{a} f(x)\,dx = \int_0^a \{f(x)+f(-x)\}\,dt$

(18) Property $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x) \\ 0 \qquad\qquad , & \text{if } f(2a-x) = -f(x) \end{cases}$

(19) Property $\int_0^{2a} f(x)\,dx = \int_0^a \{f(x) = f(2a-x)\}\,dx$

(20) Property $\int_a^b f(x)\,dx = (b-a)\int_a^1 f\{(b-a)x+a\}\,dx$

(21) Property f(x) is a periodic function with period T, then

(i) $\int_0^{nT} f(x)\,dx = n\int_0^T f(x)\,dx,\ n \in Z$

(ii) $\int_a^{a+nT} f(x)\,dx = n\int_0^T f(x)\,dx,\ n \in Z,\ a \in R$

(iii) $\int_{mT}^{nT} f(x)\,dx = (n-m)\int_0^T f(x)dx,\ m,\ n \in Z$

(iv) $\int_{nT}^{a+nT} f(x)\,dx = \int_0^a f(x)\,dx,\ n \in Z,\ a \in R$

(v) $\int_{a+nT}^{b+nT} f(x)\,dx = \int_a^b f(x)\,dx,\ n \in Z,\ a \in R$

INDEFINITE INTEGRALS

(7) Evaluate the following integral $\int \frac{\sin 2x}{\sin(x-\pi/3)\sin(x+\pi/3)}\,dx$

(11) Evaluate the following integral $\int \frac{1}{\sqrt{\sin^3 x \sin(x+a)}}\, dx$

(12) Evaluate the following integral $\int \frac{2a\sin x + b\sin 2x}{(b+a\cos x)^3}\, dx$

(13) Evaluate the following integral $\int \frac{\sin^{3/2} x + \cos^{3/2} x}{\sqrt{\sin^3 x \cos^3 x \sin(x+\theta)}}\, dx$

(14) Evaluate the following integral $\int x^{13/2}(1+x^{5/2})^{1/2}\ \text{dx}$

(15) Evaluate $\int \left(\frac{x-1}{x+1}\right)^4\ \text{dx}$

(16) Evaluate $\int \frac{1}{(x^2+2x+10)^3}\, dx$

(17) For any natural number m, evaluate $\int (x^{3m}+x^{2m}+x^m)(2x^{2m}+3x+6)^{1/m}\ dx,\ x>0$

(18) Evaluate $\int \sec^{25/13} x \cos ec^{27/13} x\ dx$

(19) $\int \sin^3 x \cos^2 2x\ dx$

(20) $\int \frac{x^7}{(1-x^2)^5}\, dx$

(21) $\int \frac{1}{\sin^3 x cs^5 x}\, dx$

(22) Evaluate $\int \frac{1}{\cos^6 x + \sin^6 x}\, dx$

(23) Evaluate $\int (\sqrt{\tan x} + \sqrt{\cot x})\, dx$

(24) Evaluate $\int \frac{\cos^3 x + \cos^5 x}{\sqrt{2}}\, dx$

(25) Evaluate $\int \frac{x-1}{(x+1)\sqrt{x^3+x^2+x}}\, dx$

(28) Prove that

$$\int \frac{x}{(x^2+a^2)\sqrt{x^2+b^2}}\, dx \begin{cases} \frac{1}{\sqrt{a^2-b^2}} \tan^{-1}\left\{\frac{\sqrt{x^2+b^2}}{a^2-b^2}\right\}, \text{ if } a^2 > b^2 \\ \frac{1}{2\sqrt{b^2-a^2}} \log\left|\frac{\sqrt{x^2+b^2}-\sqrt{b^2-a^2}}{\sqrt{x^2+b^2}+\sqrt{b^2-a^2}}\right|, \text{ if } a^2 < b^2 \end{cases}$$

(29) Evaluate $\int \sqrt{\cos ecx - 1}\, dx$

(30) $\int \frac{\sqrt{1+x}}{x}\, dx$

(31) Evaluate $\int \frac{\cos^2 x \sin x}{\sin x - \cos x}\, dx$

(32) Evaluate $\int \frac{\cos^2 x + \sin 2x}{(2\cos x - \sin x)^2}\, dx$

(33) $\int \frac{\sin^{-1} x}{(1-x^2)^{3/2}}\, dx$

(34) $\int \sin\sqrt{x}\ dx$

(35) $\int (\sin^{-1} x)^2\, dx$

(36) $\int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}$

(37) $\int \cot^{-1}(1-x+x^2)\ dx$

(38) $\int \tan^{-1}\left(\frac{2\cos^2\theta}{2-\sin 2\theta}\right)\sec^2\theta\ d\theta$

(39) $\int \sin^{-1}\frac{\sqrt{x}}{a+x}dx$

(40) $\int \frac{\cos^2 x + \sin 2x}{(2\cos x - \sin x)^2}dx$

(41) $\int e^{\sin x}\left(\frac{x\cos^3 x - \sin x}{\cos^2 x}\right)dx$

(42 $\int \cos 2x \log(1+\tan x)\ dx$

(43) If f(x) is a polynomial function of the degree prove that
$\int e^x f(x)dx = e^x\{f(x) - f'(x) + f''(x) - f'''()x \ldots (+1)^{\ n} f^{(n)})x$, where

$f^n(x) = \frac{d^n f(x)}{dx^n}$

(44) Evaluate $\int e^{x}\sin x+\cos x.\left\{\dfrac{x^{4}\cos^{3}x-x\sin x+\cos x}{x^{2}\cos^{2}x}\right\}$ dx

(45) $\int \dfrac{\tan^{-1}x}{x^{2}(1+x^{2})}dx$

(46) $\int \sec^{2}x\log(1+\sin^{2}x)\,dx$

(47) $\int \dfrac{(x+1)\{\sqrt{x^{2}+1}+1\}}{\sqrt{x^{2}+1}}\,e^{x}\,dx$

(48) $\int \dfrac{1}{\sin x(2\cos^{2}x-1)}\,dx$

(49) $\int \dfrac{1}{x^{6}-a^{6}+a^{2}x^{2}(x^{2}-a^{2})}\,dx$

(50) $\int \dfrac{x+1}{x(1+xe^{x})^{2}}\,dx$