Binomial Theorem

BINOMIAL THEOREM FOR POSITIVE INTEGRAL INDEX

Synopsis:

1. f x and a are real numbers, then for all $n \in N$,

$$(x+a)^n = {}^nC_o x^n a^o + {}^nC_1 x^{n-1}a^1 + {}^nC_2 x^{n-2}a^2 + ...$$

$$+....+ {}^nC_r x^{n-r}a^r + ...+ {}^nC_{n-1}x^1 a^{n-1} + {}^nC_n x^o a^n \text{ i.e}$$

Formula: $(x+a)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r}a^r$.

2. MIDDLE TERM IN A BINOMIAL EXPANSION

Synopsis:

If n is an even natural number, then in the binomial expansion of $(x+a)^n$, $\left(\frac{n}{2}+1\right)th$

term is the middle term.

Formula: If is odd natural number , then $\left(\frac{n+1}{2}\right)th$ and $\left(\frac{n+3}{2}\right)th$ are the middle terms

In the binomial expansion of $(x+a)^n$.

3. GREATEST TERM

Synopsis:

Let T_{r+1} and T_r be(r+1)th and rth terms respectively in the expansion of $(x+a)^n$. Then, $T_{r+1} = {}^nC_r x^{n-r} a^r$ and $T_r = {}^nC_{r-1}\, x^{n-r+1}\, a^{r-1}$.

$$\therefore \quad \frac{T_{r+1}}{T_r} = \frac{{}^nC_r\, x^{n-r}\, a^r}{{}^nC_{r-1}\, x^{n-r+1}\, a^{r-1}} = \frac{n!}{(n-r)!r!} \times \frac{(r-1)!(n-r+1)!}{n!} . \frac{a}{x} = \frac{n-r+1}{r} . \frac{a}{x}$$

CASE1 When $\frac{n+1}{1+\frac{x}{a}}$ is an integer

Let $\frac{n+1}{1+\frac{x}{a}} = m$. Then, from (i), we have

$T_{r+1} > T_r$ for r=1,2,3,…..(m -1) ……(i i)

$T_{r+1} = T_r$ for r = m ….(iii)

and, $T_{r+1} < T_r$ for r = m +1, …. n …(iv)

$\therefore$ $T_2 > T_1$, $T_3 > T_2$, $T_4 > T_3$, ……, $T_m > T_{m-1}$ [From (ii)]

$T_{m+1} = T_m$ [From (iii)]

and, $T_{m+2} < T_{m+1}$, $T_{m+3} < T_{m+2}$, … $T_{n+1} < T_n$ [From (iv)]

$\Rightarrow$ $T_1 < T_2 < \ldots. < T_{m-1} < T_m = T_{m+1} > T_{m+2} > \ldots >$ ${}_nT$

This shows that mth and (m+1)th terms are greatest terms.

Case II: When $\frac{n+1}{1+\frac{x}{a}}$ is not an integer.

Let m be the integral part of $\frac{n+1}{1+\frac{x}{a}}$. Then, from (i), we have

$$T_{r+1} > T_r \; for = 1, 2....., m \qquad \ldots(v)$$

and, $T_{r+1} < T_r \;\; for \;\; r = m+1, \; m+2, ...n$ ….(vi)

$\therefore$ $T_2 > T_1, \; T_3 > T_2, ..., T_{m+1} > T_m$ [From (v)]

and, $T_{m+2} < T_{m+1}, \; T_{m+3} < T_{m+2}, ..., T_{n+1} < T_n$ [From (vi)]

$\Rightarrow$ $T_1 < T_2 < T_3 < .... < T_m < T_{m+1} > T_{m+2} > T_{m+3...} > T_{n+1}$

$\Rightarrow$ (m+1)th term is the greatest term.

4. MULTINOMIAL THEOREM

Using binomial theorem, we have $x+a)^n = \sum_{r=0}^{n} {}^nC_r \, x^{n-r} \, a^r, n \in N$

$$= \sum_{r=0}^{n} \frac{n!}{(n-r)!r!} x^{n-r} a^r$$

$$= \sum_{r+s=n} \frac{n!}{r!s!} x^s \, a^r, \quad \text{where s=n-r.}$$

This result can be generalized in the following form :

$$(x_1 + x_2 + ... + x_k)^n = \sum_{r_1+r_2+....+r_k=n} \frac{n!}{r_1!r_2!....r_k!} x_1^{r_1} x_2^{r_2} ....x_k^{r_k}$$

The general term in the above expansion is

$$\frac{n!}{r_1!r_2!r_3!.....r_k!} x_1^{r_1} \, x_2^{r_2} \, x_3^{r_3} ....x_k^{r_k}$$

The number of terms in the above expansion is equal to the number of non-negative integral solution of the equation.

$r_1+r_2+.....+r_k = n$, because each solution of this equation gives a term in the above

expansion. The number of such solutions is ${}^{n+k-1}C_{k-1}$.

5. PARTICULAR CASES

(i) $(x+y+z)^n = \sum_{r+s+t=n} \frac{n!}{r!s!t!} x^r\ y^s\ z^t$

The above expansion has ${}^{n+3-1}C_{3-1} = {}^{n+2}C_2$ terms.

(ii) $(x+y+z+u)^n = \sum_{p+q+r+s=n} \frac{n!}{p!\,q!\,r!\,s!} x^p\ y^q\ z^r\ u^s$

There are ${}^{n+4-1}C_{4-1} = {}^{n+3}C_3$ terms in the above expansion.

REMARK The greatest coefficient in the expansion of $(x_1 + x_2 + ... + x_m)^n$ is $\frac{n!}{(q!)^{m-r}[(q+1)!]^r}$, where q and r are the quotient and remainder respectively when n is divided by m.

PROPERTIES OF THE BINOMIAL COEFFICEINT

PROPERTY I In the expansion of $(1+x)^n$ the coefficients of terms equidistant from the beginning and the end are equal.

PROPERTY II The sum of the binomial coefficients in the expansion of $(1+x)^n$ is 2^n.

i.e, $C_o + C_1 + C_2 + ... + C_n = 2^n$ or, $\sum_{r=0}^{n} {}^nC_r = 2^n$.

PROPERTY III The sum of the coefficients of the odd terms in the expansion of $(1+x)^n$ is equal to the sum of the coefficients of the even terms and each is equal to 2^{n-1}.

i.e, $C_o+C_2+C_4+\ldots. = C_1+C_3+C_5+\ldots \ldots = 2^{n-1}$.

PROPERTY IV Prove that:

${}^nC_r = \frac{n}{r}.{}^{n-1}C_{r-1} = \frac{n}{r}.\frac{n-1}{r-1}.{}^{n-2}C_{r-2}$ and so on.

PROPERTY V $C_o - C_1 + C_2 - C_3 + C_4 - \ldots + (-1)^n C_n = 0$

1. Find the coefficient of x^m in the expression $(1+x)^n + 2(1+x)^{n-1} + 3(1+x)^{n-2} + \ldots. + (n-m+1)(1+x)^m$ where $0 \le n$.

2. Find the sum of the series : $\sum_{r=0}^{n} (-1)^r\ {}^nC_r \left\{ \frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{3r}} + \frac{15^r}{2^{4r}} + ...upto\ m\ \ terms \right\}$

3. Find the sum of the series $\sum_{r=0}^{n} (-1)^r\ {}^nC_r \left\{ \frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{3r}} + \frac{15^r}{2^{4r}} + ... + to\ \infty \right\}$

4. If k and n be positive integers and $s_k = 1^k + \ldots + k^n$, then show that $\sum_{r=1}^{m} {}^{m+1}C_r\ s_r = (n+1)^{m+1} - (n+1)$

5. Prove that : $2^k\binom{n}{0}\binom{n}{k}-2^{k-1}\binom{n}{1}\binom{n-1}{k-1}+2^{k-2}\binom{n}{2}\binom{n-2}{k-2}$

$+\ldots..+\quad +(-1)^k+\ldots.+(-1)\binom{n}{k}\binom{n-k}{0}=\binom{n}{k}$, where $\binom{n}{k}={}^nC_k$

6 In the expansion of the binomial expression $(x+a)^{15}$, if the eleventh term is the geometric mean of the eighth and twelfth terms, which term in the expansion is the greatest.

7. Prove that the greatest term in the expansion of $(1+x)^{2n}$ has also the greatest coefficient, then $x\in\left(\frac{n}{n+1},\frac{n+1}{n}\right)$.

8. $1-\left(\frac{1+x}{1+nx}\right)C_1+\left(\frac{1+2x}{(1+nx)^2}\right)C_2-\left(\frac{1+3x}{(1+nx)^3}\right)C_3+\ldots.\ +(-1)^n\left(\frac{1+nx}{(1+nx)^n}\right)C_n=0$

9. If ${}^nC_o$, ${}^nC_1$, ${}^nC_1$, ${}^nC_2$, ….., ${}^nC_n$ denote the binomial coefficients in the expansion of $(1+x)^n$ and p+q=1, then prove that

(i) $\sum_{r=0}^{n} r\ {}^nC_r\ p^r\ q^{n-r}=np$ (ii) $\sum_{r=0}^{n} r^2\ {}^nC_r\ p^r\ q^{n-r}=n^2p^2+npq$

10. Evaluate $\sum_{r=0}^{n}\frac{3r+2}{r^3+6r^2+11r+6}\,{}^nC_r$, Where ${}^nC_0$, ${}^nC_1$,.....${}^nC_n$ are the binomial coefficients in the expansion of $(1+x)^n$.

11. When $32^{(32)^{(32)}}$ is divided by 7, prove that the remainder is 4.

12. A is a set containing n elements. A subset P of A is chosen at random. The set A is reconstructed by replacing the elements of P. A subset Q is again chosen at random. Find the number of ways selecting P and Q so that.

(i) P and Q are disjoint sets. (ii) $P\cap Q$ contains just one element

(iii) $P\cup Q$ contains just one element

(iv) Q is a subset of P

(v) $P\cap Q=\phi$

(vi) P and Q have equal number of elements

(vii) Q contains just one element more than P

(viii) $P\cup Q=A$ (ix) P = Q

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Numerically greatest Term in the Expansion of (x + a)n

Let T_r and T_{r+1} be r^{th} and $(r+1)^{th}$ terms respectively in the expansion of bionomial

$(x+a)^n$. Then $T_r = {}^nC_{r-1}\, x^{n-r+1}\, a^{r-1}$ and $T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$

$$\therefore \quad \frac{T_{r+1}}{T_r} = \frac{n-r+1}{r}\frac{a}{x}$$

Now, $T_{r+1} >$, = , $< T_r$ According as $\left(\frac{n-r+1}{r}\right)\frac{a}{x} >, =, < 1$, i.e. according as $\frac{n+1}{r} - 1 >, =, < \frac{x}{a}$, i.e. according as $r <, =, > \frac{n+1}{\frac{x}{a}+1}$.

So, if $\frac{n+1}{\frac{x}{a}+1}$ is an integer, say p, then $T_{r+1} > T_r$ if $r < p$ otherwise $T_{r+1} \leq T_r$

So, $T_p = T_{p+1}$ (numerically) and these are greater than any other term in the expansion.

Next, if $\frac{n+1}{\frac{x}{a}+1}$ is a non-integer, suppose m be its integral part then $T_{r+1} < T_r$ if $r \leq m$ and

$T_{r+1} < T_r$ if $r > m$.

So, T_{m+1} is the numerically greatest term among the terms of the expansion.

Again we can also write that k^{th} term is numerically greatest if $T_k > T_{k+1}$ and T_{k-1}.

1. The numerically greatest term in the expansion of $(1-2x)^8$, when x=2 is

   a) ${}^8C_6\, 4^6$ b) ${}^8C_4\, 4^4$ c) 2^{17} d) None of these

2. Magnitude wise the greatest term in the expansion of $(3, -2x)^9$ when x=1 is

   a) ${}^9C_2\, 3^7\, 2^2$ b) ${}^9C_3\, {}^36\, 2^3$ c) ${}^9C_4\, 3^5\, 2^4$ d) both (b) and (c)

3. If x > 0 and the 4^{th} term in the expansion of $\left(2+\frac{3}{8}x\right)^{10}$ has maximum value, then

   a) $2 < x < 3$ b) $3 < x < \frac{10}{3}$ c) $4 < x < 5$ d) None of these

4. If n is even positive integer, then the condition that the numerically greatest term in the expansion of $(1+x)^n$ may have the greatest coefficient also is

   a) $\frac{n}{n+2} < |x| < \frac{n+2}{n}$ b) $\frac{n+1}{n} < |x| < \frac{n}{n+1}$

   c) $\frac{n}{n+4} < |x| < \frac{n+4}{n}$ d) None of these

5. The interval in which x must lies so that the numerically greatest term in the expansion of $(1-x)^{21}$ has the greatest coefficient is, (x > 0).

a) $\left[\frac{5}{6},\frac{6}{5}\right]$ b) $\left(\frac{5}{6},\frac{6}{5}\right)$ c) $\left(\frac{4}{5},\frac{5}{4}\right)$ d) $\left[\frac{4}{5},\frac{5}{4}\right]$