Scalar or Dot Product of Vectors

DEFINITION

Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors inclined at an angle θ. Then, the scalar product of $\vec{a}$ with $\vec{b}$ is denoted by $\vec{a}.\vec{b}$ and is defined as the scalar $|\vec{a}||\vec{b}|\cos\theta$

(1) Show that the angle between two diagonals of a cube is $\cos^{-1}\left(\frac{1}{3}\right)$.

(2) The length of the sides a, b, c of a triangle ABC are related as $a^2 + b^2 = 5c^2$. Prove, using vector methods, that the medians drawn to the sides a and b are perpendicular.

(3) Determine the lengths of the diagonals of a parallelogram constructed on the vectors $\hat{a} = 2\hat{\alpha} - \hat{\beta}$ and $\hat{b} = \hat{\alpha} - 2\hat{\beta}$, where $\hat{\alpha}$ and $\hat{\beta}$ are unit vectors forming an angle of 60^o.

(4) Two points A and B are given on the curve $y=x^2$ such that $\overrightarrow{OA}.\ \hat{i} = 1$ and $\overrightarrow{OB}.\ \hat{i} = -2$. Find $|2\overrightarrow{OA} - 3\overrightarrow{OB}|$

VECTOR (CROSS) PRODUCT OF VECTORS

DEFINITION

VECTOR (CROSS) PRODUCT Let $\vec{a}, \vec{b}$ be two non – zero non-parallel vectors. Then the vector product $\vec{a} \times \vec{b}$, in that order, is defined as a vector whose magnitude is $|\vec{a}||\vec{b}|\sin\theta$ where θ is the angle between $\vec{a}$ and $\vec{b}$ and whose direction is perpendicular to the plane of $\vec{a}$ and $\vec{b}$ in such a way that $\vec{a}$, $\vec{b}$ and this direction constitute a right handed system.

PROPERTIES OF VECTOR PRODUCT

The vector product has the following properties:

(1) Vector product I not commutative i.e. if $\vec{a}$ and $\vec{b}$ are any two vectors, then $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$, however $-\vec{b} \times \vec{a}$.

(2) If $\vec{a}$ and $\vec{b}$ are two vectors and m is a scalar, then $m\vec{a} \times \vec{b} = m(\vec{a} \times \vec{b}) = \vec{a} \times m\vec{b}$

(3) If $\vec{a}, \vec{b}$ are two vectors and m, n are scalars, then $m\vec{a} \times n\vec{b} = mn(\vec{a} \times \vec{b}) = m(\vec{a} \times n\vec{b}) = n(m\vec{a} \times \vec{b})$

(4) Vector product is distributive over vector addition i.e. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$ and, $(\vec{b} + \vec{c}) \times \vec{a} = \vec{b} \times \vec{a} + \vec{c} \times \vec{a}$

(5) For any three vectors $\vec{a}, \vec{b}, \vec{c}$, we have $\vec{a} \times (\vec{b} - \vec{c}) = \vec{a} \times \vec{b} - \vec{a} \times \vec{c}$.

(6) The vector product of two non-zero vectors is she the null vector if they are collinear or parallel i.e. $\vec{a} \times \vec{b} = \vec{0} \Leftrightarrow \vec{a} \| \vec{b}$, where $\vec{a}, \vec{b}$ are non-null vectors

(7) $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \hat{0},\ \hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{i} = -\hat{k},\quad \hat{j} \times \hat{k} = \hat{i},\ k \times \hat{j} = \hat{i},\ \hat{k} \times \hat{i} = \hat{j},\ \hat{i} \times \hat{k} = -j$

(8) If $\vec{a} = a_1\hat{i} + a_2\hat{k}$, and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $\vec{a} \times \hat{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

SOME USEFUL RESULTS

RESULT I If $\vec{a}$ and $\vec{b}$ are two non-zero, non-parallel vectors, then unit vectors normal to the plane of $\vec{a}$ and $\vec{b}$ are $\pm\frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

RESULT II Vectors of magnitude λ normal to the plane of $\vec{a}$ and $\vec{b}$ are $\pm\lambda\left(\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}\right)$

RESULT III The area of the parallelogram with adjacent sides $\vec{a}$ and $\vec{b}$ is $|\vec{a}\times\vec{b}|$

RESULT IV The area of a triangle with adjacent sides $\vec{a}$ and $\vec{b}$ is $\frac{1}{2}|\vec{a}\times\vec{b}|$

RESULT V Area of $\Delta ABC = \frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$

$= \frac{1}{2}|\overrightarrow{BC}\times\overrightarrow{BA}|$

$= \frac{1}{2}|\overrightarrow{CA}\times\overrightarrow{CB}|$

RESULT VI The area of a plane quadrilateral ABCD is $\frac{1}{2}|\overrightarrow{AC}\times\overrightarrow{BD}|$, where AC and BD are its diagonals.

(1) If $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of the vertices A, B, C of a triangle ABC, show that the area of triangle ABC is $\frac{1}{2}|\vec{a}\times\vec{b}+\vec{c}+\vec{c}\times\vec{a}|$.

(2) Show that the perpendicular distance of the point $\vec{c}$ from the line joining $\vec{a}$ and $\vec{b}$ is $\frac{|\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b}|}{|\vec{b}-\vec{a}|}$

(3) If A, B, C, D be any four points in space, prove that $|\overrightarrow{AB}\times\overrightarrow{CD}+\overrightarrow{BC}\times\overrightarrow{AD}+\overrightarrow{CA}\times\overrightarrow{BD}|$ = 4 (Area of triangle ABC).

(4) Let $\overrightarrow{OA}=\vec{a}, \overrightarrow{OB}=10\vec{a}+2\vec{b},$ and $\overrightarrow{OC}=\vec{b}$ where O is origin. Let p denote the area of the quadrilateral OABC and q denote the area of the parallelogram with OA and OC as adjacent sides. Prove that p = 6q.

(5) ABCD is a quadrilateral such that $\overrightarrow{AB}=\vec{b}, \overrightarrow{AD}=\vec{d}, \overrightarrow{AC}=m\vec{b}+p\vec{d}.$ Show that the area of the quadrilateral ABCD is $\frac{1}{2}|m+p||\vec{b}\times\vec{d}|$

PRODUCT OF THREE VECTORS

SCALAR TRIPLE PRODUCT

DEFINITION Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors. Then the scalar $(\vec{a}\times\vec{b}).\vec{c}$ is called the scalar product of $\vec{a}$, $\vec{b}$ and $\vec{c}$ and is denoted by $[\vec{a}\ \vec{b}\ \vec{c}]$.

Thus, $[\vec{a}\ \vec{b}\ \vec{c}] = (\vec{a}\times\vec{b})\,.\,\vec{c}$.

PROPERTIES OF SCALAR TRIPEL PRODUCT

PROPERTY-I If $\vec{a}, \vec{b}, \vec{c}$ are cyclically permuted the value of scalar triple product remains same,

i.e $(\vec{b}\times\vec{c}).\vec{a}=(\vec{c}\times\vec{a}).\vec{b}$ (or), $[\vec{a}\ \vec{b}\ \vec{c}]=[\vec{b}\ \vec{c}\ \vec{a}]=[\vec{c}\ \vec{a}\ \vec{b}]$

PROPERTY-II The change of cyclic order of vectors in scalar triple product changes the sign of the scalar triple product but not the magnitude.

i.e. $[\vec{a}\ \vec{b}\ \vec{c}]=-[\vec{b}\ \vec{a}\ \vec{c}]=-[\vec{c}\ \vec{b}\ \vec{a}]=-[\vec{a}\ \vec{c}\ \vec{b}]$

PROPERTY-III In scalar triple product the positions of dot and cross can be interchanged provided that the cyclic order of the vectors remains same i.e.

$(\vec{a}\times\vec{b}).\ \vec{c}=\vec{a}.\ \vec{a}.\ (\vec{b}\times\vec{c})$

PROPERTY-IV The scalar triple product of three vectors is zero if any two of them are equal.

$[\lambda\vec{a}\ \ \vec{b}\ \vec{c}] = \lambda[\vec{a}\ \vec{b}\ \vec{c}]$

PROPERTY-VI The scalar triple product of three vectors is zero if any two of them are parallel or collinear.

PROPERTY-VII If $\vec{a}\ \vec{b}\ \vec{c}\ \vec{d}$, are four vectors, then $[\vec{a}+\vec{b}\ \vec{c}\ \vec{d}] = [\vec{a}\ \ \vec{c}\ \vec{d}] + [\vec{b}\ \ \vec{c}\ \vec{d}]$

PROPERTY-VIII The necessary and sufficient condition for three non-zero, non-collinear vectors $\vec{a}, \vec{b}, \vec{c}$ to be coplanar is that $[\vec{a}\ \vec{b}\ \vec{c}] = 0$. i.e. $\vec{a}, \vec{b}, \vec{c}$ are coplanar $\Leftrightarrow$ $[\vec{a}\ \vec{b}\ \vec{c}] = 0$

PROPERTY-IX For points with position vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ will be coplanar if

$[\vec{d}, \vec{b}, \vec{c}]+[\vec{d}, \vec{c}, \vec{a}]+[\vec{d}, \vec{a}, \vec{b}] = [\vec{a}, \vec{b}, \vec{c}]$

PROPERTY-X If $\vec{a} = a, \hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ are three vectors, then

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(1) If the vectors $\vec{\alpha} = a\hat{i} = a\hat{j} + c\hat{k}$, $\vec{\beta} = \hat{i} + \hat{k}$ and $\vec{\gamma} = c\hat{i} + c\hat{j} + b\hat{k}$ are coplanar, then prove that c is the geometric mean of a and b.

(2) Let $\vec{a}, \vec{b}, \vec{c}$ three non-zero vectors such that $\vec{c}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$. If the angle between $\vec{a}$ and $\vec{b}$ is $\pi/6$, prove that $[\vec{a}\ \vec{b}\ \vec{c}]^2 = \frac{1}{4}\ |\vec{a}|^2\,|\vec{b}|^2$.

(3) Consider the vectors: $\vec{A} = \hat{i} + \cos(\beta-\alpha)\hat{j} + \cos(\gamma-\alpha)\hat{k}$

$\vec{B} = \cos(\alpha-\beta)\hat{i} + \hat{j} + \cos(\gamma-\beta)\hat{k}$

and, $\vec{C} = \cos(\alpha-\gamma)\hat{i} + \cos(\beta-\gamma)\hat{j} + a\hat{k}$

where α, β are γ are different angles in $(0, \pi/2)$. If $\vec{A}, \vec{B}, \vec{C}$ are coplanar vectors, show that a is independent of α, β and γ.

(4) If $\vec{a}, \vec{b}, \vec{c}$ be three on – coplanar unit vectors equally inclined to one another at an angleθ such that $\vec{a}\times\vec{b} + \vec{b}\times\vec{c} = p\vec{a} + q\vec{b} + r\vec{c}$, find p, q, r in terms of θ. Also, prove that $p^2 + \frac{q^2}{\cos\theta} + r^2 = 2$.

(5) If $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors, prove that any vector $\vec{r}$ is expressible as

$$\vec{r} = \left\{\frac{[\vec{r}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]}\right\}\vec{a} + \left\{\frac{[\vec{r}\,\vec{c}\,\vec{a}]}{[\vec{a}\,\vec{b}\,\vec{c}]}\right\}\vec{b} + \left\{\frac{[\vec{r}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]}\right\}\vec{c}.$$

RECIPROCAL SYSTEM OF VECTORS

Let $\vec{a}, \vec{b}, \vec{c}$ be three non-coplanar vectors, vectors, so that $[\vec{a}\ \vec{b}\ \vec{c}] \neq 0$. We define another set of three vectors $\vec{a}, \vec{b}, \vec{c}$ as given below $\vec{a} = \frac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}.\vec{b} = \frac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]},\ \vec{c} = \frac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$

(1) If $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors and $\vec{a}, \vec{b}, \vec{c}$ form a reciprocal system of vectors, then prove that

(i) $\vec{a}.\vec{a} = \vec{b}.\vec{b} = \vec{c}.\vec{c} = 1$

(ii) $\vec{a}.\vec{b}=\vec{a}.\vec{c}=0;\ \vec{b}.\vec{c}=\vec{b}.\vec{a}=0;\ \vec{c}.\vec{a}=\vec{c}.\vec{b}=0$

(iii) $[\vec{a}\ \vec{b}\ \vec{c}]=\dfrac{1}{[\vec{a}\ \vec{b}\ \vec{c}]}$

(2) If $\vec{a}\ \vec{b}\ \vec{c}$ and $\vec{a}'$, $\vec{b}'$, $\vec{c}'$ be the reciprocal system of vectors, prove that

(i) $\vec{a}.\vec{a}+\vec{b}.\vec{b}+\vec{c}.\vec{c}=3$ (ii) $\vec{a}\times\vec{a}+\vec{b}\times\vec{b}+\vec{c}\times\vec{c}=\vec{0}$

(3) If $\vec{a}$ and $\vec{b}$ are two vectors such that $\vec{a}.\vec{b}\neq 0,$ then solve the vector equations $\vec{r}.\vec{a}=0,\ \vec{r}.\vec{b}=1,\ \ [\vec{r}\ \vec{a}\ \vec{b}]=1.$

(4) If $\vec{r}\times\vec{a}+(\vec{r}.\vec{b})\vec{c}=\vec{d},$ then prove that $\vec{r}=\lambda\vec{a}+\vec{a}\times\left\{\dfrac{\vec{a}\times(\vec{d}\times\vec{c}}{(\vec{a}.\vec{c})|\vec{a}|^2}\right\},$ where λ is a scalar

VOLUME OF A TETRAHEDRON

THEOREM (i) If two pairs of opposite edges of a tetrahedron are perpendicular, then the opposite edges of the third pair are also perpendicular to each other.

(ii) The sum of the squares of two opposite edges is the same for each pair of opposite edges

(iii) Any two opposite edges in a regular tetrahedron

CENTROID OF A TETRAHEDRON

THEOREM The volume V of a tetrahedron whose three coterminous edges in the right-handed system are $\vec{a},\vec{b}\ \vec{c}$ is given by $V=\dfrac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]$

(1) A tetrahedron has three of its vertices of its vertices at A, B and C whre $\overrightarrow{OA}=3\hat{i}+2\hat{j},\ \overrightarrow{OB}=\hat{i}+3\hat{j}-\hat{k};\ \overrightarrow{OC}=2\hat{j}$. Find the unit vector perpendicular to the face ABC. The fourth vertex D is such that $\overrightarrow{DA}.\overrightarrow{AB}=0=\overrightarrow{DA}.\overrightarrow{AC}.$ Find the vector equation of AD. If the volume of the tetrahedron is $3\sqrt{2}$ cubic units and D is on the same side as the origin, find the coordinates of D.

(2) The position vectors of the vertices A, B and C of a tetrahedron ABCD are $\hat{i}+\hat{j}+\hat{k},\hat{i}$ and $3\hat{i}$ respectively. The altitude from vertex D to the opposite face ABC meets the median line through A of the triangle ABC at a point E. If the length of the side AD is 4 and the volume of the tetrahedron is $\dfrac{2\sqrt{2}}{3}$ cubic units, find the position vector of the point E for all its possible positions.

(3) OABC is a regular tetrahedron. D is the circumcentre of ΔOAB and E is the middle point of the edge AC. Use vector method to find distance DE. .

(4) A pyramid with vertex at the point P whose position vector is $4\hat{i}+2\hat{j}+2\sqrt{3}\hat{k}$ has a regular hexagonal base ABCDEF. The points A and B have position vectors $\hat{i}$ and $\hat{i}+2\hat{j}$ respectively. The centre of hexagon has position vector $\hat{i}+\hat{j}+\sqrt{3}\hat{k}$. Given that the volume of the pyramid is $6\sqrt{3}$ and the perpendicular from the vertex meets the diagonal AD, locate the position vectors of the foot of this perpendicular.

(5) Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ be three non-zero vectors such that $\vec{c}$ is a unit vector perpendicular to both the vectors $\vec{a}$ and $\vec{b}$. If the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{6}$, prove that

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \frac{1}{4}(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$$

(6) If $\vec{a}, \vec{b}, \vec{c}$ are three on-coplanar vectors and $\vec{r}$ is any vector in space, then prove that

$$\vec{r} = \frac{\vec{r}.\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}(\vec{b}\times\vec{c}) + \frac{\vec{r}.\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}(\vec{c}\times\vec{a}) + \frac{\vec{r}.\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}(\vec{a}\times\vec{b})$$

(7) If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are four vectors, then prove that

(i) $(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d}) + (\vec{b}\times\vec{c}).(\vec{a}\times\vec{d}) + (\vec{c}\times\vec{a}).(\vec{b}\times\vec{d}) = 0$

(ii) $\vec{d}.[\vec{a}\times\{\vec{b}\times\vec{c}\times\vec{d})\}] = [\vec{b}.\vec{d}][\vec{a}\ \vec{c}\ \vec{d}]$