Mean Values Theorems and Some Other Applications of Derivatives

ROLE'S THEOREM

STATEMENT Let f be a real valued function defined on the closed interval [a, b] such that

(i) It is continuous on the closed interval [a, b],
(ii) it is differentiable on the open interval (a, b), and
(iii) f(a) = f(b).

LAGRANGE'S MEAN VALUE THEOREM

STATEMENT Let f(x) be a function defined on [a, b] such that
it is continuous on [a, b] (ii) it is differentiable on (a, b).

Then there exists a real number c $\in$ (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

(1) Prove that $(b-a)\sec^2 a < \tan b - \tan a < (b-a)\sec^2 b$, where $0 < a < b < \frac{\pi}{2}$.

(2) If f(x) and g(x) are continuous functions in [a, b] and they are differentiable in (a, b), then prove that there exists c $\in$ (a, b) such that $\begin{vmatrix} f(a) & f(b) \\ g(a) & f(b) \end{vmatrix} = (b-a)\begin{vmatrix} f(a) & f'(b) \\ g(a) & g'(c) \end{vmatrix}$

(3) If the function of f: [0, 4] $\rightarrow$ R is differentiable, then show that
(1) $(f(4))^2 - (f(0))^2 = 8f'(a)\ f(b)$ for a, b$\in$ (0, 4)
(2) $\int_0^4 f(t)\,dt = 2[\alpha f(\alpha^2) + \beta f(\beta)^2)]$ for all $0 < \alpha, \beta < 2$.

(4) If $2a+3b+6c=0$, then prove that the equation $ax^2+bx+c=0$ has at least one real root in (0, 1)

(5) If $a_o x^n + a_1 x^{n-1} + \ldots + a_{n-1}x = 0$ has a real positive root α, then prove that the equation $na_0 x^{n-1} + (n-1)a_1 x^{n-2} + \ldots + a_{n-1} = 0$ has a positive real root less than α