# Functions

**INTERVALS:**
Let a and b two given real numbers such that a < b. Then the set of all real numbers x such that $a \le x \le b$ is called a closed interval and is denoted by [a, b].
i.e. [a, b] = {x ∈ r |a ≤ x≤b}
For example, [1, 2] = {x ∈ R| 1 ≤ x ≤ 2} i.e., the set of all real numbers lying between 1 and 2, including the end points.

**OPEN INTERVAL:** Let a and b be two given real numbers such that a < b. Then the set of all real numbers x such that a ≤ x b is called a closed interval and is denoted by (a, b).
i.e (a, b) = {x ≤ R | a ≤ x ≤ b}

**REAL FUNCTION:**
If the domain and co-domain of a function are subsets R9st of all real numbers). It is called a real valued function or in short a real function.

**EXAMPLES:**

(i) If for non-zero x, a f(x) +b $f\left(\frac{1}{x}\right) = \frac{1}{x}$-5, where $a \neq b$ then find f(x).

(ii) Let g : $R \to R$ be given by g(x) = 4x + 3. If $g^n(x) = \underbrace{gogog....og}_{n-times}(x)$, show that

$g^n(x) = 4^n x + (4^n-1)$.

If $g^{-n}(x)$ denotes the inverse of $g^n(x)$, prove that $g^{-n}(x) = 4^{-n}x + (4^{-n} - 1)$
for all x ∈ R.

**DOMAIN:** Generally real functions in calculus are described by some formula and their domains are not explicitly stated. In such cases to find the domain of a function f (say) we use the fact that domain is the set of all real numbers x for which f (x) is a real number.

In other words, determining the domain of a function f means finding all real numbers x for which f(x) is real. For example, if f(x) = $\sqrt{2-x}$, then f(x) is real for all
x ≤ 2. For x > 2, f(x) is not real. So, domain of f(x) is the set of all real numbers less than or equal to 2 i.e. (-∞, 2]

**EXAMPLE:** Find the domain and range of function $f(x) = \frac{1}{2-\cos 3x}$.

**SOME STANDARD REAL FUNCTIONS**

**CONSTANT FUNCTION:** Let k be a fixed real number. Then a function f(x) given by f(x) =k for all $x \in R$ is called a constant function.

**GREATEST INTEGER FUNCTION:** For any real number x, we denote [x], the greatest integer less than or equal to x.

**PROPERTIES OF GREATEST INTEGER FUNCTION:**
If n is an integer and x is any real number between n and n+1, then the greatest integer function has the following properties :

(i) [-n] = -[n] (ii) [x + n] = [x] + n (iii) [-x] =-[x]-1

(iv) $[x]=[-x]=\begin{cases}-1, & if\ x \notin Z \\ 0, & if\ x \in Z\end{cases}$ (v) $\begin{cases}2[x], & if\ x \in Z \\ 2[x]+1, & if\ x \notin Z\end{cases}$

(vi) [x] ≥ n ⇒ x≥n, where n ∈ Z (vii) [x]≤n ⇒ x < n + 1, n∈ Z

(viii) [x] > n ⇒ x ≥n + 1, n∈ Z (ix) [x] < n ⇒x < n, n∈ Z

(x) [x+y] = [x] + [y=X-[X]] for all x, y ∈ R.

(xi) $[x] + \left[x+\frac{1}{n}\right]+\left[x+\frac{2}{n}\right]+....+\left[x+\frac{n-1}{n}\right]=[nx]$, n ∈ N

(1) If [x] and [x] denote respectively the fractional and integral parts of a real number x. Solve the equation 4[x] = x+[x]

(2) I f[x] and [x] denote the fractional and integral parts of x and (x) is defined as follows $(x)\begin{cases}2[x]-[x], & x<0\\ [x]+3[x], & x\geq 0\end{cases}$

then solve the equation : (x) = x + {x}

**SIGNUM FUNCTION:**

The function defined by f(x) = $\begin{cases}\frac{|x|}{x}, & x\neq 0\\ 0 \ , & x=0\end{cases}$ (Or) f(x) = $\begin{cases}1, & x>0\\ 0 \ , & x=0\\ 0 \ , & x=0\end{cases}$

is called signum function.

**RECIPROCAL FUNCTION:** The function that associates each nonzero real number x to its reciprocal 1/x is called the reciprocal function.

**LOGARITHMIC FUNCTION:** If 'a' is a positive real number, then the function that associates every positive real number to $\log_a x$ i.e. $f(x) = \log_a x$ is called the logarithmic function.

**EXPONENTIAL FUNCTION:** If a is positive real number, then the function which associates every real number x to $a^x$ i.e. $f(x) = a^x$ is called the exponential function.

**SQUARE ROOT FUNCTION:** The function that associates every positive real number x to $+\sqrt{x}$ is called the square root function, i.e., $f(x) = +\sqrt{x}$.

**POLYNOMIAL FUNCTION**: A function of the form $f(x) = a_o x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n$, where $a_o$, $a_1$, $a_2$, … $a_n$ are real numbers, $a_o \neq 0$ and $n \in N$, is called polynomial function of degree n.

The domain of a polynomial function is always R.

**RATIONAL FUNCTION:** A function of the form f(x) = $\frac{P(x)}{q(x)}$, where p(x) and q(x) are polynomials and $q(x) \neq 0$, is called a rational function.

**SUM** Let f and g be two real functions with domain $D_1$ and $D_2$ respectively. Then, we define their sum f + g as that function from $D_1 \cap D_2$ to R which associates each $x \in D_1 \cap D_2$ to the number f(x) + g(x).

Thus, f+g: D1 ∩ $D_2 \rightarrow$ R such that (f+g) (x) =f(x) + g(x) for all $x \in D_1 \cap D_2$.

Similarly, we define the difference, product and quotient as follows:

**DIFFERENCE** f-g : $D_1 \cap D_2 \rightarrow$ R such that (f-g) 9x) = f(x) – g(x) for all x∈ $D_1 \cap D_2$

**PRODUCT** $fg : D_1 \cap D_2 \rightarrow R$ such that (fg) (x)=f9x0 g(x) for all $x \in D_1 \cap D_2$

**QUOTIENT** $\frac{f}{g} : D_1 \cap D_2 - \{x \mid g(x)=0\} \rightarrow R$ such that $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ for all $x \in D_1 \cap D_2 - \{x \mid g(x)=0\}$.

**SCALAR MULTIPLE** for any real number c, the function cf is defined by

(cf) (x) = c.f(x) for all x $\in$ $D_1$.

**REMARK** Note that the above operations are defined here are true only for real functions. For general functions from one set to another, these do not make sense.

**COMPOSITION OF FUNCTIONS**: Let f and g be two functions with domain $D_1$ and $D_2$ respectively. If range (f) $\subset$ domain g (g), we define gof by the rule

(gof) (x) = g(f(x)) for all x $\in$ $D_1$.

Also, if range (g) $\subset$ domain (f), we define fog by the rule (fog) (x) = f(g(x)) for all $x \in D_2$

It follows from the above discussion that if f(x) and g(x) are two real functions with domains $D_1$ and $D_2$ respectively. Then

(i) Domain of $(f \pm g) = D_1 \cap D_2$ (ii) Domain of $(fg) = D_1 \cap D_2$

(iii) Domain of $\left(\frac{f}{g}\right) = D_1 \cap D_2 - \{x \mid g9x) = 0\}$

(1) For what real values of 'a' does the range of the function $f(x) = \frac{x-1}{a-x^2+1}$ not contain any values belonging to the interval [-1, -1/3] ?

(2) For what real values of 'a' does the range of the function f(x) = $\frac{x-1}{1-x^2-a}$ not attain any value from the interval [-1, 1]?

Fin the domains of definition of the following functions:

**PERIODIC FUNCTIONS:**

**PERIOD** If f(x) is a periodic function, then the smallest positive real number T is called the period or fundamental period of function f(x) if.

F(x+T) = f(x) for all $x \in R$.

(1) Prove that the function (x) = x-[x] is a periodic function. Also find its period.

(2) Let f(x) be a real valued function with domain R such that

f(x + p) = 1+[2-3 f(x) + 3 $(f(x))^2$ – $(f(x)^3]^{1/3}$ hold good for all $x \in R$. and some positive constant p, then prove that f(x) is a periodic function.

**SOME USEFUL RESULTS ON PERIODIC FUNCTIONS**

**RESULT 1** If f(x) is a periodic function with periodic. T and a, b, $\in$ R such that $a \neq 0$, then af(x) + b is periodic with period T.

**RESULT 2** If f(x) is a periodic function with period T and a, b $\in$ R such that a $\neq$ 0, then f(ax+b0 is periodic with period T |a|.

**RESULT 3** Let f(x) and g(x) be two periodic functions such that :

Period of f(x) = $\frac{m}{n}$, where m, n $\in$ N and m, n are co-prime.

and,

Period of g(x) = $\frac{r}{s}$, where $r \in N$ and $s \in N$ are coprime 3

Then, (f+g) (x) is periodic with period T given by T = $\frac{LCM\ of\ (m,r)}{HCF\ of\ (n,s)}$, provided that there does not exist a positive number k < T for which f(k+x) = g (x) and g(k=x)=f(x), else k will be the period of (f+g) (x).

**EXAMPLE** Prove that $f(x) = \sin^{-1}(\sin x)$ is a periodic function

**EVEN FUNCTIONS** A function f(x) is said to be an even function if f(-x) = f(x) for all x.

**ODD FUNCTION** A function f(x) is said to be an odd function if (-x) = -f(x) for all x.

(1) If f is an even function defined on the interval [-5, 5], then find the real values of x satisfying the equation $f(x) = f\left(\frac{x+1}{x+2}\right)$.

(2) Extend $f(x) = x^2 + x$ defined in [0, 3] onto the interval [-3, 3] so that f(x)
(i) even (ii) odd.