Continuity and Differentiability

CONTINUITY AT A POINT

A function f(x) is said to be continuous at a point x = a of its domain, if $\lim_{x\to a} f(x) = f(a)$

CONTINUITY ON AN OPEN INTERVAL

A function f(x) is said to be continuous on an open interval (a, b) if it is continuous at every point on the interval (a, b).

CONTINUITY ON A CLOSED INTERVAL:

A function f(x) is said to be continuous on a closed interval [a, b] if

(i) f is continuous on the open interval (a, b)

(ii) $\lim_{x\to a^+} f(x) = f(a)$ (and) (iii) $\lim_{x\to b^-} f(x) = f(b)$

In other words, f(x) is continuous on [a, b] if it is continuous on (a, b) and it is continuous at a from the right and at b from the left.

PROPERTIES OF CONTINUOUS FUNCTIONS

(I) If f, g are two continuous functions at a point a of their common domain D, then f $\pm$ g, fg are continuous at a and if g(a) $\neq 0$, then f/g is also continuous at a.

(II) If f is continuous at a and f(a) $\neq$ 0, then there exists an open interval (a-δ, a+δ) such that for all $x \in (a-\delta, a+\delta)$, $f(x)$ has the same sign as f (a)

(III) If a function f is continuous on a closed interval [a, b], then it is bounded on [a, b] i.e there exist real numbers k and K such that $k \le f(x) \le K$ for all $x \in [a, b]$.

(IV) If f is a continuous function defined on [a, b] such that f (a) and f(b) are of opposite signs, then there exists at least one solution of the equation f(x) = 0 in the open interval (a, b)

(V) If f is continuous on [a, b], then f assumes atleast once, every value between its minimum and maximum values i.e. if K is any real number between minimum and maximum values of f(x), then there exists at least one solution of the equation f(x) = K in the open interval (a, b).

(VI) If f is continuous on [a, b] and maps [a, b] into [a, b], then for some $x \in [a, b]$ we have f(x) = x.

(VII) If g is continuous at a and f is continuous at g(a), the fog is continuous at a.

(1) A function is defined as follows $f(x) = \begin{cases} x^m \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$ What condition should be imposed on m so that f(x) may be continuous at x = 0 ?

(2) If f(x) = min {|x|, |x-2, 2-|x-1|}, draw the graph of f(x) and discuss its continuity and differentiability.

(3) If f(x) = $x^2 - 2|x|$ and $g(x) \begin{cases} Min\ \{f(t) : -2 \le t \le x, -2x < 0\} \\ Max\ \{f(t) : 0 \le t \le x, 0 \le x \le 3\} \end{cases}$

(i) Draw the graph of f(x) and discuss its continuity and differentiability.

(ii) Find and draw the graph of g(x). Also, discuss the continuity and differentiability of g(x).

(4) If $y = \cos^{-1}\sqrt{\frac{\cos 3x}{\cos^3 x}}$, then show that $\frac{dy}{dx} = \sqrt{\frac{3}{\cos x \cos 3x}}$