REAL NUMBERS

This chapter comes Under paper-I. From this chapter, 2 Marks Questions-1 (1×2 = 2M), 1 Marks Questions-2, (2×1= 2M), 4 Marks Questions-2 (2×4=8M) and 1/2 mark questions-6 (6×1/2=3M) altogether 15 Marks we can score easily by spending less time for preparation. The following material will be helped alot in the public exams for S.S.C students.

Key formulae:

- $a^m.a^n = a^{m+n}$;

  $\frac{a^m}{a^n} = a^{m-n}$ if $m > n$

  $= \frac{1}{a^{n-m}}$ if $m < n$

- $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

  $(ab)^m = a^m.b^m$

- $\frac{1}{a^m} = a^{-m}; \frac{1}{a^{-m}} = a^m$
- $a^\circ = 1$, where $a \neq 0$
- $\sqrt{a} \times \sqrt{b} = \sqrt{ab}; \sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$
- $\sqrt{a} = a^{1/2}; \sqrt[n]{a} = a^{1/n}; \sqrt[n]{a^m} = a^{m/n}$
- $\sqrt[n]{\sqrt[m]{a}} = \sqrt[mn]{a}$
- $\left(a^m\right)^n = a^{mn}$
- $\left(\sqrt[n]{a}\right)^n = a; \sqrt[n]{a^n} = a$
- $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

1. Solve $2^{x+3} = 4^{y-2}$ and $3^{x-2} = 9^{3y-2x}$

sol:

$2^{x+3} = 4^{y-2}$

$2^{x+3} = (2^2)^{y-2}$

$2^{x+3} = (2)^{2(y-2)}$

$2^{x+3} = (2)^{2y-4}$

If bases are equal powers also equal

$x+3=2y-4$

$x-2y = -4-3$

$x-2y = -7$ ..... (1)

given $3^{x-2} = 9^{3y-2x}$

$3^{x-2} = (3^2)^{3y-2x}$

$3^{x-2} = 3^{2(3y-2x)}$

$3^{x-2} = 3^{6y-4x}$

BAEPSE

$x - 2 = 6y - 4x$

$x+4x - 6y = 2$

$5x-6y = 2$..... (2)

from (1) & (2)

$x - 2y = -7 \times (3)$

$5x - 6y = 2 \times (1)$

$3x - 6y = -21$

$5x - 6y = 2$

$- \quad + \quad -$

$2x = 23$

$x = 23/2$

sub x value in (1)

$x - 2y = -7$

$x = \frac{23}{2} - 2y = -7$

$-2y = -7 - \frac{23}{2}$

$-2y = \frac{-14-23}{2}$

$2y = \frac{37}{2}$

$y = -\frac{37}{2\times 2} \Rightarrow y = \frac{37}{4}$

$\therefore x = \frac{23}{2}, y = \frac{37}{4}$

2. If $a^2 = 0.04$ find a^3.

sol:

$a^2 = 0.04$

$a^2 = \frac{4}{100}$

$a = \sqrt{\frac{4}{100}}$

$a=2/10$

Cubing On Both Sides

$a^3 = (2/10)^3$

$a^3=8/1000$

$a^3=0.008$

3. If $(x^{2/3})^p = x^2$ find p?

sol:

$(x^{2/3})^p = x^2$

$x^{2p/3} = x^2$

BAEPSE

$2p/3 = 2$

$\not{2}p = \not{2} \times 3$

$p = 3$

3. If $a^x=b$, $b^y = c$, $c^z=a$, show that $xyz = 1$

sol:

$a^x=b$, $b^y = c$, $c^z=a$

consider $a^x = b$

$\Rightarrow (c^z)^x = b$

$\Rightarrow (c^{zx}) = b$

$\Rightarrow (b^y)^{zx} = b$

$\Rightarrow b^{yzx} = b^1$

$\Rightarrow yzx = 1$

$\Rightarrow xyz = 1$

4. If $a = x + \sqrt{x^2+1}$ then show that $x = \frac{1}{2}\left(a - a^{-1}\right)$

sol:

$$a = x + \sqrt{x^2+1}$$

$$a - x = \sqrt{x^2+1}$$

SOBS

$$(a-x)^2 = \left(x^2+1\right)$$

$$a^2 - 2ax + \cancel{x^2} = \cancel{x^2} + 1$$

$$-2ax = 1 - a^2$$

$$2ax = a^2 - 1$$

$$x = \frac{1}{2a}[a^2 - 1]$$

$$x = \frac{1}{2}\left[\frac{a^2}{a} - \frac{1}{a}\right]$$

$$x = \frac{1}{2}\left[a - a^{-1}\right]$$

5. If $a^{1/3} + b^{1/3} + c^{1/3} = 0$ then show that $(a+b+c)^3 = 27\,abc$.

Sol:

$a^{1/3} + b^{1/3} + c^{1/3} = 0$

$a^{1/3} + b^{1/3} = - c^{1/3}$ .....(1)

Cubing On Both Sides

$(a^{1/3} + b^{1/3})^3 = (- c^{1/3})^3$

$$\left(a^{1/\cancel{3}}\right)^{\cancel{3}} + \left(b^{1/\cancel{3}}\right)^{\cancel{3}} + 3a^{1/3}.b^{1/3}\left(a^{1/3} + b^{1/3}\right) = -\left(c^{1/\cancel{3}}\right)^{\cancel{3}}$$

$$\because (a+b)^3 = a^3 + b^3 + 3ab(a+b)$$

$$\because \text{from(1)}$$

$a + b + 3.a^{1/3}\, b^{1/3}(- c^{1/3}) = - c$

$a + b - 3a^{1/3}\, b^{1/3}\, c^{1/3} = - c$

$a + b + c = 3a^{1/3}\, b^{1/3}\, c^{1/3}$

Cubing On Both Sides

$(a+b+c)^3 = (3a^{1/3}\, b^{1/3}\, c^{1/3})^3$

$$(a+b+c)^3 = 3^3.\left(a^{1/\cancel{3}}\right)^{\cancel{3}}.\left(b^{1/\cancel{3}}\right)^{\cancel{3}}.\left(c^{1/\cancel{3}}\right)^{\cancel{3}}$$

$(a+b+c)^3 = 27\,abc$

6. If $y = \sqrt[3]{3} + \frac{1}{\sqrt[3]{3}}$ show that $3y^3 - 9y = 10$

sol:

$$y = \sqrt[3]{3} + \frac{1}{\sqrt[3]{3}}$$

$$\Rightarrow y = 3^{1/3} + \frac{1}{3^{1/3}}$$

$$\Rightarrow y = 3^{1/3} + 3^{-1/3} \text{.....(1)}$$

Cubing On Both Sides

$$y^3 = \left[3^{1/3} + 3^{-1/3}\right]^3$$

$$y^3 = \left(3^{1/\cancel{3}}\right)^{\cancel{3}} + \left(3^{-1/\cancel{3}}\right)^{\cancel{3}} + 3.3^{1/3}.3^{-1/3}\left(3^{1/3} + 3^{-1/3}\right)$$

$$\boxed{(a+b)^3 = a^3 + b^3 + 3ab(a+b)}$$

$y^3 = 3 + 3^{-1} + 3.\ 3^{\circ}.\ (y)$ $\quad \because \text{from(1)}$

$$y^3 = 3 + \frac{1}{3} + 3(1).y$$

$$y^3 = 3 + \frac{1}{3} + 3y$$

$$y^3 = \frac{9+1+9y}{3}$$

$3y^3 = 10 + 9y$

$3y^3 - 9y = 10$

7. If $l$mn=1, Prove that $\frac{1}{1+l+m^{-1}} + \frac{1}{1+m+n^{-1}} + \frac{1}{1+n+l^{-1}} = 1$

sol:

$l$mn=1$\Rightarrow$ l=1/mn $\Rightarrow$ l^{-1}=1/mn

$$\frac{1}{1+l+m^{-1}} = \frac{1}{1+l+\frac{1}{m}} \qquad \left(\because a^{-1} = \frac{1}{a}\right)$$

$$= \frac{m}{m+lm+1}$$

$$= \frac{m}{m+\frac{1}{n}+1} \qquad \left(\because lmn = 1,\ lm = \frac{1}{n}\right)$$

$$= \frac{mn}{mn+1+n} \text{.....(1)}$$

$$\frac{1}{1+m+n^{-1}}=\frac{1}{1+m+\frac{1}{n}}$$

$$=\frac{n}{n+mn+1}\text{.....(2)}$$

$$\frac{1}{1+n+l^{-1}}=\frac{1}{1+n+mn}\text{.....(3)} \qquad \left(\because lmn=1,\ mn=\frac{1}{l}=l^{-1}\right)$$

L.H.S.= (1)+(2)+(3)

$$\frac{mn}{mn+1+n}+\frac{n}{n+mn+1}+\frac{1}{1+n+mn}$$

$$=\frac{\cancel{mn+n+1}}{\cancel{mn+1+n}}$$

= 1 = R.H.S.

Modules of a Real number:

- The modules or absolute value of a real number "x" is denoted by "|x|" and defined as
  |x| = x if x > 0
  = – x if x < 0
  = 0 if x = 0
  e.g. |–3| = 3
  |7| = 7

Note: Modules of a real number is never negative
e.g. |x| = – 5 solution doesn't exists.

Absolute Value equations & Inequations:

- |x| = a solution is x = a or x = – a
- |x| ≤ a solution is –a ≤ x ≤ a
- |x| < a solution is –a < x < a
- |x| ≥ a solution is x ≥ a or x ≤ – a
- |x| > a solution is x > a or x < – a

8. Solve |2x – 5| = 7

Sol:

|2x – 5| = 7
|x| = a solution is x = a or x = – a
|2x – 5| = 7 solution is 2x – 5 = 7 or 2x – 5 = – 7
2x = 7 + 5 or 2x = – 7 + 5

$$\Rightarrow \frac{\cancel{2}}{1}x=\frac{\cancel{12}}{6} \text{ or } \frac{\cancel{2}}{1}x=-\frac{\cancel{2}^{1}}{\cancel{2}}$$

x = 6 or x = – 1

9. Solve |5x–1| ≤ 0.

Sol:

|5x–1| ≤ 0
Since Modules of a real number is never negative
|5x–1| = 0 exists
|x| = 0 solution is x = 0
∴ |5x–1| = 0 solution is 5x – 1 = 0
5x = 1
x = 1/5

10. Solve $\left|5-\frac{x}{3}\right|\le 1$

Sol:

$\left|5-\frac{x}{3}\right|\le 1$

|x|≤a solution is –a ≤ x ≤ a

$\left|5-\frac{x}{3}\right|\le 1$ solution is

$$-1\le 5-\frac{x}{3}\le 1$$

subtract 5

$$\Rightarrow -1-5\le 5-\frac{x}{3}-5\le 1-5$$

$$\Rightarrow -6\le -\frac{x}{3}\le -4$$

multiply with 3

$$\Rightarrow -6\times 3\le -\frac{x}{3}\times 3\le -4\times 3$$

⇒ – 18 ≤ – x ≤ – 12
⇒ 18 ≥ x ≥ 12
⇒ 12 ≤ x ≤ 18

11. Solve |9 – 4x| > 4

Sol:

|9 – 4x| > 4
|x|>a solution is x > a or x < – a
|9 – 4x| > 4 solution is 9 – 4x > 4 or 9 – 4x < – 4
⇒ – 4x > 4 – 9 or – 4x < – 4 –9
⇒ – 4x > – 5 or – 4x < – 13
⇒ 4x < 5 or 4x > 13
⇒ x < 5/4 or x > 13/4
⇒ x > 13/4 or x < 5/4

Limit of a function:

- f(x) is any function as x → a ⇒ f (x)→ l then 'l' is called as limit of f as x tends to 'a', This is written as $\lim_{x\to a} f(x)=l$ and read as "Limit of f as x tends to a is l"
- If the limit of function approaches to zero, then the function is called "infinitesimal"

12. Evaluate $\lim_{x\to -1}\left(4-\frac{3}{x^2}\right)$

Sol:

$$=4-\frac{3}{(-1)^2}$$

$$=4-\frac{3}{1}$$

4–3 = 1

13.Evaluate $\underset{x\to 0}{Lt}\dfrac{x^2+5x}{x}$

Sol:

$\underset{x\to 0}{Lt}\dfrac{x^2+5x}{x}$

$\dfrac{0^2+5(0)}{0}=\dfrac{0}{0}$ which is indeterminate form

$\therefore \underset{x\to 0}{Lt}\dfrac{x^2+5x}{x}=\underset{x\to 0}{Lt}\dfrac{\cancel{x}(x+5)}{\cancel{x}}=\underset{x\to 0}{Lt}\, x+5=0+5=5$

14.Find the limit of the series $1-\dfrac{1}{2}+\dfrac{1}{4}-\dfrac{1}{8}+.....\infty$

Sol:

$1-\dfrac{1}{2}+\dfrac{1}{4}-\dfrac{1}{8}+.....\infty$ is an infinitesimal Geometric Progression

Here a = 1, $r=\dfrac{t_2}{t_1}=\dfrac{-1/2}{1}=-\dfrac{1}{2}<1$

$S_\infty=\dfrac{a}{1-r}$

$=\dfrac{1}{1-\left(\dfrac{1}{2}\right)}$

$=\dfrac{1}{1+\dfrac{1}{2}}=\dfrac{1}{3/2}=\dfrac{2}{3}$

15.Evaluate $\underset{x\to 0}{Lt}\dfrac{\sqrt{1+x+x^2}-1}{x}$

Sol:

$\underset{x\to 0}{Lt}\dfrac{\sqrt{1+x+x^2}-1}{x}$

$=\dfrac{\sqrt{1+0+0^2}-1}{0}=\dfrac{0-0}{0}=\dfrac{0}{0}$ which is indeterminate form

$\therefore \underset{x\to 0}{Lt}\dfrac{\sqrt{1+x+x^2}-1}{x}\times\dfrac{\sqrt{1+x+x^2}+1}{\sqrt{1+x+x^2}+1}$

$\underset{x\to 0}{Lt}\dfrac{\left[\sqrt{1+x+x^2}\right]^2-1^2}{x\left(\sqrt{1+x+x^2}+1\right)}$

$\underset{x\to 0}{Lt}\dfrac{\cancel{1}+x+x^2-\cancel{1}}{x\left(\sqrt{1+x+x^2}+1\right)}$

$\underset{x\to 0}{Lt}\dfrac{\cancel{x}(1+x)}{\cancel{x}\left(\sqrt{1+x+x^2}+1\right)}$

$\underset{x\to 0}{Lt}\dfrac{(1+x)}{\left(\sqrt{1+x+x^2}+1\right)}$

$=\dfrac{1+0}{\sqrt{1+0+0^2}+1}$

$=\dfrac{1}{\sqrt{1}+1}=\dfrac{1}{1+1}=\dfrac{1}{2}$

16. Show that $\underset{x\to 2}{Lt}\left[\dfrac{1}{x-2}-\dfrac{2}{x(x-1)(x-2)}\right]=3/2$

Sol:

$\underset{x\to 2}{Lt}\left[\dfrac{1}{x-2}-\dfrac{2}{x(x-1)(x-2)}\right]$

$=\dfrac{1}{2-2}-\dfrac{2}{2(2-1)(2-2)}$

$\dfrac{1}{0}-\dfrac{2}{0}$

$\dfrac{1}{0}-\dfrac{2}{0}$

$=\infty-\infty$ Which is indeterminate form.

$\therefore \underset{x\to 2}{Lt}\left[\dfrac{1}{x-2}-\dfrac{2}{x(x-1)(x-2)}\right]$

$\underset{x\to 2}{Lt}\left[\dfrac{x(x-1)-2}{x(x-1)(x-2)}\right]$

$=\underset{x\to 2}{Lt}\left[\dfrac{x^2-x-2}{x(x-1)(x-2)}\right]$

$=\underset{x\to 2}{Lt}\left[\dfrac{x^2-2x+x-2}{x(x-1)(x-2)}\right]$

$=\underset{x\to 2}{Lt}\left[\dfrac{x(x-2)+1(x-2)}{x(x-1)(x-2)}\right]$

$=\underset{x\to 2}{Lt}\left[\dfrac{\cancel{(x-2)}(x+1)}{x(x-1)\cancel{(x-2)}}\right]$

$\underset{x\to 2}{Lt}\dfrac{x+1}{x(x-1)}$

$\dfrac{2+1}{2(2-1)}=\dfrac{3}{2\times 1}=\dfrac{3}{2}$

17. Evaluate $\underset{x\to a}{Lt}\dfrac{\sqrt{x+a}-\sqrt{2a}}{x-a}$

Sol:

$\underset{x\to a}{Lt}\dfrac{\sqrt{x+a}-\sqrt{2a}}{x-a}$

$\dfrac{\sqrt{a+a}-\sqrt{2a}}{a-a}$

$\dfrac{\sqrt{2a}-\sqrt{2a}}{0}=\dfrac{0}{0}$ which is indeterminate form.

$=\underset{x\to a}{Lt}\dfrac{\sqrt{x+a}-\sqrt{2a}}{x-a}\times\dfrac{\sqrt{x+a}+\sqrt{2a}}{\sqrt{x+a}+\sqrt{2a}}$

$=\underset{x\to a}{Lt}\dfrac{\left(\sqrt{x+a}\right)^2-\left(\sqrt{2a}\right)^2}{(x-a)\left(\sqrt{x+a}+\sqrt{2a}\right)}$

$=\underset{x\to a}{Lt}\dfrac{x+a-2a}{(x-a)\left(\sqrt{x+a}+\sqrt{2a}\right)}$

$=\underset{x\to a}{Lt}\dfrac{\cancel{x-a}}{\cancel{(x-a)}\left(\sqrt{x+a}+\sqrt{2a}\right)}$

$=\underset{x\to a}{Lt}\dfrac{1}{\sqrt{x+a}+\sqrt{2a}}$

$=\dfrac{1}{\sqrt{a+a}+\sqrt{2a}}$

$=\dfrac{1}{\sqrt{2a}+\sqrt{2a}}$

$=\dfrac{1}{2\sqrt{2a}}$

Note:

1. $\therefore \underset{x\to a}{Lt}\dfrac{x^n-a^n}{x-a}=n.a^{n-1}$

2. $\therefore \underset{x\to a}{Lt}\dfrac{x^m-a^m}{x^n-a^n}=\dfrac{m}{n}.a^{m-n}$

18. Evaluate $\underset{x\to a}{Lt}\dfrac{x^{1/4}-a^{1/4}}{x^4-a^4}$

Sol:

$\underset{x\to a}{Lt}\dfrac{x^{1/4}-a^{1/4}}{x^4-a^4}$

$=\dfrac{a^{1/4}-a^{1/4}}{a^4-a^4}$

0/0 which is indeterminate form.

$\therefore \underset{x\to a}{Lt}\dfrac{x^{1/4}-a^{1/4}}{x^4-a^4}=\dfrac{1}{4}.a^{\frac{1}{4}-4}$

$\because \underset{x\to a}{Lt}\dfrac{x^n-a^n}{x-a}=n.a^{n-1}$

$=\dfrac{1}{4}.a^{\frac{1-16}{4}}$

$=\dfrac{1}{4}.\ a^{-\frac{15}{4}}$

$=\dfrac{1}{4a^{15/4}}$

Note:

- as $x\to\infty\Rightarrow\dfrac{1}{x}\to 0\Rightarrow\dfrac{1}{x^n}\to 0$

19.Evaluate $\underset{x\to\infty}{Lt}\dfrac{4x-3}{2x+3}$

$=\dfrac{4(\infty)-3}{2(\infty)+3}=\dfrac{\infty-3}{\infty+3}=\dfrac{\infty}{\infty}$ which is indeterminate forms

$\therefore \underset{x\to\infty}{Lt}\dfrac{4x-3}{2x+3}$

$\therefore \underset{x\to\infty}{Lt}\dfrac{\cancel{x}(4-3/x)}{\cancel{x}(2+3/x)}$

$\therefore \underset{x\to\infty}{Lt}\dfrac{4-3.\dfrac{1}{x}}{2+3.\dfrac{1}{x}}$

as $x\to 0\Rightarrow\dfrac{1}{x}\to 0$

$=\dfrac{4-3.(0)}{2+3(0)}=\dfrac{\cancel{4}}{\cancel{2}}=2$

Note:

1. $\underset{x\to 0}{Lt}\dfrac{(1+x)^n-1}{x}=n$

2. $\underset{x\to 0}{Lt}\dfrac{(1+x)^{1/n}-1}{x}=1/n$

3. If $\left(x\sqrt{x}\right)^x=x^{x\sqrt{x}}$ then $x=9/4$

4. The limiting position of a secant of a circle is tangent

5. $\underset{x\to 0}{Lt}\dfrac{2x+3}{3x+5}=2/3$

6. $\underset{x\to 3}{Lt}\dfrac{x^3-27}{x-3}=27$

7. $a^{2/3}\left[a^{1/3}.\left(a^{1/4}\right)^4\right]=a^2$

8. $16^{1.25}=32$

9. $(a^{1/3}–b^{1/3})(a^{2/3}+a^{1/3}.b^{1/3}+b^{2/3})=a–b$

10. $|3x+1|=0$ solution is $x=-1/3$

Assignment:

1. If $x^{1/2} = 0.2$ then find $x^{3/2}$
2. If $64^x = 1/256^y$ then find the value of $3x+4y$.
3. Solve $|3-12x| = 0$
4. If $a^x=b^y=c^z = d^w$ and $ab = cd$ then S.T $\frac{1}{x}+\frac{1}{y}=\frac{1}{z}+\frac{1}{w}$
5. If $a^x = b^y = c^z$; $\frac{b}{a}=\frac{c}{b}$ then show that $\frac{y}{x}=\frac{2z}{x+z}$
6. Solve $\left|\frac{2x-1}{5}\right| \le 5$
7. Solve $\left|1-\frac{m}{3}\right| \ge 9$
8. Evaluate $\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}$
9. Evaluate $\lim_{x\to p}\frac{x^a-p^a}{x-p}$
10. Evaluate $\lim_{x\to 0}\frac{(1+x)^4-1}{(1+x)^3-1}$