PROGRESSIONS

The chapter comes under Paper-I. From this chapter 2 Marks questions - 2 (2 ×2 = 4M), 1 Mark question - 1 (1×1 = 1M), 4 Marks questions - 2 (2×4 = 8M) and 6 objective bits (6×1/2 = 3M), altogether 16 Marks easily can be scored. Even though this chapter is a bit lengthy, there are few questions which will be often repeated in the public exams. The material given below will be very much useful for S.S.C. appearing students.

Progression: An arrangement of numbers in a definite order is called "Progression".

Eg: 1) 2, 5, 8, 11, 14,.....
2) 1, 10, 100, 1000,.....
3) 1/2, 1/4, 1/6, 1/8,.....

Types of Progression: Progressions are of 3 types. They are

1. Arithmetic Progression (A.P.)
2. Geometric Progression (G.P)
3. Harmonic Progression (H.P)

1. Arithmetic Progression: A progression in which, except the first term, each term is obtained by adding a fixed number to the term immediately preceding it is called arithmetic progression.

- The fixed number is called "common difference" and it is denoted by "d".
- The terms of the progression are denoted by $t_1, t_2, t_3, t_4, ...$, first term t_1 is also denoted by "a".
- Common difference (d) = $t_2 - t_1 = t_3 - t_2 = t_4 - t_3$ .....

$$\boxed{d = t_2 - t_1}$$

- In 21, 16, 11, 6,..... common difference is –5.
  $d = t_2 - t_1$
  $= 16 - 21$
  $= -5$
- n^{th} term or general term of A.P. $\boxed{t_n = a + (n-1)d}$
- In $x, \frac{4x}{3}, \frac{5x}{3}, ...$ d = _____, t_7 = _____

  $d = t_2 - t_1$

  $$= \frac{4x}{3} - x = \frac{4x - 3x}{3} = \frac{x}{3}$$

  $$\boxed{t_n = a + (n-1)d}$$

  $$t_7 = x + (7-1)\left(\frac{x}{3}\right)$$

$= x + \cancel{6}^{2} \times \frac{x}{\cancel{3}_{1}}$

$= x + 2x$

$= 3x$

Problem: 1

(1 Mark)

Find the 17^{th} term in the series $t_n = \frac{n(n+3)}{n+2}$

Sol:

$t_n = \frac{n(n+3)}{n+2}$

$t_{17} = \frac{17(17+3)}{17+2}$

$= \frac{17 \times 20}{19} = \frac{340}{19}$

Problem: 2

(1 Mark)

Determine 'k' so that k+2, 4k–6 and 3k–2 are the three consecutive terms of an A.P.

Sol:

k+2, 4k–6, 3k–2 are in A. P.

t_1 t_2 t_3

$t_2 - t_1 = t_3 - t_2$

$\Rightarrow (4k - 6) - (k + 2) = (3k - 2) - (4k - 6)$

$\Rightarrow 4k - 6 - k - 2 = 3k - 2 - 4k + 6$

$\Rightarrow 3k - 8 = -k + 4$

$\Rightarrow 3k + k = 4 + 8$

$\Rightarrow \cancel{4}_{1} k = \cancel{12}^{3}$

$\Rightarrow k = 3$

Problem: 3

(2 Marks)

Which term in the A.P i.e 5, 2, –1,.....is –22?

Sol:

5, 2, –1, ... are in A.P

$a = 5, d = t_2 - t_1 = 2 - 5 = -3$

Let $t_n = -22$

$a+(n - 1)d = -22$

$5+(n - 1)(-3) = -22$

$5 - 3n + 3 = -22$

$-3n = -22 - 8$

$\not{-}\not{3}n = \not{-}\not{30}$

$1 \qquad 10$

$n = 10$

$\therefore t_{10} = -22$

Sum to 'n' terms of A.P:

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

or

$$S_n = \frac{n}{2}\left[a + l\right]$$

Problem: 4

(2 Marks)

How many terms of A.P $-6, \frac{-11}{2}, -5, ...$ are needed to give the sum –25?

Sol:

$-6, \frac{-11}{2}, -5, ...$ are in A.P

$$a = -6,\ d = \frac{-11}{2} - (-6) = \frac{-11+12}{2} = \frac{1}{2}$$

Let $S_n = -25$

$$\Rightarrow \frac{n}{2}\left[2a + (n-1)d\right] = -25$$

$$\Rightarrow \frac{n}{2}\left[2(-6) + (n-1)\frac{1}{2}\right] = -25$$

$$\Rightarrow \frac{n}{2}\left[-12 + \frac{n}{2} - \frac{1}{2}\right] = -25$$

$$\Rightarrow \frac{n}{2}\left[\frac{n}{2} - \frac{25}{2}\right] = -25$$

$$\Rightarrow \frac{n}{4}\left[n - 25\right] = -25$$

$\Rightarrow n^2 - 25n = -100$

$\Rightarrow n^2 - 25n + 100 = 0$

$\Rightarrow n^2 - 20n - 5n + 100 = 0$

$\Rightarrow n(n-20) - 5(n-20) = 0$

$\Rightarrow (n-20)(n-5) = 0$

$\Rightarrow n - 20 = 0 \qquad n - 5 = 0$

$\Rightarrow n = 20$ $\quad n = 5$

$\therefore S_{20}$ or $S_5 = -25$

Arithmetic Mean: If a, b, c are in A.P, then b is called arithmetic mean of a and c.

Arithmetic mean $b = \frac{a+c}{2}$

- Arithmetic mean of a and b $= \frac{a+b}{2}$

- If there are 'n' A.M's between a and b then common difference $\boxed{d = \frac{b-a}{n+1}}$

Problem: 5

(2 Marks)

Insert 5 A.M's between 4 and 22.

Sol:

Let a_1, a_2, a_3, a_4, a_5 be 5 A.M's between 4 and 22.

Then A.P is as follows

$4, a_1, a_2, a_3, a_4, a_5, 22$

$t_1 \; t_2 \; t_3 \; t_4 \; t_5 \; t_6 \; t_7$

Here $\boxed{a = 4}$

$t_7 = 22$

$a + (7-1)d = 22$

$a + 6d = 22$

$4 + 6d = 22$

$6d = 22 - 4$

$$d = \frac{\overset{3}{\cancel{18}}}{\underset{1}{\cancel{6}}}$$

$\boxed{d = 3}$

$\therefore$ Required A.M's $\quad a_1 = 4 + d = 4 + 3 = 7$

$a_2 = a_1 + d = 7 + 3 = 10$

$a_3 = a_2 + d = 10 + 3 = 13$

$a_4 = a_3 + d = 13 + 3 = 16$

$a_5 = a_4 + d = 16 + 3 = 19$

Note:

1. $\Sigma 1 = n$
2. $1 + 2 + 3 + \ldots\ldots\ldots + n$ $\quad$ ie $\quad \Sigma n = \frac{n(n+1)}{2}$
3. $1^2 + 2^2 + 3^2 + \ldots\ldots\ldots + n^2$ $\quad$ ie $\quad \Sigma n^2 = \frac{n(n+1)(2n+1)}{6}$

4. $1^3 + 2^3 + 3^3 + \ldots\ldots\ldots + n^3$ ie $\Sigma n^3 = \frac{n^2(n+1)^2}{4}$ or $[\Sigma n]^2$

Eg: 1) $\Sigma n = 10$ then $\Sigma n^3 = 100$
2) $\Sigma n^3 = 625$ then $\Sigma n = 25$

Problem: 6

(4 Marks)

If the sum of the first 'n' natural numbers is S_1, and that of their squares S_2 and cubes S_3, then show that $9S_2^2 = S_3(1 + 8S_1)$

Proof:

$$S_1 = \Sigma n = \frac{n(n+1)}{2}$$

$$S_2 = \Sigma n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$S_3 = \Sigma n^3 = \frac{n^2(n+1)^2}{4}$$

$$\text{R.H.S} \quad S_3(1+8S_1) = \frac{n^2(n+1)^2}{4}\left[1+ \cancel{8}^{4} \cdot \frac{n(n+1)}{\cancel{2}_{1}}\right]$$

$$= \frac{n^2(n+1)^2}{4}\left[1+ 4n^2 + 4n\right]$$

$$= \frac{n^2(n+1)^2}{4}\left[4n^2 + 4n + 1\right]$$

$$= \frac{n^2(n+1)^2}{4}\left[(2n)^2 + 2.(2n).1 + 1^2\right]$$

$$= \frac{n^2(n+1)^2}{4}[2n+1]^2$$

Multiply and divides with 9

$$= \frac{9}{9} \times \frac{n^2(n+1)^2}{4}(2n+1)^2$$

$$= 9\frac{n^2(n+1)^2(2n+1)^2}{36}$$

$$= 9.\left[\frac{n(n+1)(2n+1)}{6}\right]^2$$

$= 9S_2^2$ L.H.S

Hence proved

Problem: 7

(4 Marks)

The product of two numbers is 91 and their arithmetic mean is 10. Find the two numbers.

Sol:

Let the two numbers be a and b.

Given that ab = 91 _____ (1)

$$\frac{a+b}{2}=10$$

$\Rightarrow$ a + b = 20 _____ (2)

We know that

$(a-b)^2 = (a+b)^2 - 4ab$

$= 20^2 - 4\times 91$

$= 400 - 364$

$= 36$

$\Rightarrow a-b=\sqrt{36}$

$\Rightarrow$ a – b = ±6 _____ (3)

$a - b = 6$

$a + b = 20$

(i) If $------$

$2a = 26$

$a = 13$

$\boxed{a=13}$

Substitute a value in (1)

13 + b = 20

b = 20 – 13

$\boxed{b=7}$

$a - b = -6$

$a + b = 20$

(ii) If $------$

$2a = 14$

$a = 7$

$\boxed{a=7}$

Substitute a value in (1)

7 + b = 20

b = 20 – 7

$\boxed{b=13}$

$\therefore$ Required numbers are 13, 7 or 7, 13.

Geometric Progression: A progression in which except the first term each term is obtained by multiplying a fixed number to the term immediately preceding it is called "Geometric Progression". This multiplying term is called "common ratio" and it is denoted by "r".

$\rightarrow r = \frac{t_2}{t_1} = \frac{t_3}{t_2} = \frac{t_4}{t_3} = ...$

$\rightarrow \boxed{\text{common ratio}(r) = \frac{t_2}{t_1}}$

$\rightarrow$ In $-3, 1, \frac{-1}{3}, ...$ common ratio(r) = $-1/3$

$r = \frac{t_2}{t_1} = \frac{1}{-3} = \frac{-1}{3}$

$\rightarrow n^{th}$ term of G.P $t_n = a.r^{n-1}$

Problem: 8
(1 Mark)

Find the value of x so that $\frac{-2}{7}, x, \frac{-7}{2}$ are three consecutive terms of a G.P.

Sol:

$\frac{-2}{7}, x, \frac{-7}{2}$ are in G.P

$t_1 \quad t_2 \quad t_3$

$\frac{t_2}{t_1} = \frac{t_3}{t_2}$

$\Rightarrow \frac{x}{-2/7} = \frac{-7/2}{x}$

$\Rightarrow x \times x = \frac{\not{-7}^{1}}{\not{2}_{1}} \times \frac{\not{-2}^{1}}{\not{7}_{1}}$

$\Rightarrow x^2 = 1$
$\Rightarrow x = \sqrt{1}$
$\Rightarrow x = \pm 1$

Geometric Mean: If a, b, c are in G.P then 'b' is called Geometric Mean of "a and c".

ie $b = \sqrt{ac}$

$\rightarrow$ G.M of a and b = $\sqrt{ab}$

→ G.M of 4 and 16 is ±8

$G.M = \sqrt{4 \times 16} = \sqrt{64} = \pm 8$

Problem: 9

Insert 5 G.M's between 1/3 and 243.

Sol:

Let g_1, g_2, g_3, g_4, g_5 be 5 G.M's between 1/3 and 243.

The G.P is as follows

$1/3, g_1, g_2, g_3, g_4, g_5, 243.$

$t_1, \quad t_2, \quad t_3, \quad t_4, \quad t_5, \quad t_6, \quad t_7.$

$\boxed{a = 1/3}$

$t_7 = 243$

$ar^{7-1} = 243$

$1/3.r^6 = 243$

$r^6 = 243 \times 3$

$r^6 = 3^5 \times 3^1$

$r^6 = 3^6$

$\Rightarrow \boxed{r = 3}$

Required G.M are $g_1 = ar = 1/3 \times 3 = 1$

$g_2 = g_1 r = 1 \times 3 = 3$

$g_3 = g_2 r = 3 \times 3 = 9$

$g_4 = g_3 r = 9 \times 3 = 27$

$g_5 = g_4 r = 27 \times 3 = 81$

Problem: 10

(4 Marks)

If the A.M and G.M of two n numbers are 13 and 12 respectively. Find the numbers.

Sol:

Let the two numbers be a and b.

A.M = 13

$\Rightarrow \frac{a+b}{2} = 13$

$\Rightarrow a + b = 26$ _____ (1)

G.M = 12

$\Rightarrow \sqrt{ab} = 12$

$\Rightarrow ab = 144$ _____ (2)

We know that

$(a - b)^2 = (a + b)^2 - 4ab$

$= 26^2 - 4 \times 144$

$= 676 - 576$

$= 100$

$a - b = \sqrt{100}$

$a - b = \pm 10$

$a - b = 10$

$a + b = 26$

i) If $------$

$2a = 36$

$a = 18$

$\boxed{a = 18}$

Substitute a value in (1)

18 + b = 26

b = 26 – 18

$\boxed{b = 8}$

$a - b = -10$

$a + b = 26$

ii) If $------$

$2a = 16$

$a = 8$

$\boxed{a = 8}$

Substitute a value in (1)

a + b = 26

8 + b = 26

b = 26 – 8

$\boxed{b = 18}$

$\therefore$ Required numbers are 18, 8 or 8, 18.

Sum to 'n' terms of G.P:

$$S_n = \frac{a\left[1 - r^n\right]}{1 - r} \text{ if } r < 1$$

$$\text{and } S_n = \frac{a\left[r^n - 1\right]}{r - 1} \text{ if } r > 1$$

Problem: 11

(2 Marks)

How many terms of G.P $3, \frac{3}{2}, \frac{3}{4}, ...$ are needed to give a sum $\frac{3069}{512}$?

Sol:

$3, \frac{3}{2}, \frac{3}{4}, \ldots\ldots$ are in G.P

$a = 3, r = t_2/t_1 = \frac{3/2}{3} = \frac{3}{2} \times \frac{1}{3} = \frac{1}{2} < 1$

Let $S_n = \frac{3069}{512}$

$$\frac{a\left[1-r^n\right]}{1-r} = \frac{3069}{512}$$

$$\frac{3\left[1-\left(\frac{1}{2}\right)^n\right]}{1-1/2} = \frac{3069}{512}$$

$$\frac{3\left[1-\left(\frac{1}{2}\right)^n\right]}{1/2} = \frac{3069}{512}$$

$$3 \times 2\left[1-(1/2)^n\right] = \frac{3069}{512}$$

$$1-(1/2)^n = \frac{\cancel{3069}^{1023}}{512 \times \cancel{6}_{2}}$$

$$1-\left(\frac{1}{2}\right)^n = \frac{1023}{1024}$$

$$1-\frac{1023}{1024} = \left(\frac{1}{2}\right)^n$$

$$\frac{1024-1023}{1024} = \frac{1}{2^n}$$

$$\frac{1}{1024} = \frac{1}{2^n}$$

$$\frac{1}{2^{10}} = \frac{1}{2^n}$$

$\Rightarrow n = 10$

$$S_{10} = \frac{3069}{512}$$

Problem: 12

(4 Marks)

If g_1, g_2, g_3 are three Geometric Means between m and n then show that $g_1.g_3 = g_2^2$ mn.

Proof:

g_1, g_2, g_3 are three G.M's between m and n.

Then the G.P is as follows

m, g_1, g_2, g_3, n

t_1, t_2, t_3, t_4, t_5

$$\frac{t_2}{t_1} = \frac{t_3}{t_2} = \frac{t_4}{t_3} = \frac{t_5}{t_4}$$

$$\Rightarrow \frac{g_1}{m} = \frac{g_2}{g_1} = \frac{g_3}{g_2} = \frac{n}{g_3}$$

$$\Rightarrow \frac{g_1}{m} = \frac{n}{g_3}$$

$\Rightarrow g_1g_3 = mn$ _____(1)

$$\frac{g_2}{g_1} = \frac{g_3}{g_2}$$

$g_2^2 = g_1g_3$ _____(2)

From (1), (2)

$g_1g_3 = g_2^2 = mn$

Problem: 13

(4 Marks)

Find the sum to 'n' terms of the progression 7, 77, 777, ...

Sol:

7, 77, 777, ... n terms.

Sum S_n = 7 + 77 + 777+ ..... n terms.

= 7[1 + 11 + 111 + ...n terms]

Multiplying and dividing by 9

$$= \frac{7}{9} \times 9\left[1+11+111+.....\text{n terms}\right]$$

$$= \frac{7}{9}\left[9+99+999+.....\text{n terms}\right]$$

$$= \frac{7}{9}\left[(10-1)+(100-1)+(1000-1)+.....\text{n terms}\right]$$

$$= \frac{7}{9}\left[(10+100+1000+.....\text{n terms})-(1+1+1+.....\text{n terms})\right]$$

$$= \frac{7}{9}\left[(10+100+1000+.....\text{n terms})-n\right] ____ (1)$$

Consider 10, 100, 1000, ... are in G.P

$$a = 10,\ r = \frac{100}{10} = 10 > 1$$

$$S_n = \frac{a\left[r^n - 1\right]}{r - 1}$$

$$= \frac{10\left[10^n - 1\right]}{10 - 1}$$

$$= \frac{10}{9}\left[10^n - 1\right] \quad \text{——} (2)$$

Substitute (2) in (1)

$$S_n = \frac{7}{9}\left[\frac{10}{9}\left(10^n - 1\right) - n\right]$$

$$= \frac{7}{9}\frac{\left[10^{n+1} - 10 - 9n\right]}{9}$$

$$= \frac{7}{81}\left[10^{n+1} - 9n - 10\right]$$

- If a, ar, ar^2, ar^3, ... are the terms of infinite Geometric Progression.

  Then $\boxed{S_\infty = \frac{a}{1-r}}$ where r < 1

Problem: 14

(1 Mark)

The common ratio of a G.P is –4/5 and sum to infinity is 80/9, then find the first term.

Sol:

$$r = \frac{-4}{5}, \qquad S_\infty = \frac{80}{9}, a = ?$$

$$\boxed{S_\infty = \frac{a}{1-r}}$$

$$\frac{80}{9} = \frac{a}{1 - \left(-4/5\right)}$$

$$\frac{80}{9} = \frac{a}{1 + 4/5}$$

$$\frac{80}{9} = \frac{a}{9/5}$$

$$\frac{\cancel{80}^{16}}{\cancel{9}} = \frac{\cancel{5}^{1}\,a}{\cancel{9}}$$

$$\Rightarrow \boxed{a = 16}$$

Harmonic Progression: A progression is said to be Harmonic Progression if the reciprocals of these terms form an Arithmetic Progression.

Eg: 1) $\frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \frac{1}{12}$.....

2) $\frac{4}{5}, \frac{4}{7}, \frac{4}{9}, \frac{4}{11}$.....

Harmonic Mean: If a, b, c are in Harmonic Progression then b is called Harmonic Mean of a and c.

$\rightarrow$ Harmonic Mean = $\frac{2ac}{a+b}$

$\rightarrow$ Harmonic Mean of a and b = $\frac{2ab}{a+b}$

Note: A.M, G.M and H.M of any two numbers wil be in G.P ie $\frac{G}{A} = \frac{H}{G} \Rightarrow \boxed{G^2 = AH}$

Eg: A.M and G.M of two numbers is 13 and 12 respectively then H.M is _____

$G^2 = A.H \Rightarrow 12^2 = 13 \times H$

$\Rightarrow \frac{144}{13} = H$

$\Rightarrow$ Harmonic Mean = $\frac{144}{13}$

Problem: 15

(4 Marks)

If (b+c), (c+a) and (a+b) are in H.P then show that $\frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2}$ will be in H.P

Proof:

(b + c), (c + a), (a + b) are in H.P

$\Rightarrow \underset{t_1}{\frac{1}{b+c}}, \underset{t_2}{\frac{1}{c+a}}, \underset{t_3}{\frac{1}{a+b}}$ will be in A.P

$t_2 - t_1 = t_3 - t_2$

$\Rightarrow \frac{1}{c+a} - \frac{1}{b+c} = \frac{1}{a+b} - \frac{1}{c+a}$

$$\Rightarrow \frac{(b+c)-(c+a)}{(c+a)(b+c)} = \frac{(c+a)-(a+b)}{(a+b)(c+a)}$$

$$\Rightarrow \frac{b+\cancel{c}-\cancel{c}-a}{(c+a)(b+c)} = \frac{c+\cancel{a}-\cancel{a}-b}{(a+b)(c+a)}$$

$$\Rightarrow \frac{b-a}{\cancel{(c+a)}(b+c)} = \frac{c-b}{(a+b)\cancel{(c+a)}}$$

$\Rightarrow (b - a)(a + b) = (c - b)(b + c)$

$\Rightarrow (b - a)(b + a) = (c - b)(c + b)$

$\Rightarrow b^2 - a^2 = c^2 - b^2$

$\Rightarrow a^2, b^2, c^2$ are in A.P

$\Rightarrow \frac{1}{a^2}, \frac{1}{b^2}$ and $\frac{1}{c^2}$ are in H.P

Problem: 16

(4 Marks)

The A.M, G.M and H.M of two numbers are A, G, H respectively. Show that $A \geq G \geq H$.

Proof:

Let the two numbers be a and b.

A.M ie $A = \frac{a+b}{2}$

G.M ie $G = \sqrt{ab}$

H.M ie $H = \frac{2ab}{a+b}$

Consider $A - G = \frac{a+b}{2} - \sqrt{ab}$

$$= \frac{a+b-2\sqrt{ab}}{2}$$

$$= \frac{\left(\sqrt{a}\right)^2 + \left(\sqrt{b}\right)^2 - 2\sqrt{a}\times\sqrt{b}}{2}$$

$$= \frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{2} \geq 0$$

$A - G \geq 0 \Rightarrow A \geq G$ _____ (1)

$\Rightarrow \frac{A}{G} \geq 1$ _____ (2)

We know that $G^2 = AH$

$\Rightarrow \frac{G}{H} = \frac{A}{G}$

$\Rightarrow \frac{G}{H} \geq 1$ [From (2)]

$\Rightarrow G \geq H$ _____ (3)

From (1) & (3)

$A \geq G \geq H$

Objective Bits:

1. Harmonic Mean of 1/a and 1/b is _____
2. If a, b, c are in G.P then a/b = _____
3. The n^{th} term in the series – 3 + 6 – 12 + 24 – 48 +..... is _____
4. $\Sigma n^3 = 64$ then Σn = _____
5. 1 + 2 + 3 + ...+100 = _____
6. If the n^{th} term of A.P is $t_n = 2n + 5$ then first term is _____
7. Common ratio of G.P $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + ...$ is
8. If $1^3 + 2^3 + 3^3 + ...+ m^3 = 3025$ then m = _____
9. G.M of 5 and 125 is _____
10. A.M, G.M and H.M of two numbers are in _____ Progression.

Answers:

1. $\frac{2}{a+b}$
2. b/c
3. $(-1)^n.3.2^{n-1}$
4. 8
5. 5050
6. 7
7. –1/3
8. 10
9. 25
10. Geometric

Assignment:

1. If 7 times to the 7^{th} term of A.P is equals to 11 times to the 11^{th} term, then show that 18^{th} term of it is '0'.
2. Find the sum of all 3 digit numbers which leave the remainder 1 when divided by 4.
3. Which term of G.P 2, $2\sqrt{2}$, 4, ... is 64?
4. The 5^{th}, 8^{th}, and 11^{th} terms of G.P are p, q and s respectively then show that $q^2 = ps$.
5. Find the sum to 'n' terms of the series 0.5, 0.55, 0.555, 0.5555, ...
6. Express $0.\overline{68}$ in the form of p/q.
7. Insert 6 H.M's between 2/3 and 2/31.
8. In H.P the 3^{rd} term is 1/7 and the 7^{th} term is 1/5 then show that $15t^h$ term is 1.