5. BINOMIAL THEOREM

Quick Review

1. If n is a positive integer then $(x+a)^n = {}^nC_0x^n + {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 + ... + {}^nC_rx^{n-r}a^r + ... + {}^nC_na^n$.

2. The expansion of $(x+a)^n$ contains (n + 1) terms.

3. In the expansion, the sum of the powers of x and a in each term is equal to n.

4. In the expansion, the coefficients ${}^nC_0, {}^nC_1, {}^nC_2, ..., {}^nC_n$ are called ***binomial coefficients*** and these are simply denoted by $C_0, C_1, C_2, ..., C_n$.

5. In the expansion, (r + 1)th term is called the ***general term***. It is denoted by T_{r+1}. Thus $T_{r+1} = {}^nC_rx^{n-r}a^r$.

6. $(x+a)^n = \sum_{r=0}^{n} {}^nC_rx^{n-r}a^r$.

7. $(x-a)^n = \sum_{r=0}^{n} {}^nC_rx^{n-r}(-a)^r = \sum_{r=0}^{n}(-1)^n\,{}^nC_rx^{n-r}a^r = {}^nC_0x^n - {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 - ... + (-1)^n\,{}^nC_na^n$

8. $(1+x)^n = \sum_{r=0}^{n} {}^nC_rx^r = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + ... + {}^nC_nx^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$

9. Middle term(s) in the expansion of $(x + a)^n$:

   i) If n is even then $\left(\frac{n}{2}+1\right)$th term is the middle term.

   ii) If n is odd then $\frac{n+1}{2}$ th and $\frac{n+3}{2}$ th terms are the middle terms.

10. Numerically greatest term in the expansion of $(1 + x)^n$:

    i) If $\frac{(n+1)\,|x|}{|x|+1}$ = p, a positive integer then pth and (p+1)th terms are the numerically greatest terms in the expansion of $(1 + x)^n$.

    ii) If $\frac{(n+1)\,|x|}{|x|+1}$ = p+F where p is a positive integer and 0 < F < 1 then (p+1)th term is the numerically greatest term in the expansion of $(1+x)^n$.

11. If C_r denotes ${}^nC_r$ then

    i) $C_0 + C_1 + C_2 + ... + C_n = 2^n$

    ii) $C_0 + C_2 + C_4 + ... = C_1 + C_3 + C_5 + ... = 2^{n-1}$

12. i) $\sum_{r=0}^{n} {}^nC_r = 2^n$ ii) $\sum_{r=0}^{n}(-1)^r\,{}^nC_r = 0$

13. $a\cdot C_0 + (a + d)\cdot C_1 + (a + 2d)\cdot C_2 + \ldots\ldots + (a + nd)\cdot C_n = (2a + nd)\cdot 2^{n-1}$.

14. $C_1 + 2\cdot C_2x + 3\cdot C_3x^2 + ... + n\cdot C_nx^{n-1} = n(1+x)^{n-1}$

15. i) $C_1 + 2 \cdot C_2 + 3 \cdot C_3 + ... + n \cdot C_n = n \cdot 2^{n-1}$

    ii) $C_1 - 2 \cdot C_2 + 3 \cdot C_3 - ..... = 0$

16. i) $\sum_{r=0}^{n} r.{}^nC_r = n.2^{n-1}$ ii) $\sum_{r=0}^{n} (-1)^r r.{}^nC_r = 0$

17. $C_0 + \frac{C_1}{2}x + \frac{C_2}{3}x^2 + ... + \frac{C_n}{n+1}x^n = \frac{(1+x)^{n+1} - 1}{(n+1)x}$.

18. $C_0C_r + C_1C_{r+1} + C_2C_{r+2} + ... + C_{n-r}C_n = {}^{2n}C_{n-r} = \frac{(2n)!}{(n-r)!(n+r)!}$.

19. $C_0^2 + C_1^2 + C_2^2 + ... + C_n^2 = \frac{(2n)!}{(n!)^2}$.

20. **Binomial Theorem for rational index :** If n is a rational number and $|x| < 1$ then

    $1+nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ... = (1+x)^n$.

21. If $|x| < 1$ then

    i) $(1 + x)^{-1} = 1 - x + x^2 - x^3 + ... + (-1)^r x^r + ...$

    ii) $(1 - x)^{-1} = 1 + x + x^2 + x^3 + ... + x^r + ...$

    iii) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + ... + (-1)^r (r+1)x^r + ...$

    iv) $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + ... + (r+1)x^r + ...$

    v) $(1+x)^{-n} = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + ...$

    vi) $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + ...$

    vii) $(1-x)^{\frac{-p}{q}} = 1 + p\left(\frac{x}{p}\right) + \frac{p(p+q)}{2!}\left(\frac{x}{p}\right)^2 + \frac{p(p+q)(p+2q)}{3!}\left(\frac{x}{p}\right)^3 + ...$

22. If $|x| < 1$ and n is a positive integer, then

    i) $(1-x)^{-n} = 1 + {}^nC_1x + {}^{(n+1)}C_2x^2 + {}^{(n+2)}C_3x^3 + ...$

    ii) $(1+x)^{-n} = 1 - {}^nC_1x + {}^{(n+1)}C_2x^2 - {}^{(n+2)}C_3x^3 + ...$

23. Coefficient of x^m in the expansion of $\left(ax^p + \frac{b}{x^q}\right)^n$ is the coefficient of T_{r+1} where $r = \frac{np - m}{p + q}$.

24. The term independent of x in the expansion of $\left(ax^p + \frac{b}{x^q}\right)^n$ is T_{r+1} where $r = \frac{np}{p+q}$.

25. In the expansion $(x + a)^n, \frac{T_{r+1}}{T_r} = \frac{n - r + 1}{r}\frac{a}{x}$.

26. The general term in the expansion of $(x_1 + x_2 + ... + x_p)^n$ is $\frac{n!}{n_1!n_2!...n_p!}x_1^{n_1}x_2^{n_2}...x_n^{n_p}$ where

    $n_1 + n_2 + ... + n_p = n$.

27. i) The coefficient of x^{n-1} in the expansion of $(x-1)(x-2)\ldots(x-n)$ is $-n(n+1)/2$.

ii) The coefficient of x^{n-1} in the expansion of $(x+1)(x+2)\ldots(x+n)$ is $n(n+1)/2$.

28. i) If the coefficients of pth, qth terms in the expansion of $(1+x)^n$ are equal then $p+q = n+2$.

ii) If the coefficients of x^r, x^{r+1} in the expansion of $(a + x/b)^n$ are equal then $n = (r+1)(ab+1) - 1$.

29. i) If the coefficients of rth, (r+1)th, (r+2)th terms of $(1+x)^n$ are in A.P. then $n^2 - (4r+1)n + 4r^2 = 2$.

ii) If the coefficients of x^{r-1}, x^r, x^{r+1} are in A.P. then $(n-2r)^2 = n+2$.

30. If $(a+\sqrt{b})^n = I + F$ were I, n are positive integers, $0 < F < 1$, $a^2 - b = 1$, then

i) I is an odd positive integer

ii) $(I+F)(I-F) = 1$.

31. i) The number of terms in the expansion of $(x+a)^n + (x-a)^n$ is (n+2)/2 if n is even, (n+1)/2 if n is odd.

ii) The number of terms in the expansion of $(x+a)^n - (x-a)^n$ is n/2 if n is even, (n+1)/2 if n is odd.

32. Sum of the coefficients of $(ax+by)^n$ is $(a+b)^n$.

33. If $f(x) = (a_0 + a_1x + a_2x^2 + \ldots + a_mx^m)^n$ then

i) Sum of the coefficients = f(1).

ii) Sum of the coefficients of even powers of x is $\frac{f(1)+f(-1)}{2}$.

iii) Sum of the coefficients of odd powers of x is $\frac{f(1)-f(-1)}{2}$.

34. $a.C_0^{\,2} + (a+d).C_1^{\,2} + (a+2d).C_2^{\,2} + \ldots + (a+nd).C_n^2 = \frac{1}{2}(2a+nd).\,{}^{2n}C_n$.

35. $C_0C_r + C_1C_{r+1} + C_2C_{r+2} + \ldots + C_{n-r}C_n = {}^{2n}C_{n-r}$.

36. ${}^mC_0\,{}^nC_r + {}^mC_1\,{}^nC_{r-1} + {}^mC_2\,{}^nC_{r-2} + \ldots + {}^mC_r\,{}^nC_0 = {}^{(m+n)}C_r$.

37. If n is a positive integer, then $(1-x)^{-n} = 1 + {}^nC_1x + {}^{(n+1)}C_2x^2 + {}^{(n+2)}C_3x^3 + \ldots$