3. ALGEBRA OF MATRICES

Quick Review

1. If the complex numbers are arranged in the form of a rectangular array consisting of the complex numbers in horizontal and vertical lines then that arrangement is called a ***matrix***. The horizontal lines of numbers in a matrix are called ***rows*** and the vertical lines of numbers in a matrix are called ***columns***. The numbers in a matrix are called ***elements*** or ***entries***.
2. If a matrix A has m rows and n columns then the matrix is said to be a ***type*** or ***order*** or ***size*** $m \times n$ (read as m by n).
3. An $m \times n$ matrix A is usually written as $A = \begin{bmatrix} a_{11} & a_{12} & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & a_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{m1} & a_{m2} & . & . & . & a_{mn} \end{bmatrix}$ or $A = (a_{ij})_{m \times n}, \begin{matrix} 1 \le i \le m \\ 1 \le j \le n \end{matrix}$.
4. A matrix A is said to be a ***square matrix*** if the number of rows in A is equal to the number of columns in A. An $n \times n$ square matrix is called a ***square matrix of type*** n.
5. If A is a square matrix then the diagonal in A from the first element of the first row to the last element of the last row is called the ***principal diagonal*** of A.
6. If A is a square matrix then the sum of elements in the principal digonal of A is called ***trace*** of A. It is denoted by *tra* A.
7. A matrix A is said to be a ***rectangular matrix*** if the number of rows in A is not equal to the number of columns in A. i.e., A is called a ***rectangular matrix*** if A is not a square matrix.
8. A matrix A is said to be a ***zero matrix*** if every element of A is equal to zero. An $m \times n$ zero matrix is denoted by $O_{m \times n}$ or O.
9. A matrix A is said to be a ***row matrix*** if A contains only one row.
10. A matrix A is said to be a ***column matrix*** if A contains only one column.
11. A square matrix $A = (a_{ij})_{n \times n}$ is said to be an ***upper triangular matrix*** if $a_{ij} = 0$ whenever $i > j$.
12. A square matrix $A = (a_{ij})_{n \times n}$ is said to be an ***lower triangular matrix*** if $a_{ij} = 0$ whenever $i < j$.
13. A square matrix A is said to be a ***triangular matrix*** if A is either an upper triangular matrix or a lower triangular matrix.
14. A square matrix A is said to be a ***diagonal matrix*** if A is both an upper triangular matrix and a lower triangular matrix. i.e., A square matrix in which every element is equal to 0, except those of principal diagonal of the matrix is a ***diagonal matrix***.
15. A diagonal matrix A is said to be a ***scalar matrix*** if all elements in the principal diagonal are equal.
16. A diagonal matrix is said to be a ***unit matrix*** if every element in the principal diagonal is equal to unity. A unit matrix of order n is denoted by I_n.

    Now $I_n = (x_{ij})_{n \times n}$ where $\begin{matrix} x_{ij} = 1 \text{ if } i = j \\ = 0 \text{ if } i \neq j \end{matrix}$
17. Two matrices A and B are said to be ***equal*** if (i) A, B are of same type and (ii) the corresponding elements in A and B are equal. It is denoted by A = B.

18. If A and B are two matrices of same type then their ***sum*** is defined to be the matrix obtained by adding the corresponding elements of A and B. It is denoted by A + B. If $A = (a_{ij})_{m \times n}$, $B = (b_{ij})_{m \times n}$ then $A + B = (c_{ij})_{m \times n}$ where $c_{ij} = a_{ij} + b_{ij}$.
19. Matrix addition is commutative. i.e., If A and B are two matrices of same type then A + B = B + A.
20. Matrix addition is associative i.e. If A, B and C are three matrices of same type then (A + B) + C = A + (B + C).
21. If A is an m × n matrix then A + O = O + A = A. The matrix O is called ***additive identity matrix.***
22. If A is an m × n matrix then there exists as m × n matrix B such that A + B = B + A = O.
23. If A is an m × n matrix then the matrix B such that A + B = B + A = O is called ***additive inverse matrix*** of A. It is denoted by – A.
24. If A, B are two matrices of same type then A + (–B) is called the ***difference*** of B in A. It is denoted by A – B.
25. If A is a matrix of type m × n and k is a complex number then their ***product*** is defined to be the matrix obtained by multiplying every element of the matrix A by the number k. It is denoted by kA.
26. If A, B are two matrices of same type and k, *l* are complex numbers, then (i) k (A + b) = kA + kB (ii) (k + *l*)A = kA + *l*A (iii) (k *l*) A = k(*l*A) (iv) 0 A = O (v) kO = O.
27. Two matrices A and B are said to be ***conformable for multiplication*** if the number of columns in A is equal to the number of rows in B. If $A = (a_{ij})_{m \times n}$, $B = (b_{jk})_{n \times p}$ are two matrices (which are conformable for multiplication) then their product is defined to be the matrix $C = (c_{ik})_{m \times p}$ where $c_{ij} = \sum_{j=1}^{n} a_{ij} b_{jk}$. It is denoted by AB.
28. If the product AB exists then it is not necessary that the product BA will also exist.
29. Matrix multiplication is not commutative. Even if AB and BA exist, they need not be equal.
30. If AB=O, then either A or B need not be equal to O.
31. If AB = AC then B need not be equal to C even if $A \neq O$.
32. Matrix multiplication is associative, i.e., if conformability is assured for the matrices A, B and C, then (AB)C = A(BC).
33. Matrix multiplication is distributive over matrix addition i.e., If conformability is assured for the matrices A, B and C, then
    i) A(B +C) = AB + AC; (ii) (B + C)A = BA + CA
34. If A is a matrix of type m × n then $AI_n = I_m A = A$.
35. If A is a square matrix, then AI = IA = A. The matrix I is called ***multiplicative identity matrix.***
36. If A is a square matrix, then $A^m A^n = A^{m+n} = A^n A^m$.
37. A square matrix A is said to be an ***idempotent matrix*** if $A^2 = A$.
38. A square matrix A is said to be an ***involuntary matrix*** if $A^2 = I$.
39. A square matrix A is said to be a ***nipotent matrix*** if there exists a positive integer n such that $A^n = O$. If n is the least positive integer such that $A^n = O$, then n is called the ***index*** of the nilpotent matrix A.
40. The matrix obtained by changing the rows of a given matrix A into columns is called ***transpose*** of A. It is denoted by A^T or A′. If $A = (a_{ij})_{m \times n}$ then $A^T = (a'_{ji})_{m \times n}$ where $a'_{ji} = a_{ij}$.

41. If A is any matrix, then $(A^T)^T = A$.
42. If A and B are two matrices of same type, then $(A+B)^T = A^T + B^T$.
43. If A is any matrix and k is any complex number then $(kA)^T = kA^T$.
44. If A and B are two matrices for which conformability for multiplication is assured, then $(AB)^T = B^T A^T$.
45. If $A_1, A_2, \ldots, A_n$ are n matrices which are conformable for multiplication then $(A_1, A_2 \ldots A_n)^T = A_n^T A_{n-1}^T \ldots A_1^T$.
46. A square matrix A is said to be a ***symmetric matrix*** if $A^T = A$.
47. A square matrix A said to be a ***skew symmetric matrix*** if $A^T = -A$.
48. Every square matrix can be uniquely expressed as a sum of a symmetric matrix and a skew symmetric matrix.
49. If A and B are two square matrices of same type then
    (i) tra (A + B) = tra A + tra B (ii) tra (A – B) = tra A – tra B
    (iii) tra (AB) = tra BA (iv) tra (kA) = ktra A where k is a complex number.
50. **Conjugate of a matrix :** The matrix obtained from a matrix A on replacing elements by the corresponding conjugate complex numbers is called the ***conjugate*** of A and it is denoted by $\overline{A}$.
51. **Transposed conjugate of a matrix :** The transpose of the conjugate of a matrix A is called ***transposed conjugate*** of A and is denoted by A^θ or A.
52. **Hermitian matrix :** A square matrix A is said to be a ***hermitian matrix*** if $A^\theta = A$.
53. **Skew-hermitian matrix :** A square matrix A is said to be a ***skew-hermitian matrix*** if $A^\theta = -A$.

INVERSE MATRIX

1. The transpose of the matrix obtained by replacing the elements of a square matrix A by the corresponding cofactors is called the **adjoint matrix** of A. It is denoted by Adj A or adj A.
2. If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then $\text{Adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$
3. Square matrix A is said to be an **invertible matrix** if there exists a square matrix B such that AB = BA = I. The matrix B is called **inverse** of A.
4. Every square matrix need not be invertible.
5. An invertible matrix has unique inverse.
6. If A is an invertible matrix then its inverse is denoted by A^{-1}. Now $AA^{-1} = A^{-1}A = I$.
7. If I is a unit matrix then I is invertible and $I^{-1} = I$.
8. If A is an invertible matrix then A^{-1} is also invertible and $(A^{-1})^{-1} = A$.
9. In A and B are two invertible matrices of same type then AB is also invertible and $(AB)^{-1} = B^{-1}A^{-1}$.
10. If A is an invertible matrix and k is a nonzero complex number then kA is invertible and $(kA)^{-1} = k^{-1}A^{-1}$.
11. If A is an invertible matrix then A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$.
12. If A is a non-singular matrix then A is invertible and $A^{-1} = \frac{\text{Adj } A}{\det A}$.
13. If A is a square matrix, then A(Adj A) = (det A)I.

14. If A is an invertible matrix then $\det(A^{-1}) = \frac{1}{\det A}$.

15. If $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ and $ad \neq bc$ then $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

16. Let A, B, C be three square matrices of same type and A be a non-singular matrix. Then (i) $AB = AC \Rightarrow B = C$ (ii) $BA = CA \Rightarrow B = C$.

17. If A and B are non-singular matrices of same type then Adj(AB) = (Adj B) (Adj A).

18. If A is a square matrix of type n then $\det(\text{Adj } A) = (\det A)^{n-1}$.

19. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\text{Adj } A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

20. If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ and $abc \neq 0$ then

$A^{-1} = \begin{bmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{bmatrix}$.

DETERMINANTS

1. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then the number ad – bc is called ***determinant*** of A. It is denoted by det A or |A| or $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

2. If a_{ij} is an element which is in the i^{th} row and j^{th} column of a square matrix A, then the product if $(-1)^{i+j}$ and the minor of a_{ij} is called ***cofactor*** of a_{ij}. It is denoted by A_{ij}.

3. If $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, then $a_1A_1 + b_1B_1 + c_1C_1 = a_2A_2 + b_2B_2 + c_2C_2 = a_3A_3 + b_3B_3 + c_3C_3 = a_1A_1 + a_2A_2 + a_3A_3 = b_1B_1 + b_2B_2 + b_3B_3 = c_1C_1 + c_2C_2 + c_3C_3$.

4. The sum of the products of the elements of the first row of a square matrix A with their corresponding cofactors is called the ***determinant*** of the matrix A.

5. The determinant of a square matrix A is equal to the sum of the products of the elements of a row (or column) of A with their corresponding cofactors.

6. A square matrix A is said to be a (i) ***singular matrix*** if det A = 0, (ii) ***non-singular matrix*** if det $A \neq 0$.

7. If A is a square matrix, then $\det A = \det A^T$.

8. The sign of the determinant of a square matrix changes if any two rows (or columns) in the matrix are interchanged.

9. If two rows (or columns) of a square matrix are identical, the value of the determinant of the matrix is zero.

10. If all the elements of a row (or column) of a square matrix are multiplied by a number k then the value of the determinant of the matrix obtained is k times the determinant of the given matrix.

11. If the elements of a row (or column) of a square matrix are k times the elements of another row (or column), then the value of the determinant of the matrix is zero.

12. In each element in a row (or column) of a square matrix is the sum of two numbers, then its determinant can be expressed as the sum of the determinants of two square matrices.
13. If the elements of a row (or column) of a square matrix are added with k times the corresponding elements of another row (or column), then the value of the determinant of the matrix obtained is equal to the value of the determinant of the given matrix.
14. The sum of the products of the elements of any row (or column) of a square matrix with the cofactors of the corresponding elements of another row (or column) of the matrix is zero.
15. If A and B are two square matrices of same type, then det (AB) = det A . det B.
16. The determinant of a triangular matrix is the product of the elements of the principal diagonal of the matrix.
17. $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$.
18. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 3abc - a^3 - b^3 - c^3$.
19. $\begin{vmatrix} 1+a & b & c \\ a & 1+b & c \\ a & b & 1+c \end{vmatrix} = 1+a+b+c$.
20. $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = (a-b)(b-c)(c-a)$
21. $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & bc \\ 1 & b^2 & ca \\ 1 & c^2 & ab \end{vmatrix} =$

$(a-b)(b-c)(c-a)(a+b+c)$

22. $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = \begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

LINEAR EQUATIONS

1. Consider the system of simultaneous linear equations in two variables. $a_1x+b_1y = c_1$, $a_2x + b_2y= c_2$

   Let $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

   The system of equations (1) can be represented as a matrix equation AX = B.
2. If $x = \alpha$, $y = \beta$ where α, β are complex numbers, satisfy the equations of system (1), then $x = \alpha$, $y = \beta$ is called a ***solution*** of system (1).
3. Consider the system of simultaneous linear equations in three variables, $a_1x + b_1y + c_1z = d_1$; $a_2x + b_2y + c_2z = d_2 \rightarrow (2)$ $a_3x + b_3y + c_3z = d_3$

   Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$.

The system of equations (2) can be represented as a matrix equation AX = B.

4. The system of equations AX = B is said to be ***consistent*** if AX = B has a solution.
5. The system of equations AX = B is said to be ***inconsistent*** if AX = B has no solution.
6. If A is a non-singular matrix then $A^{-1}B$ is a solution of AX = B.
7. If A is a non-singular matrix then AX = B has a unique solution.
8. If A is a non-singular matrix then X = $A^{-1}B$ is the unique solution of AX = B.
9. **Cramer's Rule :** Let $a_1x + b_1y + c_1z = d_1$; $a_2x + b_2y + c_2z = d_2$; $a_3x + b_3y + c_3z = d_3 \rightarrow (1)$ be a system of linear equations. If $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$,

$\Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$,

$\Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$ then $x = \frac{\Delta_1}{\Delta}$, $y = \frac{\Delta_2}{\Delta}$, $z = \frac{\Delta_3}{\Delta}$ is the solution of (1).

10. Let $a_1x + b_1y + c_1z = d_1$; $a_2x + b_2y + c_2z = d_2$; $a_3x + b_3y + c_3z = d_3 \rightarrow (1)$ be a system of linear equations. The matrix $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is called ***coefficient matrix*** and the matrix $\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$ is called ***augmented matrix***.
11. **Elementary transformations :** An elementary ***row (column) transformation*** is an operation of any one of the following types.
    1) The interchange of any two rows (columns).
    2) The multiplication of the elements of any row (column) by a nonzero number.
    3) The addition to the elements of a row (column), the corresponding elements of any other row (column) multiplied by a number.
12. The following notation will be used to denote the elementary transformations.
    1) The interchange of i^{th} and j^{th} rows will be denoted by $R_i \leftrightarrow R_j$ (for columns $C_i \leftrightarrow C_j$).
    2) The multiplication of the elements of I^{th} row by a nonzero number k will be denoted by $R_i \rightarrow k\,R_i$ or simply kR_i (for columns $C_i \rightarrow kC_i$ or simply kC_i).
    3) The addition to the elements of i^{th} row, the corresponding elements of j^{th} row multiplied by a number k will be denoted by $R_i \rightarrow R_i + kR_j$ or simply $R_i + kR_j$ (for columns $C_i \rightarrow C_i + kC_j$ or simply $C_i + kC_j$).
13. **Gauss Jordan Method :** Let $a_1x + b_1y + c_1z = d_1$; $a_2x + b_2y + c_2z = d_2$; $a_3x + b_3y + c_3z = d_3 \rightarrow (1)$ be a system of linear equations.

    If the augmented matrix $\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$ can be reduced to the form $\begin{bmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & \beta \\ 0 & 0 & 1 & \gamma \end{bmatrix}$ by using elementary row transformations then $x = \alpha$, $y = \beta$, $z = \gamma$ is the solution of (1).
14. A matrix obtained by deleting some rows or columns (or both) of a matrix is called a ***submatrix.***
15. If B is a square submatrix of order r of a matrix A then det B is called an ***r-rowed miner*** of A.

16. (1) Suppose A is a nonzero matrix. A positive integer r is said to be the ***rank*** of A if (i) there exists a nonzero r-rowed minor of A (ii) every (r + 1)-rowed minor of A (if exists) is zero. (2) The rank of a zero matrix is defined to be zero.
17. Suppose A is a nonzero 3×3 matrix. Then (i) If A is a non-singular matrix then its rank s 3. (ii) If A is a singular matrix and if atleast one of its 2×2 submatrix is non-singular then the rank of A is 2. (iii) If A is a singular matrix and every 2×2 submatrix is also singular then the rank of A is 1.
18. Suppose A is a nonzero matrix of order 3×4 or 4×3. The rank of A is the maximum of ranks of all 3×3 submatrices of A.
19. Elementary transformations do not change the rank of a matrix.
20. The rank of a matrix remains unaltered by a finite chain of elementary transformations.
21. The system of equations AX = B is consistant iff the coefficient matrix A and the augmented matrix K are of the same rank.
22. Let AX = B be a system of equations in n unknowns, such that the rank of the coefficient matrix A is r_1 and the rank of the augmented matrix K is r_2.

    i) If $r_1 \neq r_2$ then the system AX = B is inconsistant. i.e. it has no solution.

    ii) If $r_1 = r_2 = n$ then the system AX = B is consistant and it has unique solution.

    iii) If $r_1 = r_2 < n$ then the system AX = B is consistant and it has infinitely many solutions.
23. The equations $a_1x + b_1y + c_1z = 0$; $a_2x+b_2y+c_2z=0$; $a_3x + b_3y + c_3z = 0$ are called a system of linear homogeneous equations in x, y, z.
24. x = 0, y = 0, z = 0 is always a solution of the system of linear homogeneous equations.
25. Let the augmented matrix $\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$ of AX = B can be reduced to the form $\begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ 0 & p_5 & p_6 & p_7 \\ 0 & 0 & p_8 & p_9 \end{bmatrix}$ by using elementary operations.

    1) If $p_8 \neq 0$, then AX = B has unique solution.

    2) If $p_8 = 0$, $p_9 \neq 0$, then AX = B has no solution.

    3) If $p_8 = 0$, $p_9 = 0$ and either $p_5 \neq 0$ or $p_6 \neq 0$ then AX = B has infinitely many solutions.

    4) If $p_8 = 0$, $p_9 = 0$, $p_5 = 0$, $p_6 = 0$ and $p_1 \neq 0$, then AX = B has no solution.
26. The rank of a unit matrix of order n is n.
27. The rank of a non-singular matrix of order n is n.