9. LIMITS

Synopsis :

1. Let f(x) be a function defined over a deleted neighborhood of 'a' and $l \in R$.
   i) If for each $\in > 0$ there exists $\delta>0$, such that $|f(x)-l|<\in$ when ever $0<|x-a|<\delta$ then we say that $\lim_{x\to a} f(x) = l$.
2. If limit of a function exists then it is unique.
3. If $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = l$, then $\lim_{x\to a} f(x) = l$.
4. If f(x) = c (constant), then $\lim_{x\to a} f(x) = c$.
5. $\lim_{x\to a}\left(\frac{f}{g}\right)(x) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ provided $\lim_{x\to a} g(x) \neq 0$.
6. $\lim_{x\to a} f^n(x) = \left\{\lim_{x\to a} f(x)\right\}^n$
7. $\lim_{x\to a} f(x) = \pm\infty \Rightarrow \lim_{x\to a} \frac{1}{f(x)} = 0$ the converse need not be true.
8. If $f(x) \le g(x) \le h(x)$ for every x in the deleted neighborhood of a and $\lim_{x\to a} f(x) = l = \lim_{x\to a} h(x)$ then $\lim_{x\to a} g(x) = l$.
9. $\lim_{x\to a} f(x) = l$ and $\lim_{y\to l} g(y) = m$, then $\lim_{x\to a} (gof)(x) = m$.
10. $\lim_{x\to a} \log[f(x)] = \log\left(\lim_{x\to a} f(x)\right)$, if $\lim_{x\to a} f(x)$ exists and is positive.
11. $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1 = \lim_{\theta\to 0} \frac{\tan\theta}{\theta}$ where θ is in radians.
12. $\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a$
13. $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$
14. $\lim_{x\to 0_+} e^{1/x} = \infty$, $\lim_{x\to 0^-} e^{1/x} = 0$
15. $\lim_{x\to 0^+} e^{-1/x} = 0$, $\lim_{x\to 0^-} e^{-1/x} = \infty$
16. $\lim_{n\to\infty} x^n = 0$ if $|x|<1$
17. $\lim_{n\to\infty} x^n = \infty$ if $x>1$

L'Hospials rule :

18. If $\frac{f(a)}{g(a)}$ is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f^l(a)}{g^l(a)}$.
19. $e^x = 1 + x + \frac{x^2}{\angle 2} + \frac{x^3}{\angle 3} + ...\text{to}\ \infty$

    $a^x = 1 + x + (\log_e a) + \frac{x^2}{\angle 2}(\log_e a)^2 + ...$

$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} - \ldots.$ $(-1<x\leq 1)$

$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots.$ $(-1\leq x<1)$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots.$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots.$

$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots.$

20. If $n \in Q$, then $\lim\limits_{x\to a}\frac{x^n - a^n}{x-a} = n.a^{n-1}$

21. If $m, n \in Q$ then $\lim\limits_{x\to a}\frac{x^m - a^m}{x^n - a^n} = \frac{m}{n}.a^{m-n}$

22. If f(x) and g(x) are two polynomials such that degree of f(x) = m and degree of g(x) = n then

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \begin{cases} 0, & \text{for } m < n \\ \pm\infty, & \text{for } m > n \\ \dfrac{\text{coefficient of } x^n \text{ in (numerator)}}{\text{coefficient of } x^n \text{ in (denominator)}} & \text{for } m = n \end{cases}$$

23. $\lim\limits_{x\to\infty}\left\{\sqrt{x^2+ax+b} - x\right\} = \frac{a}{2}$

24. $\lim\limits_{x\to\infty}\left\{x - \sqrt{x^2+ax+b}\right\} = \frac{-a}{2}$

25. $\lim\limits_{x\to 0}\frac{\sqrt{1+x^n} - \sqrt{1-x^n}}{x^n} = 1$

26. $\lim\limits_{x\to 0}\frac{\sqrt[n]{a+x^m} - \sqrt[n]{a-x^m}}{x^m} = \frac{2}{n}a^{\frac{1}{n}-1}$

27. $\lim\limits_{x\to 0}(1+x)^{1/x} = \lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$

28. $\lim\limits_{x\to 0}(1+ax)^{1/x} = \lim\limits_{x\to\infty}\left(1+\frac{a}{x}\right)^x = e^a$