5. PAIR OF STRAIGHT LINES

Synopsis:

1. If $h^2 \geq ab$, then $ax^2 + 2hxy + by^2 = 0$ represent a pair of straight lines passing through the origin.
2. If $h^2 < ab$, then $ax^2 + 2hxy + by^2 = 0$ represents two imaginary lines having real point of intersection, the origin.
3. If $h^2 = ab$, then $ax^2 + 2hxy + by^2 = 0$ represents coincident lines.
4. If $h^2 > ab$, then $ax^2 + 2hxy + by^2 = 0$ represents two real and different lines.
5. If $ax^2 + 2hxy + by^2 = 0$ represents a pair of lines,
   i) sum of slopes of lines is $2h/b$.
   ii) the product of slopes of lines is a/b
6. If θ is an angle between the lines represented by $ax^2 + 2hxy + by^2 = 0$, then

   i) $\cos\theta = \dfrac{a+b}{\sqrt{(a-b)^2 + 4h^2}}$

   ii) $\sin\theta = \dfrac{2\sqrt{h^2 - ab}}{\sqrt{(a-b)^2 + 4h^2}}$

   iii) $\tan\theta = \dfrac{2\sqrt{h^2 - ab}}{a+b}$
7. If θ is the acute angle between the lines represented by $ax^2 + 2hxy + by^2 = 0$, then

   i) $\cos\theta = \dfrac{|a+b|}{\sqrt{(a-b)^2 + 4h^2}}$

   ii) $\tan\theta = \dfrac{2\sqrt{h^2 - ab}}{|a+b|}$
8. If $\theta = 0$, then the two lines will be coincident $h^2 = ab$.
9. If $\theta = \pi/2$, then the two lines will be perpendicular
   $a + b = 0 \Rightarrow$ coefficient of x^2 + coefficient of $y^2 = 0$.
10. The equation to the pair of lines passing through (α, β) and parallel to $ax^2 + 2hxy + by^2 = 0$ is
    $a(x-\alpha)^2 + 2h(x-\alpha)(y-\beta) + b(y-\beta)^2 = 0$
11. The equation to the pair of lines passing through the origin and perpendicular to $ax^2 + 2hxy + by^2 = 0$ is $bx^2 - 2hxy + ay^2 = 0$.
12. The equation to the pair of lines passing through the point (α, β) and perpendicular to $ax^2 + 2hxy + by^2 = 0$ is $b(x-\alpha)^2 - 2h(x-\alpha)(y-\beta) + a(y-\beta)^2 = 0$.
13. The equation of bisectors of angles between the lines $a_1x + b_1y + c_1 = 0$. $a_2x + b_2y + c_2 = 0$ are $\dfrac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b^2_1}} \pm \dfrac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b^2_2}} = 0$ If $c_1c_2 > 0$ then the line bisecting the angle containing the origin between

the lines is $\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b^2_1}}-\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b^2_2}}=0$ and the line bisecting the other angle is $\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b^2_1}}+\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b^2_2}}=0$

14. The equation to the pair of bisectors of angles between the pair of lines $ax^2 + 2hxy + by^2 = 0$ is $h(x^2 - y^2) - (a - b)xy = 0$.
15. Every pair of lines are equally inclined to either of its angle bisectors.
16. Two pairs of concurrent lines are equally inclined to each other if both the pairs of lines have the same angle bisectors.
17. The product of the perpendiculars from (α,β) to the pair of lines $ax^2 + 2hxy + by^2 = 0$ is $\frac{\left|a\alpha^2+2h\alpha\beta+b\beta^2\right|}{\sqrt{(a-b)^2+4h^2}}$.
18. The area of the triangle formed by the lines $ax^2 + 2hxy + by^2 = 0$ and the line $lx + my + n = 0$ is $\left|\frac{n^2\sqrt{h^2-ab}}{am^2-2hlm+bl^2}\right|$.
19. The line $ax + by + c = 0$ and the pair of lines $(ax + by)^2 - 3(bx - ay)^2 = 0$ form an equilateral triangle and its area is $\frac{c^2}{\sqrt{3}(a^2+b^2)}$ sq. unit.
20. The equation to the pair of lines passing through the origin and forming an equilateral triangle with the line $ax + by + c = 0$ is $(ax + by)^2 - 3(bx - ay)^2 = 0$.
21. If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of lines then

    i) $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

    ii) $h^2 \geq ab$, $g^2 \geq ac$, $f^2 \geq bc$.

    iii) The point of intersection of the lines is $\left(\frac{hf-bg}{ab-h^2},\frac{gh-af}{ab-h^2}\right)$ where $h^2 \neq ab$.
22. The difference of the slopes of the lines represented by $ax^2 + 2hxy + by^2 = 0$ is $\frac{2\sqrt{h^2-ab}}{|b|}$.
23. The condition that the slope of the lines represented by $ax^2 + 2hxy + by^2 = 0$ is the ratio p:q is $ab(p + q)^2 = 4h^2pq$.
24. If $a_1a_2 + b_1b_2 > 0$, then the acute angle bisector of the angles between $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ is $\frac{a_1x+b_1y+c_1}{\sqrt{a^2_1+b^2_1}}+\frac{a_2x+b_2y+c_2}{\sqrt{a^2_2+b^2_2}}=0$.
25. The equation to the pair of lines passing through the origin and each is at a distance of 'd' from (x_1, y_1) is $(xy_1 - x_1y)^2 = d^2(x^2 + y^2)$.
26. The condition that the lines $ax^2 + 2hxy + by^2 = 0$, $lx + my + n = 0$ to form an isosceles triangle of vertex origin is $h(l^2 - m^2) = (a - b)lm$.
27. If $ax^2 + 2hxy + by^2 = 0$ are two sides of a parallelogram and $lx + my + n = 0$ is one diagonal, then the equation to the other diagonal is $y(bl - hm) = x(am - hl)$.

28. The length of the intercept on x – axis cut by the pair of the lines $ax^2 + 2hxy + by^2 + 2gx + 2fy+c = 0$ is $\frac{2\sqrt{g^2 - ac}}{|a|}$.

29. The length of the intercept on y – axis cut by the pair of lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is $\frac{2\sqrt{f^2 - bc}}{|b|}$.

30. The condition that the pair of lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ intersect on x – axis is $g^2 = ac$, on y – axis is $f^2 = bc$.

31. The condition that the pair of lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is $\frac{\left|a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c\right|}{\sqrt{(a-b)^2 + 4h^2}}$

32. The lines represented by $ax^2 + 2hxy + by^2 = 0$,$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ to form a
    i) parallelogram if $a + b \neq 0$, $(a - b)fg + h + h(f^2 - g^2) \neq 0$
    ii) rectangle if $a + b = 0$, $(a - b)fg + h(f^2 - g^2) \neq 0$
    iii) rhombus if $a + b \neq 0$, $(a - b)fg + h(f^2 - g^2) = 0$
    iv) square of $a + b = 0$, $(a - b)fg + h(f^2 - g^2) = 0$

35. The area of parallelogram formed by $ax^2 + 2hxy + by^2 = 0$,$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is $\frac{|c|}{2\sqrt{h^2 - ab}}$

36. The condition that one of the lines represented by $S = 0$ bisects the angle between the co – ordinate axes is $(a + b)^2 = 4h^2$.

37. Equation to the pair of angular bisectors of the lines represented by $S^I = 0$ is $\frac{(x - x_1)^2 - (y - y_1)^2}{a - b} = \frac{(x - x_1)(y - y_1)}{h}$ where (x_1, y_1) is point of intersection of lines of $S^I = 0$.

38. I) If 'θ' is the angle between the lines joining the origin to the points of intersection of the circle $x^2 + y^2 = a^2$ and the line $lx + my + n = 0$. Then $\text{Cos}\frac{\theta}{2} = \left|\frac{n}{a\sqrt{l^2 + m^2}}\right|$.

    ii) These lines are perpendicular then $a^2(l^2 + m^2) = 2n^2$.

37. The line joining the origin to the points of intersection of the line $kx + hy = 2hk$ with the curve $(x-h)^2 + (y-k)^2 = c^2$ are at right angles if $h^2+k^2 = c^2$.

38. Point of intersection of diagonals of rectangle formed by the pairs $a_1x^2 + b_1x + c_1 = 0$ and $a_2y^2 + b_2y + c_2 = 0$ is $\left(\frac{-b_1}{2a_1}, \frac{-b_2}{2a_2}\right)$.

39. If equation $ax^2 + 2hxy + by^2 = 0$ represents two sides of a triangle whose orthocentre is (c, d) then equation of third side is $(a + b)(cx + dy) = bc^2 - 2hcd + ad^2$.

40. The lines joining the origin to the points of intersection of the curves $a_1x^2 + 2h_1xy + b_1y^2 + 2g_1x = 0$ and $a_2x^2 + 2h_2xy + b_2y^2 + 2g_2x = 0$ will be at right angles if $\frac{a_1 + b_1}{a_2 + b_2} = \frac{g_1}{g_2}$.