17. COMPLEX NUMBERS

Synopsis :

1. Complex Number :

   A number which is of the form x + iy, where x, y are real numbers and i = $\sqrt{-1}$ is called a complex number. It also be written as (x, y). If z = x + iy, then x is called real part of z, y is called imaginary part of z.

2. Argand plane :

   By taking real part along x-axis, imaginary part along y-axis, complex numbers (x, y) can be plotted as points in xy plane, which is called Argand plane.

   x-axis is called real axis, y-axis is called imaginary axis.

   The points on x-axis are purely real, points on y-axis are purely imaginary, point of intersection of x and y axes is origin which represents O.

3. Modulus of a complex number :

   If z = x + iy, then $\sqrt{x^2+y^2}$ is called modulus of z and it is denoted by |z|. If z is represented by P(x, y), then |z| = OP = $\sqrt{x^2+y^2}$.

4. Amplitude or argument of a complex number :

   If OP makes an angle θ with positive direction of x-axis, then θ is called amplitude or argument of z, where z is represented by P(x, y).

   It is denoted by arg z or amp z. The value of θ, which satisfies x = r cos θ, y = r sin θ where r = |z| and lying in the interval $(-\pi, \pi)$ is called principal value of θ. If α is the principal value of θ, then $2n\pi + \alpha$, $n \in Z$ is called general value of θ.

5. Modulus amplitude form of a complex number :

   If z = x + iy, |z| = r, arg z = θ, then $\cos\theta = x/r$, $\sin\theta = y/r$ and $z = r(\cos\theta + i\sin\theta)$, which is called modulus amplitude form of z. Also $\cos\theta + i\sin\theta = e^{i\theta}$ (Euler's theorem on power series).

   Ordered relations $z_1 > z_2$ or $z_1 < z_2$ are not defined in the set of complex numbers.

6. Equality of complex numbers :

   If $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, then $z_1 = z_2 \Leftrightarrow x_1 = x_2$ and $y_1 = y_2$.

7. Addition, subtraction, multiplication and division of complex numbers.

   If $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$,

   i) $z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2)$

ii) $z_1 - z_2 = x_1 - x_2 + i(y_1 + y_2)$

iii) $z_1z_2 = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)$

iv) $\frac{z_1}{z_2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} - \frac{i(x_1y_2 - x_2y_1)}{x_2^2 + y_2^2}$

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$; $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then $z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

v) $\frac{z_1}{z_2} = \frac{r_1}{rr_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$

vi) Additive inverse of $x + iy$ is $x - iy$.

vii) Multiplicative inverse of $x + iy$ is $\frac{x}{x^2 + y^2} - \frac{iy}{x^2 + y^2}$.

viii) Multiplicative inverse of $\text{cis}\theta$ is $\text{cis}(-\theta)$.

8. **Conjugate :** If $z = x + iy$, then $\bar{z} = x - iy$ is called conjugate complex number of z.

i) $z = \bar{z} \Leftrightarrow z$ is purely real.

ii) $z = -\bar{z} \Leftrightarrow z$ is purely imaginary

iii) $z + \bar{z} = 2\,\text{Re}\,z$

iv) $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$

v) $\overline{z_1z_2} = \bar{z}_1\bar{z}_2$

vi) $\left(\frac{\bar{z}_1}{z_2}\right) = \frac{\bar{z}_1}{\bar{z}_2}$ where $z_2 \neq 0$

9. Properties of modulus of a complex number :

i) $|z_1 \pm z_2| \leq |z_1| + |z_2|$ (triangle inequality)

ii) $|z_1 \pm z_2| \geq |z_1| - |z_2|$

iii) $|z_1z_2| = |z_1| \cdot |z_2|$

iv) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

v) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$

vi) $z\bar{z} = |z|^2$

vii) $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm 2\,\text{Re}(z_1\,\bar{z}_2) =$

$|z_1|^2 + |z_2|^2 \pm 2|z_1||z_2|\cos(\theta_1 - \theta_2)$ where $\theta_1 =$

arg z_1 and $\theta_2 =$ arg z_2.

viii) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

10. Properties of amplitude :

    i) $\arg(z_1 z_2) = \arg z_1 + \arg z_2 + 2n\pi$

    ii) $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 + 2n\pi$

    iii) $\arg \bar{z} = -\arg z$

    iv) arg z is not defined if z = 0.

    v) $\arg z = \text{Tan}^{-1}\left(\frac{y}{x}\right)$, if $x > 0, y \neq 0$

    vi) $\arg z = \pi + \text{Tan}^{-1}\left(\frac{y}{x}\right)$, if $x < 0, y > 0$.

    vii) $\arg z = -\pi + \text{Tan}^{-1}\left(\frac{y}{x}\right)$, if $x < 0, y < 0$.

    viii) $\arg z = \pi/2$, if $x = 0, y > 0 = -\pi/2$ if $x = 0, y < 0$.

11. Demoivre's theorem :

    $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, where n is any integer.

    If n is rational, $\cos n\theta + i\sin n\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$.

12. Integral powers of 'i' :

    If k is any integer, $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, $i^{4k+3} = -i$.

13. Square root of a complex number :

    i) $\sqrt{x+iy} = \pm\left(\sqrt{\frac{\sqrt{x^2+y^2}+x}{2}}\right) + i\left(\sqrt{\frac{\sqrt{x^2+y^2}-x}{2}}\right)$

    ii) $\sqrt{x+iy} = p+iq \Rightarrow \sqrt{x-iy} = p-iq$

    iii) $\sqrt{-a}\sqrt{-b} = \sqrt{-1}\sqrt{a}\sqrt{-1}\sqrt{b} = -\sqrt{ab}$

14. n^{th} roots of unity :

    i) $\omega^r = e^{\frac{2\pi r}{n}i} = \cos\left(\frac{2\pi r}{n}\right) + i\sin\left(\frac{2\pi r}{n}\right)$ where r = 0,

    1, 2, ..., n –1 are n^{th} roots of unity.

    ii) Sum of n^{th} roots of unity is zero.

    iii) Product of n^{th} roots of unity is $(-1)^{n-1}$.

    iv) Cube roots of unity are 1, ω, ω^2, where $\omega = \frac{-1+i\sqrt{3}}{2}$, $\omega^2 = \frac{-1-i\sqrt{3}}{2}$

v) Also $1 + \omega + \omega^2 = 0$, $\omega^3 = 1$.

vi) Fourth roots of unity are 1, i, –1, –i.

vii) The n, n^{th} roots of unity are the vertices of an n – sided regular polygon inscribed in the circle $|z| = 1$.

viii) The three cube roots of unity are the vertices of an equilateral triangle inscribed in the circle $|z| = 1$.

15. z_1, z_2, z_3 represent points A, B, C respectively

i) If $z_1^2 + z_2^2 - z_1 z_2 = 0$ the origin, z_1 and z_2 form an equilateral triangle

ii) If z_1, z_2, z_3 are the vertices of equilateral triangle and z_0 is its centroid, then $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$.

iii) z, iz, –z, –iz in order form the vertices of a square with center at the origin.

iv) If z_1, z_2, z_3, z_4 are the consecutive vertices of a parallelogram, then $z_1 + z_3 = z_2 + z_4$.

v) If z divides the join of z_1 and z_2 in the ratio $\lambda_1 : \lambda_2$, then $z = \frac{\lambda_1 z_2 + \lambda_2 z_1}{\lambda_1 + \lambda_2}$.

vi) $|z - z_1| = r$ represents a circle with center at z_1 and radius r.

vii) $\left|\frac{z - z_1}{z - z_2}\right| = k$ represents a circle if $k \neq 1$ and a straight line if $k = 1$.

viii) Perpendicular bisector of the join of z_1, z_2 is given by $|z - z_1| = |z - z_2|$.

16. Logarithm of a complex number : $\log z = \log |z| + i \arg z$.