15. PROPERTIES OF TRIANGLES

Synopsis :

1. The perpendicular bisectors of the sides of a triangle are concurrent. The point of concurrence is called circumcentre of the triangle. If S is the circumcentre of ΔABC, then SA = SB = SC. The circle with center S and radius SA passes through the three vertices A, B, C of the triangle. This circle is called circumcircle of the triangle. The radius of the circumcircle of ΔABC is called circumradius and it is denoted by R.

2. Sine Rule : $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$

   $\therefore$ a = 2R sin A, b = 2R sin B, c = 2R sin C.

3. Cosine Rule : $a^2 = b^2 + c^2 - 2bc \cos A$, $b^2 = c^2 + a^2 - 2ca \cos B$, $c^2 = a^2 + b^2 - 2ab \cos C$.

4. $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$, $\cos B = \frac{c^2 + a^2 - b^2}{2ca}$,

   $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$.

5. Projection Rule : a = b cos C + c cos B, b = c cos A + a cos C, c = a cos B + b cos A.

6. Tangent Rule or Napier's Analogy : $\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cot\frac{A}{2}$,

   $\tan\left(\frac{C-A}{2}\right) = \frac{c-a}{c+a}\cot\frac{B}{2}$,

   $\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\frac{C}{2}$.

7. Mollweide Rule :

   $\frac{a+b}{c} = \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}}$, $\frac{a-b}{c} = \frac{\sin\left(\frac{A-B}{2}\right)}{\cos\frac{C}{2}}$

8. $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$, $\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$, $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$.

9. $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$, $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$, $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$.

10. $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$, $\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$, $\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

11. $\tan\frac{A}{2}=\frac{\Delta}{s(s-a)}=\frac{(s-b)(s-c)}{\Delta}$,

$\tan\frac{B}{2}=\frac{\Delta}{s(s-b)}=\frac{(s-c)(s-a)}{\Delta}$,

$\tan\frac{C}{2}=\frac{\Delta}{s(s-c)}=\frac{(s-a)(s-b)}{\Delta}$.

12. $\cot\frac{A}{2}=\frac{s(s-a)}{\Delta},\cot\frac{B}{2}=\frac{s(s-b)}{\Delta},\cot\frac{C}{2}=\frac{s(s-c)}{\Delta}$

13. Area of ΔABC is $\Delta=\frac{1}{2}bc\sin A=\frac{1}{2}ca\sin B=\frac{1}{2}\sin C=2R^2\sin A\sin B\sin C=$

$\frac{abc}{4R}\sqrt{s(s-a)(s-b)(s-c)}$.

14. $r=\frac{\Delta}{s}=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=(s-a)\tan\frac{A}{2}=(s-b)\tan\frac{B}{2}=(s-c)\tan\frac{C}{2}=\frac{a}{\cot\frac{B}{2}+\cot\frac{C}{2}}=\frac{b}{\cot\frac{C}{2}+\cot\frac{A}{2}}$

$=\frac{C}{\cot\frac{A}{2}+\cot\frac{B}{2}}$

15. $r_1=\frac{\Delta}{s-a}=4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=s\tan\frac{A}{2}=(s-b)\cot\frac{C}{2}=(s-c)\cot\frac{B}{2}=\frac{a}{\tan\frac{B}{2}+\tan\frac{C}{2}}$.

16. $r_2=\frac{\Delta}{s-b}=s\tan\frac{B}{2}=(s-c)\cot\frac{A}{2}=(s-a)\cot\frac{C}{2}=$

$4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}=\frac{b}{\tan\frac{C}{2}+\tan\frac{A}{2}}$.

17. $r_3=\frac{\Delta}{s-c}=s\tan\frac{C}{2}=(s-a)\cot\frac{B}{2}=(s-b)\cot\frac{A}{2}=$

$\frac{c}{\tan\frac{A}{2}+\tan\frac{B}{2}}$.

18. $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{1}{r}$.

19. $r\,r_1\,r_2\,r_3=\Delta^2$.

20. i) $\sum a^3\sin(B-C)=0$.

ii) $\sum a^3\cos(B-C)=3abc$

iii) $a^2\sin 2B+b^2\sin 2A=4\Delta$

21. i) $\cot A + \cot B + \cot C = \frac{a^2+b^2+c^2}{4\Delta}$

ii) $\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} = \frac{(a+b+c)^2}{4\Delta}$.

22. i) If a cos B = b cos A, then the triangle is isosceles.

ii) If a cos A = b cos B, then the triangle is isosceles or right angled.

iii) If $a^2 + b^2 + c^2 = 8R^2$, then the triangle is right angled.

iv) If $\cos^2 A + \cos^2 B + \cos^2 C = 1$, then the triangle is right angled.

v) If $\cos A = \frac{\sin B}{2\sin C}$, then the triangle is isosceles.

vi) If $\frac{a}{\cos A} = \frac{b}{\cos B} = \frac{c}{\cos C}$, then the triangle is equilateral.

vii) If cosA + cosB + cosC = 3/2, then the triangle is equilateral.

viii) If $\sin A + \sin B + \sin C = \frac{3\sqrt{3}}{2}$, then the triangle is equilateral.

ix) If $\cot A + \cot B + \cot C = \sqrt{3}$, then the triangle is equilateral.

23. i) If $\frac{a^2+b^2}{a^2-b^2} = \frac{\sin(A+B)}{\sin(A-B)}$, then C = 90°.

ii) If $\frac{a+b}{b+c} + \frac{b}{c+a} = 1$, then C = 60°.

iii) If $\frac{1}{a+b} + \frac{1}{a+c} = \frac{3}{a+b+c}$, then A = 60°

iv) If $\frac{b}{a^2-c^2} + \frac{c}{a^2-b^2} = 0$, then A = 60°.

i) a, b, c are In H.P. $\Leftrightarrow \sin^2\frac{A}{2}, \sin^2\frac{B}{2}, \sin^2\frac{C}{2}$ are in H.P.

ii) a, b, c are in A.P. $\Leftrightarrow \cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are in A.P.

iii) a, b, c are in A.P. $\Leftrightarrow \tan\frac{A}{2}, \tan\frac{B}{2}, \tan\frac{C}{2}$ are in H.P.

iv) a^2, b^2, c^2 are in A.P. $\Leftrightarrow$ cotA, cotB, cotC are in A.P.

v) a^2, b^2, c^2 are in A.P. $\Leftrightarrow$ tanA, tanB, tanC are in H.P