13. INVERSE TRIGONOMETRIC FUNCTIONS

Synopsis :

1. The function f : $[-\pi/2, \pi/2] \to [-1, 1]$ defined by f(x) = sinx is a bijection. The inverse of f from $[-1, 1]$ into $[-\pi/2, \pi/2]$ is also a bijection. This function is called inverse sine function or arc sine function. It is denoted by Sin^{-1} or Arc sin.

   Now $Sin^{-1}x = \theta \Leftrightarrow x = \sin\theta, \forall\ x \in [-1, 1]$

2. The domains and ranges of the inverse trigonometric functions are as follows.

S.No	Function	Domain	Range
1.	$Sin^{-1}x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
2.	$Cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
3.	$Tan^{-1}x$	R	$(-\pi/2, \pi/2)$
4.	$Cot^{-1}x$	R	$(0, \pi)$
5.	$Sec^{-1}x$	$(-\infty, -1] \cup [1, \infty) = R-(-1, 1)$	$[0, \pi/2) \cup (\pi/2, \pi)$
6.	$Cosec^{-1}x$	$(-\infty, -1] \cup [1, \infty) = R-R(-1, 1)$	$[-\pi/2, 0) \cup (0, \pi/2)$

3. i) $Sin^{-1}(-x) = -Sin^{-1}x$, for $x \in [-1, 1]$

   ii) $Cos^{-1}(-x) = \pi - Cos^{-1}x$, for $x \in [-1, 1]$

   iii) $Tan^{-1}(-x) = -Tan^{-1}x$, for $x \in R$

4. i) $Cot^{-1}(-x) = \pi - Cot^{-1}x$, for $x \in R$

   ii) $Sec^{-1}(-x) = \pi - Sec^{-1}x$, for $(-\infty, -1] \cup [1, \infty)$

   iii) $Cosec^{-1}(-x) = -Cosec^{-1}x$, for $(-\infty, -1] \cup [1, \infty)$

5. i) $Sin^{-1}(x) = Cosec^{-1}(1/x)$, for $x \in [-1, 0) \cup (0, 1]$

   ii) $Cos^{-1}(x) = Sec^{-1}(1/x)$, for $x \in [-1, 0) \cup (0, 1]$

   iii) $Tan^{-1}(x) = Cot^{-1}(1/x)$, for $x \in (0, \infty)$ and

   $Tan^{-1}x = -\pi + Cot^{-1}(1/x)$, for $x \in (-\infty, 0)$

6. If $x \in [-1, 1]$ then $\text{Cos}^{-1}x + \text{Sin}^{-1}x = \pi/2$.

7. If $x \in R$, then $\text{Tan}^{-1}x + \text{Cot}^{-1}x = \pi/2$.

8. If $x \in (-\infty, -1] \cup [1, \infty)$, then

   $\text{Sec}^{-1}x + \text{Cosec}^{-1}x = \pi/2$

9. If $0 \le x \le 1$, $0 \le y \le 0$, then

   i) $\text{Sin}^{-1}x + \text{Sin}^{-1}y = \text{Sin}^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$, for $x^2 + y^2 \le 1$

   ii) $\text{Sin}^{-1}x + \text{Sin}^{-1}y = \pi - \text{Sin}^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$, for $x^2 + y^2 > 1$

10. If $0 \le x \le 1$, $0 \le y \le 1$, then $\text{Sin}^{-1}x - \text{Sin}^{-1}y = \text{Sin}^{-1}(x\sqrt{1-y^2} - y\sqrt{1-x^2})$

11. If $0 \le x \le 1$, $0 \le y \le 1$, then

    i) $\text{Cos}^{-1}x + \text{Cos}^{-1}y = \text{Cos}^{-1}(xy - \sqrt{1-x^2}\sqrt{1-y^2})$, for $x^2 + y^2 \le 1$

    ii) $\text{Cos}^{-1}x + \text{Cos}^{-1}y = \pi - \text{Cos}^{-1}(\sqrt{1-x^2}\sqrt{1-y^2} - xy)$, for $x^2 + y^2 > 1$

12. If $0 \le x \le 1$, $0 \le y \le 1$, then $\text{Cos}^{-1}x - \text{Cos}^{-1}y = \text{Cos}^{-1}(xy + \sqrt{1-x^2}\sqrt{1-y^2})$

13. $\text{Tan}^{-1}x + \text{Tan}^{-1}y = \text{Tan}^{-1}\left(\frac{x+y}{1-xy}\right)$, for $x>0$, $y>0$, $xy<1$

    $= \pi + \text{Tan}^{-1}\left(\frac{x+y}{1-xy}\right)$, for $x>0$, $y > 0$, $xy > 1$.

14. $\text{Tan}^{-1}x + \text{Tan}^{-1}y = \text{Tan}^{-1}\left(\frac{x+y}{1-xy}\right)$, for $x<0$, $y<0$, $xy<1$

    $= -\pi + \text{Tan}^{-1}\left(\frac{x+y}{1-xy}\right)$, for $x < 0$, $y < 0$, $xy > 1$.

15. If $x>0$, $y>0$, then $\text{Tan}^{-1}x - \text{Tan}^{-1}y = \text{Tan}^{-1}\left(\frac{x-y}{1+xy}\right)$.

16. If $x<0$, $y<0$, then $\text{Tan}^{-1}x - \text{Tan}^{-1}y = \text{Tan}^{-1}\left(\frac{x-y}{1+xy}\right)$

17. i) $3\text{Sin}^{-1}x = \text{Sin}^{-1}(3x - 4x^3)$, for $0 \le x \le 1/2$

    ii) $3\text{Cos}^{-1}x = \text{Cos}^{-1}(4x^3 - 3x)$, for $\sqrt{3}/2 \le x \le 1$

    iii) $3\text{Tan}^{-1}x = \text{Tan}^{-1}\left(\frac{3x - x^3}{1-3x^2}\right)$, for $0 \le x \le 1/\sqrt{3}$

18. i) If $\text{Cos}^{-1}x + \text{Cos}^{-1}y + \text{Cos}^{-1}z = \pi$, then

$x^2 + y^2 + z^2 + 2xyz = 1$.

ii) If $\text{Sin}^{-1}x + \text{Sin}^{-1}y + \text{Sin}^{-1}z = \pi/2$, then

$x^2 + y^2 + z^2 + 2xyz = 1$.

iii) If $\text{Sin}^{-1}x + \text{Sin}^{-1}y + \text{Sin}^{-1}z = \pi$, then $x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$.

iv) If $\text{Tan}^{-1}x + \text{Tan}^{-1}y + \text{Tan}^{-1}z = \pi/2$, then

$xy + yz + zx = 1$.

v) If $\text{Tan}^{-1}x + \text{Tan}^{-1}y + \text{Tan}^{-1}z = \pi$, then

$x + y + z = xyz$.

19. i) If $\text{Sin}^{-1}x + \text{Sin}^{-1}y = \theta$, then $\text{Cos}^{-1}x + \text{Cos}^{-1}y = \pi - \theta$

ii) If $\text{Cos}^{-1}x + \text{Cos}^{-1}y = \theta$, then $\text{Sin}^{-1}x + \text{Sin}^{-1}y = \pi - \theta$

20. i) If $\text{Tan}^{-1}x + \text{Tan}^{-1}y = \pi/2$, then $xy = 1$.

ii) If $\text{Cot}^{-1}x + \text{Cot}^{-1}y = \pi/2$, then $xy = 1$.

21. i) If $\text{Sin}^{-1}\frac{a}{x} + \text{Sin}^{-1}\frac{b}{x} = \frac{\pi}{2}$, then $x = \sqrt{a^2 + b^2}$

ii) If $\text{Cos}^{-1}\frac{a}{x} + \text{Cos}^{-1}\frac{b}{x} = \frac{\pi}{2}$, then $x = \sqrt{a^2 + b^2}$

iii) If $\text{Tan}^{-1}\frac{a}{x} + \text{Tan}^{-1}\frac{b}{x} = \frac{\pi}{2}$, then $x = \sqrt{ab}$

22. If $\text{Cos}^{-1}\frac{x}{a} + \text{Cos}^{-1}\frac{y}{b} = \theta$, then $\frac{x^2}{a^2} - \frac{2xy}{ab}\cos\theta + \frac{y^2}{b^2} = \sin^2\theta$