6. TRIPLE PRODUCT AND PRODUCT OF FOUR VECTORS

Synopsis:

1. For any vector $\bar{a}$, $(\bar{a}\,x\bar{i})^2 + (\bar{a}x\bar{j})^2 + (\bar{a}x\bar{k})^2 = 2|\,\bar{a}\,|^2$
2. $\bar{a}.\bar{b}x\bar{c}$ or $\bar{a}x\bar{b}.\bar{c}$ is called the scalar product of three vectors and is written as $[\bar{a}\,\bar{b}\,\bar{c}]$.
3. i) In scalar triple product dot and cross are interchangeable i.e., $\bar{a}.\bar{b}x\bar{c} = \bar{a}x\bar{b}.\bar{c}$
   ii) If any two vectors are parallel or equal in scalar triple product its value becomes zero. i.e., $[\bar{a}\,\bar{b}\,\bar{b}] = [\bar{a}\,\bar{a}\,\bar{b}] = [\bar{a}\,\bar{c}\,\bar{c}] = 0$.
   iii) The value of scalar triple product remains unaltered unless the cyclic order is not changed. i.e.,
   (a) $[\bar{a}\,\bar{b}\,\bar{c}] = [\bar{b}\bar{c}\,\bar{a}] = [\bar{c}\,\bar{a}\,\bar{b}]$,
   (b) $[\bar{a}\,\bar{c}\,\bar{b}] = [\bar{b}\,\bar{a}\,\bar{c}] = [\bar{c}\,\bar{b}\,\bar{a}] = -[\bar{a}\,\bar{b}\,\bar{c}]$
   iv) $[\bar{a}+\bar{b}\;\;\bar{b}+\bar{c}\;\;\bar{c}+\bar{a}] = 2[\bar{a}\,\bar{b}\,\bar{c}]$
   v) $[\bar{a}x\bar{b}\;\;\bar{b}x\bar{c}\;\;\bar{c}x\bar{a}] = [\bar{a}\,\bar{b}\,\bar{c}]^2$
4. For unit orthonormal vectors $\bar{i}, \bar{j}, \bar{k}$
   i) $[\bar{i}\,\bar{j}\,\bar{k}] = [\bar{j}\,\bar{k}\,\bar{i}] = [\bar{k}\,\bar{i}\,\bar{j}] = 1$
   ii) $[\bar{j}\,\bar{i}\,\bar{k}] = [\bar{i}\,\bar{k}\,\bar{j}] = [\bar{k}\,\bar{j}\,\bar{i}] = -1$
5. Necessary and sufficient condition for the three non collinear and nonzero vectors $\bar{a}, \bar{b}$ and $\bar{c}$ to be coplanar is $[\bar{a}\,\bar{b}\;\;\bar{c}] = 0$.
6. If $[\bar{a}\,\bar{b}\;\;\bar{c}] \neq 0$, then the three vectors are non–coplanar.
7. Let $\bar{a}, \bar{b}$ and $\bar{c}$ be any three vectors
   i) If $[\bar{a}\,\bar{b}\,\bar{c}] > 0$, then they form a right handed system.
   ii) If $[\bar{a}\,\bar{b}\,\bar{c}] < 0$, then they form a left handed system.
8. If $\bar{a}, \bar{b}, \bar{c}$ represent coterminous edges of a prallelopiped then volume of the parallelopiped = $|[\bar{a}\,\bar{b}\,\bar{c}]|$.
9. The volume of the parallelopiped with vertices A, B, C, D is $\left|\left[\overrightarrow{DA}\,\overrightarrow{DB}\,\overrightarrow{DC}\right]\right|$ cu.units.
10. If $\bar{a}, \bar{b}, \bar{c}$ are concurrent edges of a tetrahedron, then the volume of the tetrahedron is equal to $\frac{1}{6}|[\bar{a}\,\bar{b}\,\bar{c}]|$.
11. The volume of the tetrahedron with vertices A, B, C, D is $\frac{1}{6}|[\overrightarrow{DA}\,\overrightarrow{DB}\,\overrightarrow{DC}]|$.
12. If $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$, $\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$ and $\bar{c} = c_1\bar{i} + c_2\bar{j} + c_3\bar{k}$, then $\bar{a}.(\bar{b}x\bar{c}) = [\bar{a}\,\bar{b}\,\bar{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.
13. If $\bar{p}, \bar{q}, \bar{r}$ are any three non–coplanar vectors and $\bar{a} = a_1\bar{p} + a_2\bar{q} + a_3\bar{r}$, $\bar{b} = b_1\bar{p} + b_2\bar{q} + b_3\bar{r}$ and $\bar{c} = c_1\bar{p} + c_2\bar{q} + c_3\bar{r}$, then $[\bar{a}\,\bar{b}\,\bar{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}[\bar{p}\,\bar{q}\,\bar{r}]$.
14. $[\lambda\bar{a}\,\mu\bar{b}\,\gamma\bar{c}] = \lambda\mu\gamma[\bar{a}\,\bar{b}\,\bar{c}]$

15. i) If $\bar{a}, \bar{b}, \bar{c}$ are any three non–coplanar vectors, then the vectors $\bar{a}', \bar{b}', \bar{c}'$ defined by $\bar{a}' = \frac{\bar{b} \times \bar{c}}{[\bar{a}\ \bar{b}\ \bar{c}]}$, $\bar{b}' = \frac{\bar{c} \times \bar{a}}{[\bar{a}\ \bar{b}\ \bar{c}]}$ $\bar{c}' = \frac{\bar{a} \times \bar{b}}{[\bar{a}\ \bar{b}\ \bar{c}]}$ are called the reciprocal system to the vectors $\bar{a}, \bar{b}, \bar{c}$.

ii) $\bar{a}.\bar{a}' = \bar{b}.\bar{b}' = \bar{c}.\bar{c}' = 1$

iii) $[\bar{a}'\ \bar{b}'\ \bar{c}'] = \frac{1}{[\bar{a}\ \bar{b}\ \bar{c}]}$

iv) $\bar{a}.\bar{b}' = \bar{b}.\bar{c}' = \bar{c}.\bar{a}' = \ldots = 0$

v) Any vector $\bar{r}$ can be expressed as
$\bar{r} = (\bar{r}.\bar{a}')\bar{a} + (\bar{r}.\bar{b}')\bar{b} + (\bar{r}.\bar{c}')\bar{c}$

16. i) The vector equation of the plane containing three non collinear points with position vectors $\bar{a}, \bar{b}, \bar{c}$ is $\bar{r}.(\bar{b} \times \bar{c} + \bar{c} \times \bar{a} + \bar{a} \times \bar{b}) = [\bar{a}\ \bar{b}\ \bar{c}]$

ii) Equation of plane passing through $\bar{a}, \bar{b}$ and parallel to $\bar{c}$ is $\bar{r}.(\bar{b} \times \bar{c} + \bar{c} \times \bar{a}) = [\bar{a}\ \bar{b}\ \bar{c}]$

iii) Equation of plane passing through $\bar{a}$ and parallel to $\bar{b}$ and $\bar{c}$ is $\bar{r}.(\bar{b} \times \bar{c}) = [\bar{a}\ \bar{b}\ \bar{c}]$.

iv) A unit vector perpendicular to the plane passing through $\bar{a}, \bar{b}, \bar{c}$ is $\frac{\bar{b} \times \bar{c} + \bar{c} \times \bar{a} + \bar{a} \times \bar{b}}{|\ \bar{b} \times \bar{c} + \bar{c} \times \bar{a} + \bar{a} \times \bar{b}\ |}$.

v) Length of the perpendicular from the origin to the plane passing through $\bar{a}, \bar{b}, \bar{c}$ is $\frac{|\ [\bar{a}\ \bar{b}\ \bar{c}]\ |}{|\ \bar{b} \times \bar{c} + \bar{c} \times \bar{a} + \bar{a} \times \bar{b}\ |}$

17. $[\bar{a}\ \bar{b}\ \bar{c}][\bar{p}\ \bar{q}\ \bar{r}] = \begin{vmatrix} \bar{a}.\bar{p} & \bar{b}.\bar{p} & \bar{c}.\bar{p} \\ \bar{a}.\bar{q} & \bar{b}.\bar{q} & \bar{c}.\bar{q} \\ \bar{a}.\bar{r} & \bar{b}.\bar{r} & \bar{c}.\bar{r} \end{vmatrix}$

18. The shortest distance between two non–intersecting lines given by $\bar{r} = \bar{a} + t\bar{b}$ and $\bar{r} = \bar{c} + s\bar{d}$ is $\frac{|\ [\bar{a} - \bar{c}\ \ \bar{b}\ \ \bar{d}]\ |}{|\ \bar{b} \times \bar{d}\ |}$

19. Vector product of three vectors $\bar{a}, \bar{b}$ and $\bar{c}$ is $\bar{a} \times (\bar{b} \times \bar{c})$ or $(\bar{a} \times \bar{b}) \times \bar{c}$.

20. If $\bar{a}, \bar{b}$ and $\bar{c}$ are three vectors, then

i) $\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a}.\bar{c})\bar{b} - (\bar{a}.\bar{b})\bar{c}$

ii) $(\bar{a} \times \bar{b}) \times \bar{c} = (\bar{a}.\bar{c})\bar{b} - (\bar{b}.\bar{c})\bar{a}$

21. i) If $\bar{a} \times (\bar{b} \times \bar{c}) \neq (\bar{a} \times \bar{b}) \times \bar{c}$ i.e., vector product is not associative.

ii) $\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \times \bar{b}) \times \bar{c}$ then $\bar{c}$ and $\bar{a}$ are collinear.

22. For any three vectors $\bar{a}, \bar{b}, \bar{c}$ $\bar{a} \times (\bar{b} \times \bar{c}) + \bar{b} \times (\bar{c} \times \bar{a}) + \bar{c} \times (\bar{a} \times \bar{b}) = \bar{0}$

23. Scalar product of four vectors is $(\bar{a} \times \bar{b}).(\bar{c} \times \bar{d})$ and is equal to $\begin{vmatrix} \bar{a}.\bar{c} & \bar{b}.\bar{c} \\ \bar{a}.\bar{d} & \bar{b}.\bar{d} \end{vmatrix}$.

24. Vector product of four vectors is $(\bar{a} \times \bar{b}) \times (\bar{c} \times \bar{d})$.

25. $(\bar{a} \times \bar{b}) \times (\bar{c} \times \bar{d}) = [\bar{a}\ \bar{b}\ \bar{d}]\bar{c} - [\bar{a}\ \bar{b}\ \bar{c}]\bar{d}$
$= [\bar{a}\ \bar{c}\ \bar{d}]\bar{b} - [\bar{b}\ \bar{c}\ \bar{d}]\bar{a}$

26. Four points $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ are coplanar if any only if $[\bar{b}\ \bar{c}\ \bar{d}] + [\bar{c}\ \bar{a}\ \bar{d}] + [\bar{a}\ \bar{b}\ \bar{d}] = [\bar{a}\ \bar{b}\ \bar{c}]$.