# 3. ADDITION OF VECTORS

## Synopsis:

1. Scalar : The quantity having only magnitude but no direction is called a scalar.
2. Vector : The quantity having both magnitude and direction is called a vector.

   Representation : Vectors are generally represented by directed line segments. If a vector is represented by $\overrightarrow{AB}$ then the direction from A to B gives its direction and the distance from A to B gives its magnitude. A is called the initial point and B is called its terminal point.

   Notation : Vectors are generally denoted by letters with a bar over it. i.e., $\bar{a}, \bar{b}, \bar{c}$ …. (or) by bold faced letters a, b, c …. and its magnitude by II or simply a.
3. Unit vector : A vector of unit magnitude is called a unit vector. The unit vector in the direction of $\bar{a}$ is denoted by $\hat{a}$.
4. Null vector : Vector of zero magnitude and indefinite direction is called a null vector (i.e., direction of a null vector is indeterminate). For this vector, initial and terminal points coincide.
5. Negative vector : A vector having same magnitude and opposite direction as that of $\bar{a}$ is called negative vector of $\bar{a}$ and is denoted by $-\bar{a}$.
6. Equal vectors : Two vectors are said to be equal if they have same magnitude and same direction. (irrespective of their initial points).
7. Like and unlike vectors : Vectors having the same direction are called like vectors and having opposite directions are called unlike vectors.
8. Addition of vectors : Let $\overrightarrow{AB} = \bar{a}$ and $\overrightarrow{BC} = \bar{b}$ be two vectors. The vector $\overrightarrow{AC}$ is defined as the sum of $\bar{a}$ and $\bar{b}$ i.e., $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \bar{a} + \bar{b}$. This is called the triangle law of vectors.
9. If $\bar{a}, \bar{b}, \bar{c}$ are three vectors then

   i) $\bar{a} + (\bar{b} + \bar{c}) = (\bar{a} + \bar{b}) + \bar{c}$ (vector addition is associative).

   ii) $\bar{a} + \bar{b} = \bar{b} + \bar{a}$ (vector addition is commutative).

   iii) $\bar{a} + \bar{0} = \bar{0} + \bar{a} = \bar{a}$ (null vector is the identity element in the set of vectors under addition)

   iv) $\bar{a} + (-\bar{a}) = (-\bar{a}) + \bar{a} = \bar{0}$ ($-\bar{a}$ is the additive inverse of $\bar{a}$ in the set of vectors)

   v) In any triangle ABC, $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$.
10. Position vector : Let ‘O’ be the origin of vectors and P be a point in space. Then $\overrightarrow{OP}$ is called the position vector of the point P w.r.to O.

11. To find the vector $\overrightarrow{AB}$, subtract the position vector of A from B. i.e., $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$.
12. Free vector : A vector which is independent of its (initial point) position is called a free vector.
13. Localised vector : If a vector is associated with a given point in space, then it is called a localized vector.
14. Scalar multiplication of a vector : The product of a vector $\bar{a}$ by a scalar k is a vector whose magnitude is |k| times the magnitude of $\bar{a}$ and whose direction is same as that of $\bar{a}$ when k > 0 and opposite to that of $\bar{a}$ when k < 0.
15. If $k_1$, $k_2$ are two scalars and $\bar{a}, \bar{b}$ are two vectors then

    i) $(k_1 + k_2)\bar{a} = k_1\bar{a} + k_2\bar{a}$

    ii) $k(\bar{a} + \bar{b}) = k_1\bar{a} + k_2\bar{b}$

    iii) $(k_1k_2)\bar{a} = k_1(k_2\bar{a}) = k_2(k_1\bar{a})$
16. Collinear vectors : Vectors which lie on a line or which are parallel are called collinear vectors (whatever be their magnitudes)
17. Two vectors $\bar{a}$ and $\bar{b}$ are collinear if and only if $\bar{a} = m\bar{b}$ or $\bar{a} = n\bar{a}$, where m, n are scalars (real numbers).
18. Coplanar vectors : Vectors which lie on a plane or which are parallel to a plane are called coplanar vectors. The vectors $\bar{a}, \bar{b}, \bar{c}$ are coplanar if

    i) $x\bar{a} + y\bar{b} + z\bar{c} = 0$ and ii) the scalars x, y, z are not all zero.
19. The position vector of a point which divides the join of two given points in the ratio m : n is $\frac{m\bar{a} \pm n\bar{b}}{m \pm n}$, + ve sign is taken for internal division and –ve sign is taken for external division.
20. Position vector of the mid point of $\bar{a}$ and $\bar{b}$ is $\frac{\bar{a} + \bar{b}}{2}$.
21. The centroid of a triangle with vertices $\bar{a}, \bar{b}, \bar{c}$ is $\frac{\bar{a} + \bar{b} + \bar{c}}{3}$
22. If G is the centroid of the triangle ABC, then $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \bar{0}$.
23. The orthocentre of the triangle is $\frac{(a\sec A)\bar{a} + (b\sec B)\bar{b} + (c\sec C)\bar{c}}{a\sec A + b\sec B + c\sec C}$ where a, b, c are the lengths of the sides of the triangle and A, B, C are the angles of the triangle.
24. The circumcentre of the triangle is $\frac{\bar{a}\sin 2A + \bar{b}\sin 2B + \bar{c}\sin 2C}{\sec 2A + \sec 2B + \sec 2C}$.

25. The position vector of the centroid of a tetrahedron ABCD is $\frac{\bar{a}+\bar{b}+\bar{c}+\bar{d}}{4}$ where $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ are the position vectors of A, B, C, D respectively.
26. If $\overrightarrow{OA} = \bar{a}$ and $\overrightarrow{OB} = \bar{b}$ be two non zero vectors, then the value of $\angle$AOB lies between 0 and $\pi$ is defined as the angle between $\bar{a}$ and $\bar{b}$ and is denoted by $(\bar{a}, \bar{b})$.
27. If $(\bar{a}, \bar{b}) = \theta$, then
    i) $(-\bar{a}, -\bar{b}) = (-\bar{b}, -\bar{a}) = (x\bar{a}, y\bar{b}) = (y\bar{b}, x\bar{a}) = \theta$ for all x, y > 0
    ii) $(\bar{a}, -\bar{b}) = (-\bar{a}, \bar{b}) = (x\bar{a}, -y\bar{b}) = (-x\bar{a}, y\bar{b}) = 180 - \theta$ for all x, y > 0.
    iii) $(\bar{a}, \bar{b}) = 0 \Rightarrow \bar{a}$ and $\bar{b}$ are like vectors.
    iv) $(\bar{a}, \bar{b}) = \pi \Rightarrow \bar{a}$ and $\bar{b}$ are unlike vectors.
    v) $(\bar{a}, \bar{b}) = \pi/2 \Rightarrow \bar{a}$ and $\bar{b}$ are orthogonal vectors.
28. The vectors $x_1\bar{a}_1 + x_2\bar{a}_2 + x_3\bar{a}_3 + .... + x_n\bar{a}_n$ is called the linear combination of the system of vectors $\bar{a}_1, \bar{a}_2, \bar{a}_3, .....\bar{a}_n$ where $x_1$, $x_2$, $x_3$, …$x_n$ are scalars.
29. A system of vectors $\bar{a}_1, \bar{a}_2, \bar{a}_3, .....\bar{a}_n$ is said to be linearly dependent if $x_1\bar{a}_1 + x_2\bar{a}_2 + x_3\bar{a}_3 + .... + x_n\bar{a}_n = \bar{0} \Rightarrow$ atleast one of the scalars $x_1$, $x_2$, $x_3$, …. $x_n$ not equal to zero.
30. A system of vectors $\bar{a}_1, \bar{a}_2, \bar{a}_3, .....\bar{a}_n$ is said to be linearly independent if $x_1\bar{a}_1 + x_2\bar{a}_2 + x_3\bar{a}_3 + ... + x_n\bar{a}_n = \bar{0} \Rightarrow x_1 = x_2 = ..... = x_n = 0$.
31. Any two collinear vectors are linearly dependent vectors.
32. Any set containing the zero vector is linearly dependent.
33. Any three coplanar vectors are linearly dependent and any three non coplanar vectors are linearly independent.
34. If $\bar{a}, \bar{b}, \bar{c}$ be three non coplanar vectors then any other vector $\bar{r}$ can be uniquely expressed as $\bar{r} = x\bar{a} + y\bar{b} + z\bar{c}$, where x, y, z are scalar.
35. Let $\bar{a}, \bar{b}, \bar{c}$ be three non coplanar vectors. If the angle measured from $\bar{a}$ to $\bar{b}$ in the anti clockwise direction is less than $180^o$ by observing from $\bar{c}$, then $\bar{a}, \bar{b}, \bar{c}$ are said to form a right handed system. Otherwise they are said to form a left handed system.
36. The system of three mutually perpendicular unit vectors forming a right handed system is called orthonormal triad of unit vectors. These are generally denoted by $\bar{i}, \bar{j}, \bar{k}$ and are taken along the three rectangular co–ordinate axes namely x–axis, y–axis, z–axis respectively.

37. If $\alpha$, $\beta$, $\gamma$ be the angles made by the line OP with X, Y, Z axes respectively, then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called the direction cosines (d.cs) of the line OP. These are generally denoted by l, m, n.

38. If l, m, n are the d.c's of a line, then $l^2 + m^2 + n^2 = 1$.

39. If l, m, n are the d.c's of a line, then –l, –m, –n are also the d.c's of same line.

40. The direction cosines of the vector $\overrightarrow{OP}$ are the components of a unit vector parallel to $\overrightarrow{OP}$ i.e., $OP$.

41. The direction cosines of the vectors $\bar{i}, \bar{j}, \bar{k}$ respectively are (1, 0, 0), (0, 1, 0), (0, 0, 1).

42. If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, then $|\bar{r}| = \sqrt{x^2 + y^2 + z^2}$.

43. For any two vectors $\bar{a}$ and $\bar{b}$

    i) $|\bar{a}| - |\bar{b}| \le |\bar{a} + \bar{b}| \le |\bar{a}| + |\bar{b}|$

    ii) $|\bar{a} - \bar{b}| \ge |\bar{a}| - |\bar{b}|$

44. Vector equation of a straight line passing through the point $\bar{a}$ and parallel to the vector $\bar{b}$ is $\bar{r} = \bar{a} + t\bar{b}$, where t is a scalar.

45. The Cartesian equation of the straight line passing through the point $(a_1, a_2, a_3)$ and parallel to the vector $b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$ is $\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3}$.

46. Vector equation of a straight line passing through two points having position vectors $\bar{a}$ and $\bar{b}$ is $\bar{r} = (1 - t)\bar{a} + t\bar{b}$, where t is a scalar.

47. The Cartesian equation of the straight line passing through the two given points $(a_1, a_2, a_3)$ and $(b_1, b_2, b_3)$ is $\frac{x - a_1}{b_1 - a_1} = \frac{y - a_2}{b_2 - a_2} = \frac{z - a_3}{b_3 - a_3}$.

48. Vector equation of a plane :

i) The vector equation of the plane through the points $\bar{a}, \bar{b}, \bar{c}$ is $\bar{r} = (1 - s - t)\bar{a} + s\bar{b} + t\bar{c}$, where s and t are parameters. In Cartesian form, it is $\begin{vmatrix} x - a_1 & y - a_2 & z - a_3 \\ b_1 - a_1 & b_2 - a_2 & b_3 - a_3 \\ c_1 - a_1 & c_2 - a_2 & c_3 - a_3 \end{vmatrix} = 0$, where $\bar{a} = (a_1, a_2, a_3)$; $\bar{b} = (b_1, b_2, b_3)$ and $\bar{c} = (c_1, c_2, c_3)$

ii) The vector equation of the plane through the point whose position vector is $\bar{a}$ and parallel to the vectors $\bar{b}$ and $\bar{c}$ is $\bar{r} = \bar{a} + s\bar{b} + t\bar{c}$, where s and t are parameters. In Cartesian form, it is $\begin{vmatrix} x - a_1 & y - a_2 & z - a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$.

iii) Vector equation of the plane through the points whose position vectors are $\bar{a}$, $\bar{b}$ and parallel to vector $\bar{c}$ is $\bar{r} = (1-s)\bar{a} + s\bar{b} + t\bar{c}$ where s and t are parameters $\begin{vmatrix} x-a_1 & y-a_2 & z-a_3 \\ b_1-a_1 & b_2-a_2 & b_3-a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0.$

49. To show that three points A, B, C are collinear.

1st Method : Find the equation of the line passing through two points and show that it is satisfied by the third point.

2nd Method : Show that there exist scalars not all zero such that $x\bar{a} + y\bar{b} + z\bar{c} = \bar{0}$ and x + y + z = 0.

3rd Method : Find $\overrightarrow{AB}, \overrightarrow{AC}$ and express any one of them as a scalar multiple of other.

50. If the vectors $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$, $\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$ and $\bar{c} = c_1\bar{i} + c_2\bar{j} + c_3\bar{k}$ are linearly dependent then

$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0.$

51. If the above three vectors are linearly independent then $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \neq 0$.

52. Bisector of an angle : Let $\overrightarrow{OA} = \bar{a}$, $\overrightarrow{OB} = \bar{b}$ be the two vectors. Then

i) internal bisector of $\angle$AOB is $\lambda\left(\frac{\bar{a}}{|\bar{a}|} + \frac{\bar{b}}{|\bar{b}|}\right)$

ii) external bisector of $\angle$AOB is $\lambda\left(\frac{\bar{a}}{|\bar{a}|} - \frac{\bar{b}}{|\bar{b}|}\right)$ where $\lambda$ is any real.

53. Let A, B, C be the P.Vs of the vertices of a $\Delta$ABC. Then the internal bisector of the angle A divides the opposite side in the ratio AB : AC and its P.V is $\frac{|\overline{AB}|\bar{C} + |\overline{AC}|\bar{B}}{|\overline{AB}| + |\overline{AC}|}$ (similarly other bisectors).

54. Let $\overrightarrow{OA} = \bar{a}$ and $\overrightarrow{OB} = \bar{b}$ be two vectors, then the point $\bar{c} = m\bar{a} + n\bar{b}$ lies

i) inside the $\Delta$OAB if m, n > 0 and m + n < 1.

ii) outside the $\Delta$OAB but inside the angle $\angle$AOB if m, n > 0 and m + n > 1.

iii) outside the $\Delta$OAB but inside the angle $\angle$OAB if m < 0, n > 0 and m + n < 1.

iv) outside the $\Delta$OAB but inside the angle $\angle$OBA if m > 0, n < 0 and m + n < 1.