EXERCISE - 8(D)

I. Evaluate the following integrals.

1. $\int \frac{dx}{\sqrt{2x-3x^2+1}}$

Sol. $\int \frac{dx}{\sqrt{2x-3x^2+1}}$

$$= \int \frac{dx}{\sqrt{3\left(\frac{2x}{3}-x^2+\frac{1}{3}\right)}} = \frac{1}{\sqrt{3}} \int \frac{dx}{\sqrt{\left(\frac{2}{3}\right)^2-\left(x-\frac{1}{3}\right)^2}}$$

$$= \frac{1}{\sqrt{3}} \sin^{-1}\left(\frac{x-\frac{1}{3}}{\frac{2}{3}}\right)+C = \frac{1}{\sqrt{3}} \sin^{-1}\left(\frac{3x-1}{2}\right)+C$$

2. $\int \frac{\sin\theta}{\sqrt{2-\cos^2\theta}} d\theta$

Sol. $\int \frac{\sin\theta}{\sqrt{2-\cos^2\theta}} d\theta$

put $\cos\theta = t \Rightarrow -\sin\theta\, d\theta = dt$

$$= \int -\frac{dt}{\sqrt{2-t^2}} = -\int \frac{dt}{\sqrt{(\sqrt{2})^2-t^2}} = -\sin^{-1}\left(\frac{t}{\sqrt{2}}\right)+C = -\sin^{-1}\left(\frac{\cos\theta}{\sqrt{2}}\right)+C$$

3. $\int \frac{\cos x}{\sin^2 x+4\sin x+5} dx$

Sol. $\int \frac{\cos x}{\sin^2 x+4\sin x+5} dx$

put $\sin x = t \Rightarrow \cos x dx = dt$

$$= \int \frac{dt}{t^2+4t+5} = \int \frac{dt}{(t+2)^2+1} = \tan^{-1}(t+2)+C = \tan^{-1}(\sin x+2)+C$$

4. $\int \frac{dx}{1+\cos^2 x}$

Sol. $\int \frac{dx}{1+\cos^2 x} = \int \frac{\sec^2 dx}{\sec^2 x+1} = \int \frac{\sec^2 x dx}{\tan^2 x+2}$

Let $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$$= \int \frac{dt}{t^2 + (\sqrt{2})^2} = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + C \qquad = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right) + C$$

5. $\int \frac{dx}{2\sin^2 x + 3\cos^2 x}$

Sol. $\int \frac{dx}{2\sin^2 x + 3\cos^2 x} = \int \frac{\sec^2 x dx}{2\tan^2 x + 3}$

Let $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$$= \int \frac{dt}{2t^2 + 3} = \frac{1}{2} \int \frac{dt}{t^2 + \left(\sqrt{\frac{3}{2}}\right)^2} = \frac{2}{2\sqrt{\frac{3}{2}}} \tan^{-1}\left(\frac{\sqrt{2}t}{\sqrt{3}}\right) + C$$

$$= \frac{1}{2\sqrt{3}} \tan^{-1}\left(\frac{\sqrt{2}\tan x}{\sqrt{3}}\right) + C = \frac{1}{\sqrt{6}} \tan^{-1}\left(\sqrt{\frac{2}{3}}\tan x\right) + C$$

6. $\int \frac{1}{1 + \tan x} dx$

Sol. $\int \frac{1}{1 + \tan x} dx$

$$= \int \frac{1}{1 + \frac{\sin x}{\cos x}} dx = \int \frac{\cos x\,dx}{\sin x + \cos x} = \frac{1}{2} \int \frac{2\cos x\,dx}{\sin x + \cos x}$$

$$= \frac{1}{2} \int \frac{(\cos x + \sin x) + (\cos x - \sin x)}{\sin x + \cos x} dx = \frac{1}{2} \int dx + \frac{1}{2} \int \frac{\cos x - \sin x}{\sin x + \cos x} dx$$

$$= \frac{1}{2} x + \frac{1}{2} \log|\sin x + \cos x| + C$$

7. $\int \frac{1}{1 - \cot x} dx$

Sol. $\int \frac{1}{1 - \cot x} dx = \int \frac{1}{1 - \frac{\cos x}{\sin x}} dx = \int \frac{\sin x dx}{\sin x - \cos x}$

$$= \frac{1}{2} \int \frac{(\sin x - \cos x) + (\cos x + \sin x)}{\sin x - \cos x} dx = \frac{1}{2} \int dx + \frac{1}{2} \int \frac{\cos x + \sin x}{\sin x - \cos x} dx$$

$$= \frac{1}{2}x + \frac{1}{2}\log|\sin x - \cos x| + C$$

II. Evaluate the following integrals.

1. $\int\sqrt{1+3x-x^2}\,dx$

Sol. $\int\sqrt{1+3x-x^2}\,dx = \int\sqrt{1-(x^2-3x)}\,dx$

$$= \int\sqrt{1-(x-\frac{3}{2})^2+\frac{9}{4}}\,dx$$

$$= \int\sqrt{\left(\frac{\sqrt{13}}{2}\right)-\left(x-\frac{3}{2}\right)^2} = \frac{\left(x-\frac{3}{2}\right)\sqrt{1+3x-x^2}}{2} + \frac{13}{8}\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{13}}{2}}\right) + C$$

$$= \frac{(2x-3)\sqrt{1+3x-x^2}}{2} + \frac{13}{8}\sin^{-1}\left(\frac{2x-3}{\sqrt{13}}\right) + C$$

2. $\int\frac{9\cos x - \sin x}{4\sin x + 5\cos x}\,dx$

Sol. $\int\frac{9\cos x - \sin x}{4\sin x + 5\cos x}\,dx$

let $9\cos x - \sin x = A\,\frac{d}{dx}(4\sin x + 5\cos x) + B(4\sin x + 5\cos x)$

$9\cos x - \sin x = A\,(4\cos x - 5\sin x) + B(4\sin x + 5\cos x)$

Comparing the coefficients of sin and cos , we get

$9 = 4A+5B$ and $-5 = -5A+4B$

Solving these equations, A =1 and B=1.

$\therefore\ 9\cos x - \sin x = 1(4\cos x - 5\sin x) + 1(4\sin x + 5\cos x)1(4\cos x - 5\sin x) + 1(4\sin x + 5\cos x)$

$$\int\frac{9\cos x - \sin x}{4\sin x + 5\cos x}\,dx = \int\frac{(4\sin x + 5\cos x) + (4\cos x - 5\sin x)}{4\sin x + 5\cos x}\,dx$$

$$= \int dx + \int\frac{4\cos x - 5\sin x}{4\sin x + 5\cos x}\,dx$$

$$= x + \log|4\sin x + 5\cos x| + C$$

3. $\int \frac{2\cos x + 3\sin x}{4\cos x + 5\sin x} dx$

ans $= \frac{23}{41}x - \frac{2}{41}\log|4\cos x + 5\sin x| + C$

4. $\int \frac{dx}{1+\sin x + \cos x}$

Sol. $\int \frac{dx}{1+\sin x + \cos x}$

$= \int \frac{dx}{\left[1 + \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} + \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right]}$

$= \int \frac{\sec^2\frac{x}{2}dx}{1+\tan^2\frac{x}{2} + 2\tan\frac{x}{2} + 1 - \tan^2\frac{x}{2}}$

$= \int \frac{\sec^2\frac{x}{2}}{2+2\tan\frac{x}{2}}$ put $\tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}dx = dt$

$= 2\int \frac{dt}{2+2t} = \int \frac{dt}{1+t}\log|1+t| + C$

$= \log\left|1+\tan\frac{x}{2}\right| + C$

5. $\int \frac{dx}{3x^2 + x + 1}$

Sol. $\int \frac{dx}{3x^2 + x + 1} = \int \frac{dx}{3\left(x^2 + \frac{1}{3}x + \frac{1}{3}\right)}$

$= \frac{1}{3}\int \frac{dx}{\left(x+\frac{1}{6}\right)^2 + \frac{1}{3} - \frac{1}{36}} = \frac{1}{3}\int \frac{dx}{\left(x+\frac{1}{6}\right)^2 + \left(\frac{\sqrt{11}}{6}\right)^2}$

$= \frac{1}{3}\cdot\frac{1}{\frac{\sqrt{11}}{6}}\tan^{-1}\left(\frac{x+(1/6)}{(\sqrt{11}/6)}\right) + C$

$$= \frac{2}{\sqrt{11}} \tan^{-1}\left(\frac{6x+1}{\sqrt{11}}\right) + C$$

6. $\int \frac{dx}{\sqrt{5-2x^2+4x}}$

Sol. $\int \frac{dx}{\sqrt{5-2x^2+4x}}$

$5 - 2x^2 + 4x$

$$= -2\left[x^2 - 2x - \frac{5}{2}\right] = -2\left[(x-1)^2 - 1 - \frac{5}{2}\right]$$

$$= -2\left[(x-1)^2 - \frac{7}{2}\right] = 2\left[\frac{7}{2} - (x-1)^2\right]$$

Now $\int \frac{1}{\sqrt{5-2x^2+4x}} dx$

$$= \int \frac{1}{\sqrt{2}\left\{\frac{7}{2} - (x-1)^2\right\}} dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\left(\sqrt{\frac{7}{2}}\right)^2 - (x-1)^2}} dx$$

$$= \frac{1}{\sqrt{2}} \sin^{-1} \frac{(x-1)}{\sqrt{7/2}} + C$$

$$= \frac{1}{\sqrt{2}} \sin^{-1} \sqrt{\frac{2}{7}}(x-1) + C$$

III. Evaluate the following integrals.

1. $\int \frac{x+1}{\sqrt{x^2-x+1}} dx$

Sol.

take $x+1 = A\frac{d}{dx}\left(x^2 - x + 1\right) + B$

$x+1 = A\left(2x-1\right) + B$

Comparing the coefficients of like terms,

$2A = 1$ and $B\text{-}A = 1$

$\Rightarrow A = \frac{1}{2}$ and $b = \frac{3}{2}$

$\therefore x+1 = \frac{1}{2}(2x-1) + \frac{3}{2}$

$$\int \frac{x+1}{\sqrt{x^2-x+1}}dx = \int \frac{\frac{1}{2}(2x-1)+\frac{3}{2}}{\sqrt{x^2-x+1}}dx$$

$$= \frac{1}{2}\int \frac{(2x-1)dx}{\sqrt{x^2-x+1}} + \frac{3}{2}\int \frac{dx}{\sqrt{x^2-x+1}}$$

$$= \sqrt{x^2-x+1} + \frac{3}{2}\int \frac{dx}{\sqrt{\left(x-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}}$$

$$= \sqrt{x^2-x+1} + \frac{3}{2}\sinh^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$$

$$= \sqrt{x^2-x+1} + \frac{3}{2}\sinh^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

2. $\int (6x+5)\sqrt{6-2x^2+x}\,dx$

Sol.

let $6x+5 = A\,\frac{d}{dx}\left(6-2x^2+x\right) + B$

$\Rightarrow$ **$6x + 5 = A(1 - 4x) + B$**

Equating the coefficients

$$6 = -4A \Rightarrow A = \frac{-3}{2}$$

Equating the constants

$A + B = 5$

$B = 5 - A =$ $5 + \frac{3}{2} = \frac{13}{2}$

$$\int (6x+5)\sqrt{6-2x^2+x}\,dx$$

$$= -\frac{3}{2}\int (1-4x)\sqrt{6-2x^2+x} + \frac{13}{2}\sqrt{6-2x^2+x}\,dx$$

$$= -\frac{3}{2}\frac{(6-2x^2+x)^{3/2}}{3/2} + \frac{13}{2}\sqrt{2}\int\sqrt{3-x^2+\frac{x}{2}}dx$$

$$= -(6-2x^2+x)^{3/2} + \frac{13}{\sqrt{2}}\int\sqrt{\left(\frac{7}{4}\right)^2-\left(x-\frac{1}{4}\right)^2}\,dx$$

$$= -(6-2x^2+x)^{3/2} + \frac{13}{\sqrt{2}}\left(\frac{\left(x-\frac{1}{4}\right)\sqrt{3-x^2+\frac{x}{2}}}{2} + \frac{49}{32}\sin^{-1}\frac{\left(x-\frac{1}{4}\right)}{\left(\frac{7}{4}\right)}\right)+C$$

$$= -(6-2x^2+x)^{3/2} + \frac{13}{\sqrt{2}}\left[\frac{(4x-1)\sqrt{6-2x^2+x}}{16\times 2} + \frac{49}{32}\sin^{-1}\left(\frac{4x-1}{7}\right)\right]+C$$

$$= -(6-2x^2+x)^{3/2} + \frac{13}{16}(4x-1)\sqrt{6-2x^2+x} + \frac{637}{32\sqrt{2}}\sin^{-1}\left(\frac{4x-1}{7}\right)+C$$

3. $\int\frac{dx}{4+5\sin x}$

Sol. $\int\frac{dx}{4+5\sin x} = \int\frac{dx}{4+5\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}$

$$\text{put } \tan\frac{x}{2} = t \Rightarrow \sec^2\frac{x}{2}\cdot\frac{1}{2}dx = dt$$

$$\Rightarrow dx = \frac{2dt}{\sec^2\frac{x}{2}} = \frac{2dt}{1+t^2}$$

G.I. $= 2\int\frac{dt}{\frac{1+t^2}{4+5\frac{2t}{1+t^2}}} = 2\int\frac{dt}{4+4t^2+10t}$

$$= \frac{1}{2}\int\frac{dt}{t^2+\frac{5t}{2}+1} = \frac{1}{2}\int\frac{dt}{\left(t+\frac{5}{4}\right)^2-\left(\frac{3}{4}\right)^2}$$

$$= \frac{1}{2}\frac{1}{2\cdot\frac{3}{4}}\log\left|\frac{t+\frac{5}{4}-\frac{3}{4}}{t+\frac{5}{4}+\frac{3}{4}}\right| + C$$

$$= \frac{1}{3}\log\left|\frac{4t+2}{4t+8}\right| + C = \frac{1}{3}\log\left|\frac{2t+1}{2t+4}\right| + C$$

$$= \frac{1}{3}\log\left|\frac{2\tan\frac{x}{2}+1}{2\left(\tan\frac{x}{2}\right)+2}\right| + C$$

4. $\int\frac{1}{2-3\cos 2x}dx$

Sol. $\int\frac{1}{2-3\cos 2x}dx = \int\frac{dx}{2-3\frac{1-\tan^2 x}{1+\tan^2 x}}$

put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$$dx = \frac{dt}{1+t^2}$$

$$GI = \int\frac{\frac{dt}{1+t^2}}{2-3\frac{1-t^2}{1+t^2}} = \int\frac{dt}{2+2t^2-3+3t^2}$$

$$= \int\frac{dt}{5t^2-1} = \frac{1}{5}\int\frac{dt}{t^2-\left(\frac{1}{\sqrt{5}}\right)^2}$$

$$= \frac{1}{5}\frac{(1/2)}{\sqrt{5}}\log\left|\frac{t-\frac{1}{\sqrt{5}}}{t+\frac{1}{\sqrt{5}}}\right| + C$$

$$= \frac{1}{2\sqrt{5}}\log\left|\frac{\sqrt{5}t-1}{\sqrt{5}t+1}\right| + C$$

$$= \frac{1}{2\sqrt{5}}\log\left|\frac{\sqrt{5}\tan x-1}{\sqrt{5}\tan x+1}\right| + C$$

5. $\int x\sqrt{1+x-x^2}\,dx$

Ans $= -\frac{1}{3}(1+x-x^2)^{3/2} + \frac{(2x-1)\sqrt{1+x-x^2}}{8} + \frac{5}{16}\sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right) + C$

6. $\int \frac{dx}{(1+x)\sqrt{3+2x-x^2}}$

Sol. $\int \frac{dx}{(1+x)\sqrt{3+2x-x^2}} = \int \frac{dx}{(1+x)\sqrt{(3-x)(1+x)}}$

Put $1+x = t^2 \Rightarrow dx = 2t\,dt$

$$\text{G.I.} = \int \frac{2t\,dt}{t^2\sqrt{t^2(4-t^2)}} = \int \frac{2dt}{t^2\sqrt{4-t^2}} = \int \frac{2}{t^3}\frac{dt}{\sqrt{\frac{4}{t^2}-1}}$$

Put $\frac{4}{t^2}-1 = y^2 \Rightarrow -\frac{8}{t^3}dt = 2y\,dy$

$$\Rightarrow \frac{2}{t^3}dt = -\frac{y}{4}dy$$

$$\text{G.I.} = 2\int -\frac{y}{4}\frac{dy}{\sqrt{y^2}} = -\frac{1}{2}\int dy = -\frac{1}{2}y + C$$

$$= -\frac{1}{2}\sqrt{\frac{4}{t^2}-1} + C$$

$$= -\frac{1}{2}\sqrt{\frac{4}{1+x}-1} + C - \frac{1}{2}\sqrt{\frac{3-x}{3+x}} + C$$

7. $\int \frac{dx}{4\cos x + 3\sin x}$

Sol. $\int \frac{dx}{4\cos x + 3\sin x} = \int \frac{dx}{4\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} + 3\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}$

Put $\tan\frac{x}{2} = t \Rightarrow dx = \frac{2dt}{1+t^2}$

$$I=\int\frac{\frac{2dt}{1+t^2}}{4\frac{(1-t^2)}{1+t^2}+\frac{3\cdot 2t}{1+t^2}}=2\int\frac{dt}{4-4t^2+6t}$$

$$=-\frac{1}{2}\int\frac{dt}{t^2-\frac{3}{2}t-1}=-\frac{1}{2}\int\frac{dt}{\left(t-\frac{3}{4}\right)^2-\left(\frac{5}{4}\right)^2}$$

$$=-\frac{1}{2}\frac{1}{2\cdot\frac{5}{4}}\log\left|\frac{t-\frac{3}{4}-\frac{5}{4}}{t-\frac{3}{4}+\frac{5}{4}}\right|+C$$

$$=-\frac{1}{5}\log\left|\frac{t-2}{t+(1/2)}\right|+C=-\frac{1}{5}\log\left|\frac{2t-4}{2t+1}\right|+C$$

$$=-\frac{1}{5}\log\left|\frac{2\left(\tan\frac{x}{2}-2\right)}{2\tan\frac{x}{2}+1}\right|+C$$

8. $\int\frac{1}{\sin x+\sqrt{3}\cos x}dx$

Ans $=\frac{1}{2}\log\left|\frac{\sqrt{3}\tan\frac{x}{2}+1}{\sqrt{3}\left(\sqrt{3}-\tan\frac{x}{2}\right)}\right|+c$

9. $\int\frac{dx}{5+4\cos 2x}$

Ans $=\frac{1}{3}\tan^{-1}\left(\frac{\tan x}{3}\right)+C$

10. $\int\frac{2\sin x+3\cos x+4}{3\sin x+4\cos x+5}dx$

Sol.

$$\int\frac{2\sin x+3\cos x+4}{3\sin x+4\cos x+5}dx$$

Let $2\sin x+3\cos x+4=A(3\sin x+4\cos x+5)+B\frac{d}{dx}(3\sin x+4\cos x+5)+C$

2 sinx + 3 cos x + 4

= A(3sinx + 4cosx + 5) + B(3cosx – 4sinx) + C

Equating the coefficients of

sin x and cos x,

we get $3A - 4B = 2$ and $4A + 3B = 3$

Solving these equations,

$A = \frac{18}{25}$, $B = \frac{1}{25}$

Equating the constants

$4 = 5A + C$

C = 4 – 5A = 4 – $5 \cdot \frac{18}{25} = \frac{2}{5}$

$2\sin x + 3\cos x + 4 = \frac{18}{25}(3\sin x + 4\cos x + 5) + \frac{1}{25}(3\cos x - 4\sin x) + \frac{2}{5}$

$$\therefore \int \frac{2\sin x + 3\cos x + 4}{3\sin x + 4\cos x + 5} dx$$

$$= \frac{18}{25}\int dx + \frac{1}{25}\int \frac{3\cos x - 4\sin x}{3\sin x + 4\cos x + 5} + \frac{2}{5}\int \frac{dx}{3\sin x + 4\cos x + 5}$$

$$= \frac{18}{25}x + \frac{1}{25}\log|3\sin x + 4\cos x + 5| + \frac{2}{5}\int \frac{dx}{3\sin x + 4\cos x + 5} \quad ...(1)$$

Let $I = \int \frac{dx}{3\sin x + 4\cos x + 5}$

Put $\tan\frac{x}{2} = t \Rightarrow dx = \frac{2dt}{1+t^2}$ $I = \int \frac{\frac{2dt}{1+t^2}}{\frac{3-2t}{1+t^2} + \frac{4(1+t^2)}{1+t^2} + 5}$

$$= 2\int \frac{dt}{6t + 4 - 4t^2 + 5 + 5t^2} = 2\int \frac{dt}{t^2 + 6t + 9}$$

$$= 2\int \frac{dt}{(t+3)^2} = -\frac{2}{t+3} = -\frac{2}{3 + \tan\frac{x}{2}}$$

Substituting in (1)

$$I = \frac{18}{25} \cdot x + \frac{1}{25}\log|3\sin x + 4\cos x + 5| - \frac{4}{5\left(3 + \tan\frac{x}{2}\right)} + C$$

INTEGRATION OF RATIONAL FUNCTIONS USING PARTIAL FRACTIONS

EXERCISE – 8(E)

I. Evaluate the following integrals.

1. $\int \frac{(x-1)dx}{(x-2)(x-3)}$

Sol.

$$\frac{(x-1)}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3} = \frac{A(x-3)+B(x-2)}{(x-2)(x-3)} \quad x-1 = A(x-3)+B(x-2)$$

Put x= 3, then B = 2 and put x=2 then A= -1

$$\therefore \frac{(x-1)}{(x-2)(x-3)} = \frac{-1}{x-2} + \frac{2}{x-3}$$

$$\therefore \int \frac{(x-1)dx}{(x-2)(x-3)} = \int\left(\frac{-1}{x-2} + \frac{2}{x-3}\right)dx$$

$$= = 2\log(x-3) - \log(x-2) + c$$

2. $\int \frac{x^2}{(x+1)(x+2)^2} dx$

Sol. let

$$\frac{x^2}{(x+1)(x+2)^2} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

$$\Rightarrow x^2 = A(x+2)^2 + B(x+1)(x+2) + C(x+1)$$

Put x = –2 in (1)

$(-2)^2 = A(0) + B(0) + C(-2+1) \Rightarrow C = -4$

Put x = –1 in (1)

$(-1)^2 = A(-1+2)^2 + B(0) + C(0) \Rightarrow A = 1$

Equating coefficients of x^2 in (1)

$1 = A + B \Rightarrow B = 1 - A = 1 - 1 = 0$

$$\therefore \frac{x^2}{(x+1)(x+2)^2} = \frac{1}{x+1} + \frac{0}{x+2} + \frac{-4}{(x+2)^2}$$

$$\therefore \int \frac{x^2}{(x+1)(x+2)^2} dx$$

$$= \int \frac{1}{x+1} dx - 4\int \frac{1}{(x+2)^2} dx$$

$$= \log|x+1| - 4\left(\frac{-1}{x+2}\right)$$

$$= \log|x+1| + \frac{4}{x+2} + C$$

3. $\int \frac{x+3}{(x-1)(x^2+1)}dx$

Sol. Let $\frac{x+3}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$

$\Rightarrow (x+3) = A(x^2+1) + (Bx+C)(x-1)$ **…(1)**

Put x = 1 in (1)

Then 4 = A(1 + 1) + 0 $\Rightarrow$ A = 2

Put x = 0 in (1)

3 = A(1) + C(–1)

$\Rightarrow$ A – C = 3 $\Rightarrow$ C = A – 3 = 2 – 3 = 1

Equating coefficient of x^2 in (1)

0 = A + B $\Rightarrow$ B = –A = –2

$\therefore \frac{x+3}{(x-1)(x^2+1)} = \frac{+2}{(x-1)} + \frac{-2x-1}{x^2+1}$

$\int \frac{x+3}{(x-1)(x^2+1)}dx$

$= 2\int \frac{1}{x-1}dx - \int \frac{2x}{x^2+1}dx - \int \frac{1}{x^2+1}dx$

$= 2\log|x-1| - \log|x^2+1| - \tan^{-1}(x) + C$

4. $\int \frac{dx}{(x^2+a^2)(x^2+b^2)}$

Sol. From partial fractions

$$\frac{1}{(x^2+a^2)(x^2+b^2)} = \frac{1}{(b^2-a^2)}\left(\frac{1}{(x^2+a^2)} - \frac{1}{(x^2+b^2)}\right)$$

$\therefore \int \frac{dx}{(x^2+a^2)(x^2+b^2)}$

$= \frac{1}{(b^2-a^2)}\left[\int \frac{1}{(x^2+a^2)}dx - \int \frac{1}{(x^2+b^2)}dx\right]$

$$= \frac{1}{(b^2 - a^2)}\left[\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) - \frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)\right] + C$$

5. $\int \frac{dx}{e^x + e^{2x}}$

Sol. $\frac{1}{e^x + e^{2x}} = \frac{1}{e^x(1+e^x)} = \left(\frac{1}{e^x} - \frac{1}{1+e^x}\right)$

$$\therefore \int \frac{1}{e^x + e^{2x}}dx = \int \frac{1}{e^x}dx - \int \frac{1}{1+e^x}dx$$

$$= \int e^{-x}dx - \int \frac{e^x}{e^x(1+e^x)}dx$$

$$= \int e^{-x}dx + \int e^x\left\{\frac{1}{e^x} - \frac{1}{1+e^x}\right\}dx$$

$$= \int e^{-x}dx - \int 1dx + \int \frac{e^x}{1+e^x}dx$$

$$= \frac{e^x}{(-1)} - x + \log|1+e^x| + C$$

$$= -e^x + \log(1+e^x) - \log(e^x) + C \ [\because x = \log e^x]$$

$$= -e^{-x} + \log\left(\frac{1+e^x}{e^x}\right) + C$$

6. $\int \frac{dx}{(x+1)(x+2)}$

Ans $= \log\left|\frac{x+1}{x+2}\right| + C$

7. $\int \frac{1}{e^x - 1}dx$

Sol. $\int \frac{1}{e^x - 1}dx = \frac{e^x - (e^x - 1)}{e^x - 1}dx = \int \frac{e^x dx}{e^x - 1} - \int dx$

$$= \log(e^x - 1) - x = \log(e^x - 1) - \log e^x + C$$

$$= \log\left|\frac{e^x - 1}{e^x}\right| + C$$

8. $\int \frac{1}{(1-x)(4+x^2)}dx$

Ans: $= -\frac{1}{5}\log|1-x| + \frac{1}{10}\log|4+x^2| + \frac{1}{10}\tan^{-1}\left(\frac{x}{2}\right) + C$

9. $\int \frac{2x+3}{x^3+x^2-2x}dx$

Sol. $\frac{2x+3}{x(x^2+x-2)} = \frac{2x+3}{x(x+2)(x-1)}$

Let $\frac{2x+3}{x(x+2)(x-1)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-1}$

$\Rightarrow 2x+3 = A(x+2)(x-1) + Bx(x-1) + C(x)(x+2)$ ...(1)

Put x = 0 in (1), then

$3 = A(2)(-1) + B(0) + C(0) \Rightarrow A = -3/2$

Put x = 1 in (1), then

$2 + 3 = A(0) + B(0) + C(1)(3) \Rightarrow C = 5/3$

Put x = –2 in (1), then

$2(-2) + 3 = A(0) + B(-2)(-2-1) + C(0)$

$\Rightarrow -1 = 6B \Rightarrow B = -1/6$

$\therefore \frac{2x+3}{x^3+x^2-2x} = \frac{2x+3}{x(x+2)(x-1)}$

$= \frac{-\frac{3}{2}}{x} + \frac{-\frac{1}{6}}{x+2} + \frac{\frac{5}{3}}{x-1}$

Now $\int \frac{2x+3}{x^3+x^2-2x}dx =$

$= -\frac{3}{2}\int \frac{1}{x}dx - \frac{1}{6}\int \frac{1}{x+2}dx + \frac{5}{3}\int \frac{1}{x-1}dx$

$= -\frac{3}{2}\log|x| - \frac{1}{6}\log|x+2| + \frac{5}{3}\log|x-1| + C$

II. Evaluate the following integrals.

1. $\int \frac{dx}{6x^2-5x+1}$

Sol.

$\mathbf{6x^2 - 5x + 1 = (3x - 1)(2x - 1)}$

Let $\frac{1}{6x^2-5x+1} = \frac{A}{3x-1} + \frac{B}{2x-1}$

$\Rightarrow 1 = A(2x-1) + B(3x-1)$

Put x = 1/3, $1 = A\left(\frac{2}{3}-1\right) \Rightarrow A = -3$

Put x = $\frac{1}{2} \Rightarrow 1 = B\left(\frac{3}{2}-1\right) \Rightarrow B = 2$

$\therefore \frac{1}{6x^2-5x+1} = \frac{-3}{3x-1} + \frac{2}{2x-1}$

$\int \frac{1}{6x^2-5x+1} dx = -3\int \frac{dx}{3x-1} + 2\int \frac{dx}{2x-1}$

$= -3\frac{\log|3x-1|}{3} + 2\frac{\log|2x-1|}{2}$

$= \log\left|\frac{2x-1}{3x-1}\right| + C$

2. $\int \frac{dx}{x(x+1)(x+2)}$

ANS $= \frac{1}{2}\log|x| - \log|x+1| + \frac{1}{2}\log|x+2| + C$

3. $\int \frac{3x-2}{(x-1)(x+2)(x-3)} dx$

Sol. $\frac{3x-2}{(x-1)(x+2)(x-3)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x-3}$

$\mathbf{3x - 2 = A(x + 2)(x - 3) + B(x - 1)(x - 3) + C(x - 1)(x + 2)}$

Put x = 1

$\mathbf{3(1) - 2 = A(1+2)(1-3) + B(0) + C(0) \Rightarrow A = -\frac{1}{6}}$

Put x = 3

$\mathbf{3(3) - 2 = A(0) + B(0) + C(3-1)(3+2) \Rightarrow C = \frac{7}{10}}$

Put x = –2

$3(-2) - 2 = A(0) + B(-2-1)(-2-3) + C(0) - 8$

$= 15B \Rightarrow B = \frac{-8}{15}$

$$\therefore \frac{3x-2}{(x-1)(x+2)(x-3)}$$

$$= \frac{-1}{6}\cdot\frac{1}{x-1} - \frac{8}{15}\cdot\frac{1}{x+2} + \frac{7}{10}\cdot\frac{1}{x-3}$$

$$\int \frac{3x-2}{(x-1)(x+2)(x-3)}dx = -\frac{1}{6}\log|x-1| - \frac{8}{15}\log|x+2| + \frac{7}{10}\log|x-3| + C$$

4. $\int \frac{7x-4}{(x-1)^2(x+2)}dx$

Sol. $\frac{7x-4}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x+2)}$

$\Rightarrow 7x - 4 = A(x-1)(x+2) + B(x+2) + C(x-1)^2 \ldots(1)$

Put x = 1 in (1)

$7 - 4 = A(0) + B(1+2) + C(0) \Rightarrow B = 1$

Put x = –2 in (1)

$7(-2) - 4 = A(0) + B(0) + C(-2-1)^2 \Rightarrow C = -2$

Equating coefficients of x^2 in (1)

$0 = A + C \Rightarrow A = -C \Rightarrow A = 2$

$$\therefore \frac{7x-4}{(x-1)^2(x+2)} = \frac{2}{x-1} + \frac{1}{(x-1)^2} - \frac{2}{x+2}$$

$$\therefore \int \frac{7x-4}{(x-1)^2(x+2)}dx = 2\int\frac{dx}{x-1} + \int\frac{dx}{(x-1)^2} - 2\int\frac{dx}{x+2}$$

$$= 2\log|x-1| - \left(\frac{1}{x-1}\right) - \log|x+2| + C$$

III. Evaluate the following integrals.

1. $\int \frac{1}{(x-a)(x-b)(x-c)}$ **dx**

Sol. $\frac{1}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$

$= \frac{A(x-b)(x-c) + B(x-a)(x-c) + C(x-a)(x-b)}{(x-a)(x-b)(x-c)}$

$\Rightarrow 1 = A(x-b)(x-c) + B(x-a)(x-c) + C(x-a)(x-b)$ …(1)

Put x = a, we get

$1 = A(a-b)(a-c) \Rightarrow A = \frac{1}{(a-b)(a-c)}$

Put x = b, we get $B = \frac{1}{(b-a)(c-b)}$

Similarly $C = \frac{1}{(c-a)(c-b)}$

$\therefore \frac{1}{(x-a)(x-b)(x-c)} =$

$= \frac{\frac{1}{(a-b)(a-c)}}{x-a} + \frac{\frac{1}{(b-a)(b-c)}}{x-b} + \frac{\frac{1}{(c-a)(c-b)}}{x-c}$

$\therefore \int \frac{1}{(x-a)(x-b)(x-c)} dx$

$= \frac{1}{(a-b)(a-c)} \int \frac{1}{x-a} dx + \frac{1}{(b-a)(b-c)}$

$\int \frac{1}{x-b} dx + \frac{1}{(c-a)(c-b)} \int \frac{1}{x-c} dx$

2. $\int \frac{2x+3}{(x+3)(x^2+4)} dx$

Sol. Let $\frac{2x+3}{(x+3)(x^2+4)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+4}$

$2x + 3 = A(x^2 + 4) + (Bx + C)(x + 3)$

$x = -3 \Rightarrow -3 = A(9 + 4) = 13A \Rightarrow A = -\frac{3}{13}$

Equating the coefficients of x^2

$0 = A + B \Rightarrow B = -A = \frac{3}{13}$

Equating the constants

$3 = 4A + 3C$

$3C = 3 - 4A = 3 + \frac{12}{13} = \frac{39+12}{13} = \frac{51}{13} \Rightarrow C = \frac{17}{13}$

$$\frac{2x+3}{(x+3)(x^2+4)} = \frac{-3}{13} \cdot \frac{1}{x+3} + \frac{3x+17}{13(x^2+4)}$$

$$\int \frac{2x+3}{(x+3)(x^2+4)} dx$$

$$= \frac{-3}{13} \int \frac{dx}{x+3} + \frac{3}{26} \int \frac{2xdx}{x^2+4} + \frac{17}{13} \int \frac{dx}{x^2+4}$$

$$= \frac{-3}{13} \log|x+3| + \frac{3}{26} \log|x^2+4| + \frac{17}{26} \tan^{-1}\left(\frac{x}{2}\right) + C$$

3. $\int \frac{2x^2+x+1}{(x+3)(x-2)^2} dx$

Sol. $\frac{2x^2+x+1}{(x+3)(x-2)^2} = \frac{A}{x+3} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$

$2x^2 + x + 1 = A(x-2)^2 + B(x+3)(x-2) + C(x+3)$

$x = 2 \Rightarrow 8 + 2 + 1 = C(2+3) = 5C \Rightarrow C = \frac{11}{5}$

$x = -3 \Rightarrow 18 - 3 + 1 = A(-5)^2 = 25A \Rightarrow A = \frac{16}{25}$

Equating the coefficients of x^2

$2 = A + B \Rightarrow B = 2 - A = 2 - \frac{16}{25} = \frac{34}{25}$

$$\int \frac{2x^2+x+1}{(x+3)(x-2)^2} = \frac{16}{25} \int \frac{dx}{x+3} + \frac{34}{25} \int \frac{dx}{x-2} + \frac{11}{5} \int \frac{1}{(x-2)^2} dx$$

$$= \frac{16}{25} \log|x+3| + \frac{34}{25} \log|x-2| - \frac{11}{5(x-2)} + C$$

4. $\int \frac{dx}{x^3+1}$

Sol. $\frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)}$

Let $\frac{1}{x^3+1} = \frac{A}{(x+1)} + \frac{Bx+C}{x^2-x+1}$

$\Rightarrow 1 = A(x^2 - x + 1) + (Bx + C)(x + 1)$ …(1)

Put x = –1 in (1)

$1 = A(1 + 1 + 1) + (–B + C)(0)$

$\Rightarrow 3A = 1 \Rightarrow A = 1/3$

Put x = 0 in (1)

$1 = A(1) + C(1) \Rightarrow C = 1 – A = 1 – \frac{1}{3} = \frac{2}{3}$

Equating the coefficient of x^2

$0 = A + B \Rightarrow B = –A = –1/3$

$\frac{1}{x^3+1} = \frac{1}{3(x+1)} + \frac{-x+2}{3|x^2-x+1|}$

$\int \frac{1}{x^3+1}dx = \int\left(\frac{1}{3(x+1)} + \frac{-x+2}{3\left(x^2-x+1\right)}\right)dx$

In the second integral of rhs,

Let $-x+2 = A\frac{d}{dx}\left(x^2-x+1\right)+B = A(2x-1)+B$

Comparing the coefficients,

2A = -1 and B-A =2

A = - 1/2, B =3/2

$-x+2 = \frac{-1}{2}(2x-1)+\frac{3}{2}$

$\int \frac{dx}{x^3+1} = \frac{1}{3}\int \frac{dx}{(x+1)} + \int \frac{-\frac{1}{2}(2x-1)+\frac{3}{2}}{x^2-x+1}dx$

$= \frac{1}{3}\log|x+1| - \frac{1}{6}\int \frac{2x-1}{x^2-x+1}dx + \frac{1}{2}\int \frac{dx}{x^2-x+1}$

$= \frac{1}{3}\log|x+1| - \frac{1}{6}\log|x^2-x+1| + \frac{1}{2}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$

$$= \frac{1}{3}\log|x+1| - \frac{1}{6}\log|x^2-x+1| + \frac{1}{2}\cdot\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+C$$

$$= \frac{1}{3}\log|x+1| - \frac{1}{6}\log|x^2-x+1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+C$$

5. $\int \frac{\sin x \cos x}{\cos^2 x + 3\cos x + 2}\,dx$

Sol.

Put $\cos x = t \Rightarrow -\sin x\,dx = dt$

$$\int \frac{\sin x \cos x}{\cos^2 x + 3\cos x + 2}\,dx = \int \frac{-t\,dt}{t^2+3t+2}$$

$$= -\int \frac{t}{t^2+3t+2}\,dt \qquad \ldots(1)$$

Let $\frac{t}{t^2+3t+2} = \frac{t}{(t+1)(t+2)} = \frac{A}{t+1} + \frac{B}{t+2}$

$\Rightarrow t = A(t+2) + B(t+1) \ldots(2)$

Put $t = -1$ in (2)

$-1 = A(-1+2) \Rightarrow A = -1$

Put $t = -2$ in (2)

$-2 = B(-2+1) \Rightarrow B = 2$

$$\therefore \frac{t}{t^2+3t+2} = \frac{-1}{t+1} + \frac{2}{t+2} \qquad \ldots(3)$$

$\therefore$ From (1) and (3)

$$\int \frac{\sin x \cos x}{\cos^2 x + 3\cos x + 2}\,dx$$

$$= -\left[\int \frac{-1}{t+1}\,dt + 2\int \frac{1}{t+2}\,dt\right]$$

$$= \int \frac{1}{t+1}\,dt - 2\int \frac{1}{t+2}\,dt$$

$$= \log|t+1| - 2\log|t+2| + C$$

$$= \log|1+\cos x| - 2\log|2+\cos x| + C$$

$$= \log|1+\cos x| - \log(2+\cos x)^2 + C$$

$$= \log\left|\frac{1+\cos x}{(2+\cos x)^2}\right| + C$$

REDUCTION FORMULAE THEOREM

If $I_n = \int x^n e^{ax}\,dx$**, then** $I_n = \frac{e^{ax}}{a}x^n - \frac{n}{a}I_{n-1}$ **where n is a positive integer.**

Proof :

$$I_n = \int x^n e^{ax}\,dx = x^n \frac{e^{ax}}{a} - \int \frac{e^{ax}}{a} n\,x^{n-1}\,dx$$

$$= \frac{e^{ax}}{a}x^n - \frac{n}{a}\int e^{ax}x^{n-1}dx = \frac{e^{ax}}{a}x^n - \frac{n}{a}I_{n-1}$$

THEOREM

If $I_n = \int \sin^n x\,dx$ **then** $I_n = \frac{-\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}I_{n-2}$**, where n is a positive integer.**

THEOREM

If $I_n = \int \cos^n x\,dx$ **then** $I_n = \frac{\cos^{n-1}x\sin x}{n} + \frac{n-1}{n}I_{n-2}$**.**

Proof :

$$I_n = \int \cos^{n-1}x\cos x\,dx$$

$$= \cos^{n-1}x\sin x - \int \sin x(n-1)\cos^{n-2}x(-\sin x)dx$$

$$= \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x(1-\cos^2 x)dx$$

$$= \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x\,dx - (n-1)\int \cos^n x\,dx$$

$$= \cos^{n-1}x\sin x + (n-1)I_{n-2} - (n-1)I_n$$

$$I_n(1+n-1) = \cos^{n-1}x\sin x + (n-1)I_{n-2}$$

$$\therefore I_n = \frac{\cos^{n-1}x\sin x}{n} + \frac{n-1}{n}I_{n-2}$$

THEOREM

If $I_n = \int \tan^n x\,dx$ **then** $I_n = \frac{\tan^{n-1}x}{n-1} - I_{n-2}$**.**

Proof :

$$I_n = \int \tan^{n-2}x\tan^2 x\,dx = \int \tan^{n-2}(\sec^2 x - 1)dx$$

$$= \int \tan^{n-2}x\sec^2 x\,dx - \int \tan^{n-2}dx = \frac{\tan^{n-1}dx}{n-1} - I_{n-2}$$

THEOREM

If $I_n = \int \cot^n x\,dx$ **then** $I_n = \dfrac{-\cot^{n-1} x}{n-1} - I_{n-2}$.

Proof :

$$I_n = \int \cot^{n-2} x \cot^2 x\,dx = \int \cot^{n-2} x(\csc^2 x - 1)dx$$

$$= \int \cot^{n-2} x \csc^2 x\,dx - \int \cot^{n-2} x\,dx = -\frac{\cot^{n-1} x}{n-1} - I_{n-2}$$

THEOREM

If $I_n = \int \sec^n x\,dx$ **then** $I_n = \dfrac{\sec^{n-2} x \tan x}{n-1} + \dfrac{n-2}{n-1} I_{n-2}$.

Proof :

$$I_n = \int \sec^n x\,dx = \int \sec^{n-2} x \sec^2 x\,dx$$

$$= \sec^{n-2} x \tan x - \int \tan x(n-2)\sec^{n-3} x \sec x \tan x\,dx$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \tan^2 x\,dx$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x(\sec^2 x - 1)dx$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2} x\,dx$$

$$= \sec^{n-2} x \tan x - (n-2)I_n + (n-2)I_{n-2}$$

$$I_n(1 + n - 2) = \sec^{n-2} x \tan x + (n-2)I_{n-2}$$

$$\therefore I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

THEOREM

If $I_n = \int \csc^n x\,dx$ **then** $I_n = \dfrac{-\csc^{n-2} x \cot x}{n-1} + \dfrac{n-2}{n-1} I_{n-2}$.

Proof :

$$I_n = \int \csc^{n-2} x \csc^2 x\,dx$$

$$= -\csc^{n-2} x \cot x - \int -\cot x(n-2)\csc^{n-3} x(-\csc x \cot x)\,dx$$

$$= -\csc^{n-2} x \cot x - (n-2)\int \csc^{n-2} x \cot^2 x\,dx$$

$$= -\csc^{n-2} x \cot x - (n-2)\int \csc^{n-2} x(\csc^2 x - 1)dx$$

$$= -\csc^{n-2} x \cot x - (n-2)\int \csc^n x\,dx + (n-2)\int \csc^{n-2} x\,dx$$

$$= -\csc^{n-2} x \cot x - (n-2)I_n + (n-2)I_{n-2}$$

$$I_n(1+n-2) = -\csc^{n-2} x \cot x + (n-2) I_{n-2}$$

$$I_n = \frac{-\csc^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

THEOREM

If $I_n = \int (\log x)^n dx$ **then** $I_n = x(\log x)^n - nI_{n-1}$.

Proof :

$$I_n = \int (\log x)^n dx = x(\log x)^n - \int x\, n(\log x)^{n-1} \frac{1}{x} dx$$

$$x(\log x)^n - n\int (\log x)^{n-1} dx = x(\log x)^n - nI_{n-1}.$$

THEOREM

If $I_{m,n} = \int \sin^m x \cos^n x\, dx$ **then**

i) $$I_{m,n} = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n} I_{m,n-2}$$

ii) $$I_{m,n} = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n}$$

Proof : i)

$$I_{m,n} = \int \sin^m x \cos^n x\, dx = \int (\sin^m x \cos^{n-1} x) \cos x\, dx$$

$$= \sin^m x \cos^{n-1} x \sin x - \int \left[\begin{matrix} \sin^m x (n-1) \cos^{n-2} x(-\sin x) + \\ \cos^{n-1} x\, m \sin^{m-1} x \cos x \end{matrix} \right] \sin x\, dx$$

$$= \sin^{m+1} x \cos^{n-1} x + (n-1) \int \sin^m x \cos^{n-2} x \sin^2 x dx - m \int \sin^m x \cos^n x\, dx$$

$$= \sin^{m+1} x \cos^{n-1} x + (n-1) \int \sin^m x \cos^{n-2} x (1-\cos^2 x) dx - m I_{m,n}$$

$$= \sin^{m+1} x \cos^{n-1} x + (n-1) \int \sin^m x \cos^{n-2} x\, dx (n-1) \int \sin^m x \cos^n x\, dx - m I_{m,n}$$

$$= \sin^{m+1} x \cos^{n-1} x + (n-1) I_{m,n-2} - (n-1) I_{m,n} - m I_{m,n}$$

$$\Rightarrow (m+n) I_{m,n} = \sin^{m+1} x \cos^{n-1} x + (n-1) I_{m,n-2}$$

$$\Rightarrow I_{m,n} = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n} I_{m,n-2}$$

ii) $$I_{m,n} = \int \sin^m x \cos^n x\, dx$$

$$= \int \sin^{m-1} x \cos^n x \sin x\, dx$$

$$= \sin^{m-1} x \cos^n x(-\cos x) - \int [\sin^{m-1} x\, n \cos^{n-1} x(-\sin x) - \cos^n x (m-1) \sin^{m-2} x \cos x](-\cos x) dx$$

$$= -\sin^{m-1} x \cos^{n+1} x - n \int \sin^{m} x \cos^{n} x \, dx + (m-1) \int \sin^{m-2} x \cos^{n} x \cos^{2} x \, dx$$

$$= -\sin^{m-1} x \cos^{n+1} x - nI_{m,n} + (m-1) \int \sin^{m-2} x \cos^{n} x (1 - \sin^{2} x) dx$$

$$= -\sin^{m-1} x \cos^{n+1} x - nI_{m,n} + (m-1) \int \sin^{m-2} x \cos^{n} x \, dx - (m-1) \int \sin^{m} x \cos^{n} x \, dx$$

$$= -\sin^{m-1} x \cos^{n+1} x - nI_{m,n} + (m-1) I_{m-2,n} - (m-1) I_{m,n}$$

$$\Rightarrow (m+n) I_{m,n} = -\sin^{m-1} x \cos^{n+1} x + (m-1) I_{m-2,n}$$

$$\Rightarrow I_{m,n} = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n} \cdot$$