EXERCISE -- 8(B)

I. Evaluate the following integrals.

1. $\int e^{2x}dx,\ x \in R$ **.**

Sol. $\int e^{2x}dx = \frac{e^{2x}}{2} + C$

2. $\int \sin 7x\, dx, x \in R$

Sol. $\int \sin 7x\, dx = -\frac{\cos 7x}{7} + C$

3. $\int \frac{x}{1+x^2}dx,\ x \in R$

Sol. $\int \frac{x}{1+x^2}dx = \frac{1}{2}\int \frac{2x\,dx}{1+x^2} = \frac{1}{2}\log(1+x^2) + C$

4. $\int 2x \sin(x^2+1)dx, x \in R$

Sol. $\int 2x \sin(x^2+1)dx$

Put $x^2+1 = t \Rightarrow 2x\,dx = dt$

$\int 2x \cdot \sin(x^2+1)dx = \int \sin t\, dt = -\cot t + C = -\cos(x^2+1) + C$

5. $\int \frac{(\log x)^2}{x}dx$ **.**

Sol. $\int \frac{(\log x)^2}{x}dx$

put $\log x = t \Rightarrow dt = \frac{1}{x}dx$

$\int \frac{(\log x)^2}{x}dx = \int t^2 \cdot dt = \frac{t^3}{3} + C = \frac{(\log x)^3}{3} + C$

6. $\int \frac{e^{\text{Tan}^{-1}x}}{1+x^2}dx$ **on I $\subset$ (0, ∞).**

Sol. $\int \frac{e^{\text{Tan}^{-1}x}}{1+x^2}dx$

$$\text{put } \tan^{-1} x = t \Rightarrow \frac{dx}{1+x^2} = dt$$

$$\int \frac{e^{\text{Tan}^{-1}x}}{1+x^2} dx = \int e^t \cdot dt = e^t + C = e^{\tan^{-1} x} + C$$

7. $\int \frac{\sin(\text{Tan}^{-1}x)}{1+x^2} dx, x \in R$

Sol. $\int \frac{\sin(\text{Tan}^{-1}x)}{1+x^2} dx$

$$\text{put } \tan^{-1} x = t \Rightarrow \frac{dx}{1+x^2} = dt$$

$$\int \frac{\sin(\text{Tan}^{-1}x)}{1+x^2} dx = \int \sin t \, dt$$

$$= -\cos t + t = -\cos(\tan^{-1} x) + C$$

8. $\int \frac{1}{8+2x^2} dx$ **on R.**

Sol. $\int \frac{1}{8+2x^2} dx = \frac{1}{2} \int \frac{dx}{x^2 + 2^2}$

$$= \frac{1}{2} \cdot \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + C = \frac{1}{4} \tan^{-1}\left(\frac{x}{2}\right) + C$$

9. $\int \frac{3x^2}{1+x^6}$ **dx on R.**

Sol. $\int \frac{3x^2}{1+x^6}$ **dx**

$$\text{put } x^3 = t \Rightarrow 3x^2 dx = dt$$

$$\int \frac{3x^2 dx}{1+x^6} = \int \frac{dt}{1+t^2}$$

$$= \tan^{-1}(t) + C = \tan^{-1}(x^3) + C$$

10. $\int \frac{2}{\sqrt{25+9x^2}}dx$ **on R.**

Sol. $\int \frac{2}{\sqrt{25+9x^2}}dx = \frac{2}{3}\int \frac{dx}{\sqrt{x^2+\left(\frac{5}{3}\right)^2}}$

$= \frac{2}{3}\sinh^{-1}\left(\frac{x}{5/3}\right)+C$

$= \frac{2}{3}\sinh^{-1}\left(\frac{3x}{5}\right)+C$

11. $\int \frac{3}{\sqrt{9x^2-1}}dx$ on $\left(\frac{1}{3},\infty\right)$

Sol. $\int \frac{3}{\sqrt{9x^2-1}}dx = \int \frac{dx}{\sqrt{x^2-\left(\frac{1}{3}\right)^2}}$

$= \cosh^{-1}\left(\frac{x}{1/3}\right)+C = \cosh^{-1}(3x)+C$

$\therefore \int \frac{1}{\sqrt{x^2-a^2}}dx = \cosh^{-1}\left(\frac{x}{a}\right)+C$

II. Evaluate the following integrals.

1. $\int (3x-2)^{1/2}\ dx$

Sol. Given integral = $\int (3x-2)^{1/2}\ dx$

put 3x – 2=t ⇒ 3 dx = dt

$\int (3x-2)^{1/2}dx = \frac{1}{3}\int t^{1/2}dt$

$= \frac{1}{3}\frac{t^{3/2}}{3/2}+C$

$= \frac{2}{9}(3x-2)^{3/2}+C$

2. $\int \frac{1}{7x+3}dx$ **on** $I \subset R \setminus \left\{-\frac{3}{7}\right\}$

Sol. $\int \frac{1}{7x+3}dx$

Put 7x + 3=t $\Rightarrow$ 7 dx = dt

$$\int \frac{1}{7x+3}dx = \frac{1}{7}\int \frac{dt}{t}$$

$$= \frac{1}{7}\log|t| + C$$

$$= \frac{1}{7}\log|7x+3| + C$$

3. $\int \frac{\log(1+x)}{1+x}dx$ **on (–1, ∞).**

Sol. $\int \frac{\log(1+x)}{1+x}dx$

Put 1 + x = t $\Rightarrow$ dx= dt

$$\int \frac{\log(1+x)}{1+x}dx = \int \frac{\log t}{t}\cdot dt = \frac{(\log t)^2}{2} + C = \frac{1}{2}\left[\log(1+x)^2\right] + C$$

4. $\int (3x^2-4)x\,dx$ on R **.**

Sol. $\int (3x^2-4)x\,dx$

put $3x^2 - 4 = t \Rightarrow 6x\,dx = dt$

$$\int (3x^2-4)x\,dx = \frac{1}{6}\int t\,dt = \frac{1}{6}\cdot\frac{t^2}{2} + C = \frac{(3x^2-4)^2}{12} + C$$

5. $\int \frac{dx}{\sqrt{1+5x}}dx$ on $\left(-\frac{1}{5}, \infty\right)$

Sol. $\int \frac{dx}{\sqrt{1+5x}}$

Put 1 + 5x = t^2 ; 5dx = 2t dt, dx = $\frac{2}{5}t\,dt$

$$\int \frac{dx}{\sqrt{1+5x}} = \frac{2}{5}\int \frac{t\,dt}{t} = \frac{2}{5}\int dt = \frac{2}{5}t + C = \frac{2}{5}\sqrt{1+5x} + C$$

6. $\int(1-2x^3)x^2dx$ **on R.**

Sol. $\int(1-2x^3)x^2dx$

put $1-2x^3=t \Rightarrow -6x^2dx=dt$

$$\int(1-2x^3)x^2dx=-\frac{1}{6}\int t\,dt$$

$$=-\frac{1}{6}\cdot\frac{t^2}{2}+C=\frac{-(1-2x^3)^2}{12}+C$$

7. $\int\frac{\sec^2 x}{(1+\tan x)^3}dx$ **on** $I\subset R\setminus\left\{n\pi-\frac{\pi}{4}:n\in Z\right\}$

Sol. $\int\frac{\sec^2 x}{(1+\tan x)^3}dx$

put $1+\tan x=t \Rightarrow \sec^2 x\,dx=dt$

$$\int\frac{\sec^2 x}{(1+\tan x)^3}dx=\int\frac{dt}{t^3}=\int t^{-3}dt$$

$$=\frac{t^{-2}}{(-2)}+C=-\frac{1}{2t^2}+C$$

$$=-\frac{1}{2(1+\tan x)^2}+C$$

8. $\int x^3\sin x^4dx$ on R

Sol. $\int x^3\sin x^4dx$

Put $x^4=t \Rightarrow 4x^3dx=dt$

$$\int x^3\sin x^4dx=\frac{1}{4}\int\sin t\cdot dt=-\frac{1}{4}\cos t+C=-\frac{1}{4}\cdot\cos x^4+C$$

9. $\int\frac{\cos x}{(1+\sin x)^2}dx$ **on** $I\subset R\setminus\left\{2n\pi+\frac{3\pi}{2}:n\in Z\right\}$

Sol. $\int\frac{\cos x}{(1+\sin x)^2}dx$

Put $1+\sin x=t \Rightarrow \cos x dx=dt$

$$\int\frac{\cos x}{(1+\sin x)^2}dx=\int\frac{dt}{t^2}=-\frac{1}{t}+C=-\frac{1}{1+\sin x}+C$$

10. $\int \sqrt[3]{\sin x}\cos x\,dx$ **on [2nπ, (2n + 1)π, (n ∈ Z)].**

Sol. $\int \sqrt[3]{\sin x}\cos x\,dx$

Put sin x=t ⇒ cos x dx =dt

$$\int \sqrt[3]{\sin x}\cos x\,dx = \int \sqrt[3]{t}\cdot dt = \frac{t^{4/3}}{(4/3)} + C = \frac{3}{4}t^{4/3} + C = \frac{3}{4}(\sin x)^{4/3} + C$$

11. $\int 2x\,e^{x^2}\,dx$ **on R.**

Sol. $\int 2x\,e^{x^2}\,dx$

Let x^2 =t⇒ 2x dx =dt

$$\int 2x\,e^{x^2}\,dx = \int e^t\,dt = e^t + C = e^{x^2} + C$$

12. $\int \frac{e^{\log x}}{x}dx$ **on (0, ∞)**

Sol. $\int \frac{e^{\log x}}{x}dx$

Put log x= t ⇒ $\frac{1}{x}dx = dt$

$$\int \frac{e^{\log x}}{x}dx = \int e^t\cdot dt = e^t + C = e^{\log x} + C = x + C$$

13. $\int \frac{x^2}{\sqrt{1-x^6}}dx$ **on I = (–1, 1).**

Sol. $\int \frac{x^2}{\sqrt{1-x^6}}dx$

Put x^3 =t ⇒ 3x^2 dx =dt

$$\int \frac{x^2}{\sqrt{1-x^6}}dx = \frac{1}{3}\int \frac{dt}{\sqrt{1-t^2}} = \frac{1}{3}\sin^{-1}t + C = \frac{1}{3}\sin^{-1}(x^3) + C$$

14. $\int \frac{2x^3}{1+x^8}dx$ **on R.**

Sol. let $x^4 = t \Rightarrow 4x^3\,dx = dt$

$$\int \frac{2x^3}{1+x^8}dx = \frac{1}{2}\int \frac{dt}{1+t^2} = \frac{1}{2}\tan^{-1} t + C = \frac{1}{2}\tan^{-1}(x^4) + C$$

15. $\int \frac{x^8}{1+x^{18}}dx$

Sol. $\int \frac{x^8}{1+x^{18}}dx = \int \frac{x^8}{1+\left(x^9\right)^2}dx$ **on R.**

Put $x^9 = t \Rightarrow 9x^8\,dx = dt$

$$\int \frac{x^8}{1+x^{18}}dx = \int \frac{x^8}{1+(x^9)^2}dx = \frac{1}{9}\int \frac{dt}{1+t^2} = \frac{1}{9}\tan^{-1} t + C = \frac{1}{9}\tan^{-1}(x^9) + C$$

16. $\int \frac{e^x(1+x)}{\cos^2(xe^x)}dx$ **on** $I \subset R \setminus \{x \in R : \cos(xe^x) = 0\}$

Sol. $\int \frac{e^x(1+x)}{\cos^2(xe^x)}dx$

Put $x\,e^x = t$

$$(x \cdot e^x + e^x)dx = e^x(1+x)dx = dt$$

G.I. $= \int \frac{e^x(1+x)}{\cos^2(xe^x)}dx = \int \frac{dt}{\cos^2 t} = \int \sec^2 t\ dt$

$$= \tan t + C = \tan(x \cdot e^x) + C$$

17. $\int \frac{\csc^2 x}{(a+b\cot x)^5}dx$ **on $I \subset R \setminus \{x \in R : a + b\cot x = 0\}$, where $a, b \in R, b \neq 0$.**

Sol.

G.I. $= \int \frac{\csc^2 x}{(a+b\cot x)^5}dx$

Put $a + b\cot x = t \Rightarrow -b\csc^2 x\,dx = dt$

$$\int \frac{\csc^2 x}{(a+b\cot x)^5}dx = -\frac{1}{b}\int \frac{dt}{t^5} = -\frac{1}{b}\int t^{-5}dt$$

$$= -\frac{1}{b}\frac{t^{-4}}{-4} + C = \frac{1}{4bt^4} + C = \frac{1}{4b(a+b\cot x)^4} + C$$

18. $\int e^x \sin e^x dx$ **on R.**

Sol. $\mathbf{e^x = t \Rightarrow \ e^x\, dx = dt}$

$$\int e^x \sin e^x dx = \int \sin t\, dt$$

$$= -\cos t + C = -\cos(e^x) + C$$

19. $\int \frac{\sin(\log x)}{x} dx$ **on (0, ∞)**

Sol. $\int \frac{\sin(\log x)}{x} dx$

put log x =t⇒ dt = $\frac{1}{x} dx$ **=dt**

$$\int \frac{\sin(\log x)}{x} dx = \int \sin t\, dt$$

$$= -\cos t + C = -\cos(e^x) + C$$

20. $\int \frac{1}{x \log x} dx$ **on (0, ∞)**

Sol. $\int \frac{1}{x \log x} dx$

Put log x =t⇒ dt = $\frac{1}{x} dx$ **=dt**

$$\int \frac{1}{x \log x} dx = \int \frac{1}{t} dt = \log t + C = \log(\log x + C)$$

21. $\int \frac{(1+\log x)^n}{x} dx$ **on (0, ∞), n ≠ –1.**

Sol. $\int \frac{(1+\log x)^n}{x} dx$

Put 1 + log x= t , $\Rightarrow \frac{1}{x} dx$ **=dt**

$$\int \frac{(1+\log x)^n}{x} dx = \int t^n dt = \frac{t^{n+1}}{n+1} + C$$

$$= \frac{(1+\log x)^{n+1}}{n+1} + C$$

22. $\int \frac{\cos(\log x)}{x} dx$ **on (0, ∞)**

Sol. $\int \frac{\cos(\log x)}{x} dx$

Put log x =t⇒ dt = $\frac{1}{x} dx$ **=dt**

$$\int \frac{\cos(\log x)}{x} dx = \int \cos t \, dt = \sin t + C = \sin(\log x) + C$$

23. $\int \frac{\cos\sqrt{x}}{\sqrt{x}} dx$ **on (0, ∞)**

Sol. $\text{let}\sqrt{x} = t \Rightarrow \frac{1}{2\sqrt{x}} dx = dt \Rightarrow 2dt = \frac{dx}{\sqrt{x}}$

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}} dx = 2\int \cos t \, dx$$

$$= 2\sin t + C = 2\sin\sqrt{x} + C$$

24. $\int \frac{2x+1}{x^2+x+1} dx$ **on R.**

Sol. $\int \frac{2x+1}{x^2+x+1} dx$

put $x^2 + x + 1 = t \Rightarrow (2x + 1)dx = dt$

$$\int \frac{2x+1}{x^2+x+1} dx = \int \frac{dt}{t}$$

$$= \log|t| + C = \log|x^2 + x + 1| + C$$

25. $\int \frac{ax^{n-1}}{bx^n + C} dx$ **, where n ∈ N, a, b, c are real numbers, b ≠ 0 and**

$$x \in I \subset \left\{ x \in R : x^n \neq -\frac{c}{b} \right\}$$

Sol. $\int \frac{ax^{n-1}}{bx^n + C} dx$

let $bx^n + C = t \Rightarrow$ $nbx^{n-1}dx = dt$, $x^{n-1}dx = \frac{1}{nb} dt$

$$\int \frac{ax^{n-1}}{bx^n + C}dx = \frac{a}{nb}\int \frac{dt}{t} = \frac{a}{nb}\log|t| + dt$$

$$= \frac{a}{nb}\log|bx^n + c| + k$$

26. $\int \frac{1}{x \log x[\log(\log x)]}dx$ **on (1, ∞)**

Sol. G.I. $\int \frac{1}{x \log x[\log(\log x)]}dx$

Put log(log x)=t, $\frac{1}{\log x}\cdot\frac{1}{x}dx = dt$

$$\int \frac{1}{x \log x[\log(\log x)]}dx = \int \frac{dt}{t}$$

$$= \log|t| + C = \log|\log(\log x)| + C$$

27. $\int \coth x dx$ **on R.**

Sol. sinh x=t ⇒ cosh x dx= dt

$$\int \coth x dx = \int \frac{dt}{t} = \log|t| + C$$

$$= \log|\sinh x| + C$$

28. $\int \frac{1}{\sqrt{1-4x^2}}dx$ on $\left(-\frac{1}{2}, \frac{1}{2}\right)$

Sol. $\int \frac{1}{\sqrt{1-4x^2}}dx = \frac{1}{2}\int \frac{dx}{\sqrt{(1/2)^2 - x^2}}$

$$= \frac{1}{2}\sin^{-1}\left(\frac{x}{1/2}\right) + C = \frac{1}{2}\sin^{-1}(2x) + C$$

29. $\int \frac{dx}{\sqrt{25+x^2}}$ **on R**

Sol. $\int \frac{dx}{\sqrt{25+x^2}} = \int \frac{dx}{\sqrt{x^2+5^2}} = \sinh^{-1}\left(\frac{x}{5}\right) + C$

30. $\int \frac{1}{(x+3)\sqrt{x+2}}dx$ **on I ⊂ (–2, ∞)**

Sol. put x + 2 = t^2 , dx = 2t dt

$$\int\frac{1}{(x+3)\sqrt{x+2}}dx=\int\frac{2t\,dt}{t(t^2+1)}=2\int\frac{dt}{t^2+1}$$

$$=2\tan^{-1}(t)+C=2\tan^{-1}(\sqrt{x+2})+C$$

31. $\int\frac{1}{1+\sin 2x}dx$ **on**

$$I\subset R\setminus\left\{\frac{n\pi}{2}+(-1)^n\frac{\pi}{4}:n\in Z\right\}$$

Sol. $\int\frac{1}{1+\sin 2x}dx=\int\frac{dx}{1+\frac{2\tan x}{1+\tan^2 x}}$

$$=\int\frac{(1+\tan^2 x)dx}{1+\tan^2 x+2\tan x}=\int\frac{\sec^2 x dx}{(1+\tan x)^2}$$

put $1+\tan x=t\Rightarrow\sec^2 x dx=dt$

$$\int\frac{1}{1+\sin 2x}dx=\int\frac{dt}{t^2}=-\frac{1}{t}+C=-\frac{1}{1+\tan x}+C$$

III. Evaluate the following integrals.

1. $\int\frac{\sin 2x}{a\cos^2 x+b\sin^2 x}dx$ **on**

$$I\subset R\setminus(x\in R\mid a\cos^2 x+b\sin^2 x=0\}$$

Sol. put $a\cos^2 x+b\sin^2 x=t$

$$\big(a(2\cos x)(-\sin x)+b(2\sin x\cos x)\big)dx=dt$$

$$=\sin 2x(b-a)dx$$

$$\sin 2x\cdot dx=\frac{1}{(b-a)}dt$$

$$\int\frac{\sin 2x}{a\cos^2 x+b\sin^2 x}dx=\frac{1}{(b-a)}\int\frac{dt}{t}=\frac{1}{(b-a)}\log|t|+C$$

$$=\frac{1}{(b-a)}\log|a\cos^2 x+b\sin^2 x|+C$$

2. $\int\frac{1-\tan x}{1+\tan x}dx$ **for** $x\in I\subset R\setminus\left\{n\pi-\frac{\pi}{4}:n\in Z\right\}$

Sol. $\int\frac{1-\tan x}{1+\tan x}dx=\int\frac{1-\frac{\sin x}{\cos x}}{1+\frac{\sin x}{\cos x}}dx$

$$=\int\frac{\cos x-\sin x}{\cos x+\sin x}dx$$

$\cos x+\sin x=t$

$\Rightarrow dt=-\sin x+\cos x\,dx$

$$\int\frac{1-\tan x}{1+\tan x}dx=\int\frac{dt}{t}=\log|t|+C$$

$$=\log|\cos x+\sin x|+C$$

3. $\int\frac{\cot(\log x)}{x}dx$**,** $x\in I\subset(0,\infty)\setminus\{e^{n\pi}:n\in Z\}$**.**

Sol. Put log x =t⇒ dt = $\frac{1}{x}dx$ **=dt**

$$\int\frac{\cot(\log x)}{x}dx=\int\cot t\,dt=\log(\sin t)+C$$

$$=\log(\sin(\log x))+C$$

4. $\int e^x\cdot\cot e^x dx,\ x\in I\subset R\setminus\{\log n\pi:n\in Z\}$

Sol. Put $e^x=t\Rightarrow$ $e^x\,dx=dt$

$$\int e^x\cdot\cot e^x dx=\int\cot t\,dt=\log|\sin t|+C\qquad=\log(\sin e^x)+C$$

5. $\int\sec x(\tan x)\sec^2 x\,dx$**, on**

$I\subset\left\{x\in E:\tan x\neq\frac{(2k+1)\pi}{2}\text{ for any }k\in Z\right\}$ **where** $E=R/\left\{\frac{(2n+1)\pi}{2}:n\in Z\right\}$

Sol. tan x=t ⇒ $\sec^2 x\,dx=dt$

$$\int\sec x(\tan x)\sec^2 x\,dx=\int\sec t\cdot dt$$

$$=\log\tan\left(\frac{\pi}{4}+\frac{t}{2}\right)+C=\log\left(\tan\left(\frac{\pi}{4}+\frac{\tan x}{2}\right)\right)+C$$

6. $\int \sqrt{\sin x}\cos x\,dx$ **on** $[2n\pi,(2n+1)\pi]$**,** $\mathbf{n \in Z}$**.**

Sol. t = sin x $\Rightarrow$ **dt = cosx dx**

$$\int \sqrt{\sin x}\cdot\cos x\,dx = \int \sqrt{t}\ dt = \frac{2}{3}t^{3/2} + C$$

$$= \frac{2}{3}(\sin x)^{3/2} + C$$

7. $\int \tan^4 x \sec^2 x\,dx, x \in I \subset R \setminus$

$$\left\{\frac{(2n+1)\pi}{2} : n \in Z\right\}$$

Sol. $\tan x = t \Rightarrow \sec^2 x dx = dt$

$$\int \tan^4 x \sec^2 x\,dx = \int t^4 dt$$

$$= \frac{t^5}{5} + C = \frac{(\tan x)^5}{t} + C$$

8. $\int \frac{2x+3}{\sqrt{x^2+3x-4}}dx$**,** $x \in I \subset R \setminus [-4,1]$**.**

Sol. Let $x^2 + 3x - 4 = t \Rightarrow$ $(2x + 3)dx = dt$

$$\int \frac{2x+3}{\sqrt{x^2+3x-4}}dx = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} + C \quad = 2\sqrt{x^2+3x-4} + C$$

9. $\int \csc^2 x\sqrt{\cot x}\,dx$ on $\left(0, \frac{\pi}{2}\right)$

Sol. put $\cot x = t \Rightarrow -\csc^2 x\,dx = dt$

$$\int \csc^2 x\sqrt{\cot x}\,dx = -\int \sqrt{t}\,dt = -\frac{2}{3}t\sqrt{t} + C = -\frac{2}{3}(\cot x)^{3/2} + C$$

10. $\int \sec x \log(\sec x + \tan x)dx$ on $\left(0, \frac{\pi}{2}\right)$

Sol. $\log(\sec x + \tan x) = t$

$$\Rightarrow \frac{(\sec x\cdot\tan x + \sec^2 x)dx}{(\sec x + \tan x)} = dt = \textbf{sec x dx}$$

$$\int \sec x \cdot \log(\sec x + \tan x)dx = \int t\,dt$$

$$= \frac{t^2}{2} + C = \frac{(\log(\sec x + \tan x))^2}{2} + C$$

11. $\int \sin^3 x\,dx$ **on R.**

Sol. since $\sin 3x = 3\sin x - 4\sin^3 x$

$$\sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$$

$$\int \sin^3 x\,dx = \frac{3}{4}\int \sin x - \frac{1}{4}\int \sin 3x\,dx$$

$$= -\frac{3}{4}\cos x + \frac{1}{12}\cos 3x + C$$

$$= \frac{1}{12}(\cos 3x - 9\cos x) + C$$

12. $\int \cos^3 x dx$ **on R.**

Sol. since $\cos 3x = 4\cos^3 x - 3\cos x$

$$\cos^3 x = \frac{1}{4}(3\cos x + \cos 3x)$$

$$\int \cos^3 x\,dx = \frac{3}{4}\int \cos x\,dx + \frac{1}{4}\int \cos 3x\,dx$$

$$= \frac{3}{4}\sin x + \frac{1}{12}\sin 3x + C = \frac{1}{12}(9\sin x + \sin 3x) + C$$

13. $\int \cos x \cos 2x\,dx$ **on R.**

Sol. $\cos 2x \cos x = \frac{1}{2}(2\cos 2x \cdot \cos x)$

$$\int \cos x = \cos 2x\,dx = \frac{1}{2}\int (\cos 3x + \cos x)dx$$

$$= \frac{1}{2}\int \cos 3x dx + \frac{1}{2}\int \cos x dx = \frac{1}{2}\left(\frac{\sin 3x}{3} + \sin x\right) + C = \frac{\sin 3x + 3\sin x}{6} + C$$

14. $\int \cos x \cos 3x\,dx$ **on R.**

Sol. $\cos 3x \cos x = \frac{1}{2}(2\cos 3x \cdot \cos x) =$ $\frac{1}{2}(\cos 4x + \cos 2x)$

$$\int \cos x \cos 3x\,dx = \frac{1}{2}\int \cos 4x\,dx + \frac{1}{2}\cos 2x\,dx$$

$$= \frac{1}{2}\left(\frac{\sin 4x}{4} + \frac{\sin 2x}{2}\right) + C = \frac{1}{8}\left(\sin 4x + 2\sin 2x\right) + C$$

15. $\int \cos^4 x\, dx$ **on R.**

Sol. $\cos^4 x = (\cos^2 x)^2 = \left(\frac{1+\cos 2x}{2}\right)^2$

$$= \frac{1}{4}(1 + 2\cos 2x + \cos^2 2x)$$

$$= \frac{1}{4}\left(1 + 2\cos 2x + \frac{1+\cos 4x}{2}\right)$$

$$= \frac{1}{8}(2 + 4\cos 2x + 1 + \cos 4x)$$

$$= \frac{1}{8}(3 + 4\cos 2x + \cos 4x)$$

$$= \frac{1}{8}\left(3\int dx + 4\int \cos 2x\, dx + \int \cos 4x\, dx\right)$$

$$= \frac{1}{8}\left(3x + 4\frac{\sin 2x}{2} + \frac{\sin 4x}{4}\right) + C$$

$$= \frac{1}{32}\left(12x + 8\sin 2x + \sin 4x\right) + C$$

16. $\int x\sqrt{4x+3}\, dx$ **on** $\left(-\frac{3}{4}, \infty\right)$**.**

Sol. put $4x + 3 = t^2 \Rightarrow 4dx = 2t\, dt$

$$dx = \frac{1}{2}t\, dt \Rightarrow x = \frac{t^2 - 3}{4}$$

$$\int x\sqrt{4x+3}\, dx = \int \frac{t^2-3}{4} \cdot t \cdot \frac{1}{2} t\, dt$$

$$= \frac{1}{8}\int (t^4 - 3t^2) dt = \frac{1}{8}\left(\frac{t^5}{5} - t^3\right) + C$$

$$= \frac{(4x+3)^{5/2}}{40} - \frac{1}{8}(4x+3)^{3/2} + C$$

17. $\int \frac{dx}{\sqrt{a^2-(b+cx)^2}}$ **on** $\{x \in R : |b+cx| < a\}$**, where a, b, c are real numbers** $c \neq 0$ **and** $a > 0$**.**

Sol. $\int \frac{dx}{\sqrt{a^2-(b+cx)^2}} = \int \frac{dx}{c\sqrt{\left(\frac{a}{c}\right)^2-\left(\frac{b}{c}-x\right)^2}}$

$$= \frac{1}{c}\sin^{-1}\left(\frac{\left(\frac{b}{c}+x\right)}{\left(\frac{a}{c}\right)}\right)+K = \frac{1}{c}\sin^{-1}\left(\frac{b+cx}{a}\right)+K$$

18. $\int \frac{dx}{a^2+(b+cx)^2}$ **on R, where a, b, c are real numbers,** $c \neq 0$ **and** $a > 0$**.**

Sol. $\int \frac{dx}{a^2+(b+cx)^2} = \frac{1}{c^2}\int \frac{dx}{\left(\frac{a}{c}\right)^2+\left(\frac{b}{c}+x\right)^2}$

$$= \frac{1}{a^2\cdot\frac{a}{c}}\tan^{-1}\left(\frac{\frac{b}{c}+x}{\frac{a}{c}}\right)+C = \frac{1}{ac}\tan^{-1}\left(\frac{b+cx}{a}\right)+C$$

19. $\int \frac{dx}{1+e^x}, x \in R$

Sol. $\int \frac{dx}{1+e^x} = \int\left(\frac{1+e^x-e^x}{1+e^x}\right)dx$

$$= \int\left(1-\frac{e^x}{1+e^x}\right)dx = x-\log(1+e^x)+C$$

20. $\int \frac{x^2}{(1+bx)^2}dx$**,** $x \in I \subset R \setminus \left\{-\frac{a}{b}\right\}$**, where a, b are real numbers,** $b \neq 0$**.**

Sol. **Put** $a + bx = t$**,** $\Rightarrow b\,dx = dt \Rightarrow dx = \frac{1}{b}\cdot dt$

$$\int \frac{x^2}{(a+bx)^2}dx = \frac{1}{b}\int \frac{\left(\frac{t-a}{b}\right)^2}{t^2}dt$$

$$= \frac{1}{b^3}\int \frac{t^2 - 2at + a^2}{t^2}\,dt$$

$$= \frac{1}{b^3}\int\left(1 - \frac{2a}{t} + \frac{a^2}{t^2}\right) + C$$

$$= \frac{1}{b^3}\left(t - 2a\log|t| - \frac{a^2}{t}\right) + C$$

$$= \frac{1}{b^3}\left[(a+bx) - 2a\log|a+bx| - \frac{a^2}{(a+bx)}\right] + C$$

21. $\int \frac{x^2}{\sqrt{1-x}}\,dx, x \in (-\infty, 1)$

Sol. Put $1 - x = t^2, -dx = 2t\,dt$

$$\int \frac{x^2}{\sqrt{1-x}}\,dx = \int (1-t^2)^2 \cdot \frac{-2t}{t}\,dt$$

$$= 2\int (1 - 2t^2 + t^4)dt = -2\left(t - \frac{2}{3}t^3 + \frac{t^5}{5}\right) + C$$

$$= -2\left(\sqrt{1-x} - \frac{2}{3}(1-x)^{3/2} + \frac{1}{5}(1-x)^{5/2}\right) + C$$

INTEGRATION BY PARTS THEOREM

If f(x) and g(x) are two integrable functions then

$\int f(x)\cdot g(x)dx = f(x)\int g(x)dx - \int f'(x)\left[\int g(x)dx\right]dx$ **.**

Proof :

$$\frac{d}{dx}\left[f(x)\cdot\int g(x)dx\right] = f(x)\frac{d}{dx}\left[\int g(x)dx\right] + \int g(x)dx \cdot \frac{d}{dx}[f(x)]$$

$$= f(x)g(x) + \left[\int g(x)dx\right]f'(x)$$

$$\therefore \int\left[f(x)g(x) + f'(x)\int g(x)dx\right]dx = f(x)\int g(x)\,dx$$

$$\Rightarrow \int f(x)g(x)dx + \int f'(x)\left[\int g(x)dx\right]dx = f(x)\int g(x)\,dx$$

$$\therefore \int f(x)g(x)dx = f(x)\int g(x)\,dx - \int f'(x)\left[\int g(x)dx\right]dx$$

Note 1 : If u and v are two functions of x then $\int u\,dv = uv - \int v\,du$.

Note 2 : If u and v are two functions of x; u′, u″, u‴ …… denote the successive derivatives of u and v_1, v_2, v_3, v_4, v_5,……. the successive integrals of v then the extension of integration by pairs is $\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots$.

Note 3 : In integration by parts, the first function will be taken as the following order.

Inverse functions, Logarithmic functions, Algebraic functions, Trigonometric functions and Exponential functions. (To remember this a phrase ILATE).

THEOREM

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c$$

Proof :

Let I = $\int e^{ax}\cos bx\,dx = \cos bx\int e^{ax}dx - \int\left[\,d(\cos bx)\int e^{ax}dx\right]dx$

$$= \cos bx\frac{e^{ax}}{a} - \int(-b\sin bx)\frac{e^{ax}}{a}dx$$

$$= \frac{e^{ax}}{a}\cos bx + \frac{b}{a}\int e^{ax}\sin bx\,dx$$

$$= \frac{e^{ax}}{a}\cos bx + \frac{b}{a}\left[\sin bx\frac{e^{ax}}{a} - \int b\cos bx\frac{e^{ax}}{a}dx\right]$$

$$= \frac{e^{ax}}{a}\cos bx + \frac{b}{a^2}e^{ax}\sin bx - \frac{b^2}{a^2}I$$

$$\Rightarrow I\left(1+\frac{b^2}{a^2}\right) = \frac{1}{a^2}e^{ax}\left[a\cos bx + b\sin bx\right]$$

$$\Rightarrow I\left(\frac{a^2+b^2}{a^2}\right) = \frac{1}{a^2}e^{ax}\left[a\cos bx + b\sin bx\right]$$

$$\therefore I = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right] + c$$

THEOREM

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

Proof : Let I =

$\int e^{ax}\sin bx\,dx = \sin bx\int e^{ax}dx - \int\left[\,d(\sin bx)\int e^{ax}dx\right]dx$

$$= \sin bx \frac{e^{ax}}{a} - \int b \cos bx \frac{e^{ax}}{a} dx$$

$$= \frac{1}{a} e^{ax} \sin bx - \frac{b}{a} \int e^{ax} \cos bx \, dx$$

$$= \frac{1}{a} e^{ax} \sin bx - \frac{b}{a} \left[\cos bx \frac{e^{ax}}{a} - \int (-b \sin bx) \frac{e^{ax}}{a} dx \right]$$

$$= \frac{1}{a} e^{ax} \sin bx - \frac{b}{a^2} e^{ax} \cos bx - \frac{b^2}{a^2} \int e^{ax} \sin bx \, dx$$

$$= \frac{1}{a} e^{ax} \sin bx - \frac{b}{a^2} e^{ax} \cos bx - \frac{b^2}{a^2} I$$

$$\Rightarrow I \left(1 + \frac{a^2}{b^2} \right) = \frac{1}{a^2} e^{ax} \left[a \sin bx - b \cos bx \right]$$

$$\Rightarrow I \left(\frac{a^2 + b^2}{b^2} \right) = \frac{e^{ax}}{a^2} \left[a \sin bx - b \cos bx \right]$$

$$\therefore I = \frac{e^{ax}}{a^2 + b^2} \left[a \sin bx - b \cos bx \right] + c$$

THEOREM $\int e^x \left[f(x) + f'(x) \right] dx = e^x f(x) + c$

Proof :

$$\int e^x \left[f(x) + f'(x) \right] dx = \int e^x f(x) dx + \int e^x f'(x) dx$$

$$= f(x) \int e^x dx - \int \left[d[f(x)] \int e^x dx \right] dx + \int e^x f'(x) dx$$

$$= f(x) e^x - \int f'(x) e^x dx + \int e^x f'(x) dx = e^x f(x) + c$$

Note : $\int e^{-x} \left[f(x) - f'(x) \right] dx = -e^{-x} f(x) + c$

EXERCISE – 8 (C)

I. Evaluate the following integrals.

1. $\int x\sec^2 x\,dx$ **on** $I \subset R \setminus \left\{\frac{(2n+1)\pi}{2} : n \text{ is an integer}\right\}$

Sol. $\int x\sec^2 x\,dx = x(\tan x) - \int \tan x\,dx \qquad = x\tan x - \log|\sec x| + C$

2. $\int e^x\left(\tan^{-1}x + \frac{1}{1+x^2}\right)dx$**, x ∈ R.**

Sol.

Let $f(x) = \tan^{-1}x$ **so that** $f'(x) = \frac{1}{1+x^2}$

$\therefore \int e^x\left(\tan^{-1}x + \frac{1}{1+x^2}\right)dx = e^x\tan^{-1}x + C \left(\because \int e^x[f(x)+f'(x)]dx = e^x\cdot f(x) + C\right)$

3. $\int \frac{\log x}{x^2}dx$ **on (0, ∞).**

Sol. $\int \frac{\log x}{x^2}dx = (\log x)\left(-\frac{1}{x}\right) + \int \frac{1}{x}\cdot\frac{1}{x}dx \qquad = -\frac{1}{x}\log x - \frac{1}{x} + C$

4. $\int (\log x)^2 dx$ **on (0, ∞).**

Sol. $\int (\log x)^2 dx = (\log x)^2 x - \int x\cdot 2\log x\cdot\frac{1}{x}dx$

$= x(\log x)^2 - 2\int \log x\,dx = x(\log x)^2 - 2\left(x\log x - \int x\frac{1}{x}dx\right)$

$= x(\log x)^2 - 2x\cdot\log x + x + c$

5. $\int e^x(\sec x + \sec x\tan x)dx$ **on** $I \subset R \setminus \left\{(2n+1)\frac{\pi}{2} : n \in Z\right\}$

Sol. $\int e^x(\sec x + \sec x\tan x)dx = e^x\cdot\sec x + C$

$\left(\because \int e^x[f(x)+f'(x)]dx = e^x f(x) + C\right)$

6. $\int e^x \cos x\,dx$ **on R.**

Sol. $I = \int e^x \cos x\,dx = e^x \sin x - \int \sin x \cdot e^x dx$

$= e^x \cdot \sin x + e^x \cdot \cos x - \int e^x \cdot \cos x\,dx$

$= e^x(\sin x + \cos x) - I$

$2I = e^x(\sin x + \cos x)$

$I = \frac{e^x}{2}(\sin x + \cos x) + C$

7. $\int e^x(\sin x + \cos x)dx$ **on R.**

Sol. $\int e^x(\sin x + \cos x)dx$

$f(x) = \sin x \Rightarrow f'(x) = \cos x$

$\therefore \int e^x(\sin x + \cos x)dx = e^x \cdot \sin x + C$

8. $\int (\tan x + \log \sec x)e^x dx$ **on** $\left(\left(2n - \frac{1}{2}\right)\pi, \left(2n + \frac{1}{2}\right)\pi\right) n \in Z$

Sol. let $f = \log|\sec x| \Rightarrow f'(x) = \frac{1}{\sec x} \cdot \sec x \cdot \tan x \cdot$

$= \tan x$

$\int (\tan x + \log \sec x)e^x dx = e^x \cdot \log|\sec x| + C \left(\because \int e^x[f(x) + f'(x)]dx = e^x f(x) + C\right)$

II. Evaluate the following integrals.

1. $\int x^n \log x\,dx$ **on (0, ∞), n is a real number and n ≠ –1.**

Sol. $\int x^n \log x\,dx = (\log x)\frac{x^{n+1}}{n+1} - \frac{1}{n+1}\int x^{n+1}\frac{1}{x}dx$

$= \frac{x^{n+1}(\log x)}{n+1} - \frac{1}{n+1}\int x^n dx$

$= \frac{x^{n+1}(\log x)}{n+1} - \frac{x^{n+1}}{(n+1)^2} + C$

$= \frac{x^{n+1}}{(n+1)^2}[(n+1)\log x - 1] + C$

2. $\int \log(1+x^2)dx$ **on R.**

Sol. $\int \log(1+x^2)dx = \int 1.\log(1+x^2)dx$ =

$$= \log(1+x^2)\cdot x - \int x\frac{1}{1+x^2}2x\,dx$$

$$= x\log(1+x^2) - 2\int\frac{1+x^2-1}{1+x^2}dx$$

$$= x\log(1+x^2) - 2\int dx + 2\int\frac{dx}{1+x^2}$$

$$= x\log(1+x^2) - 2x + 2\tan^{-1}x + C$$

3. $\int\sqrt{x}\log x\,dx$ **on (0, ∞).**

Sol. $\int\sqrt{x}\log x\,dx$ =

$$= \log x\cdot\frac{2}{3}x^{3/2} - \frac{2}{3}\int x^{3/2}\cdot\frac{1}{x}dx$$

$$= \frac{2}{3}x^{3/2}(\log x) - \frac{2}{3}\int x^{1/2}dx$$

$$= \frac{2}{3}x^{3/2}(\log x) - \frac{2}{3}\frac{x^{3/2}}{3/2} + C$$

$$= \frac{2}{3}x^{3/2}\log x - \frac{4}{9}x^{3/2} + C$$

4. $\int e^{\sqrt{x}}dx$ **on (0, ∞).**

Sol. **let** $\sqrt{x} = t \Rightarrow x = t^2,\ dx = 2t\,dt$

$$\int e^{\sqrt{x}}dx = 2\int t\,e^t dt = 2\left[te^t - \int e^t dt\right]$$

$$= 2(te^t - e^t) + C$$

$$= 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$$

5. $\int x^2\cos x\,dx$ **on R.**

Sol. $\int x^2\cos x\,dx = x^2(\sin x) - \int\sin x(2x\,dx)$

$$= x^2\sin x + 2\int x(-\sin x)dx$$

$$= x^2\cdot\sin x + 2[x\cos x - \int\cos x\,dx]$$

$$= x^2\sin x + 2x\cos x - 2\sin x + c$$

6. $\int x \sin^2 x\,dx$ **on R.**

Sol. $\int x \sin^2 x\,dx = \frac{1}{2}\int x(1-\cos 2x)dx$

$$= \frac{1}{2}\left[\int x dx - \int x\cos 2x\,dx\right]$$

$$= \frac{1}{2}\left[\frac{x^2}{2} - \left\{x\cdot\frac{\sin 2x}{2} - \frac{1}{2}\int \sin 2x\,dx\right\}\right]$$

$$= \frac{x^2}{4} - \frac{x}{4}\sin 2x + \frac{1}{4}\int \sin 2x\,dx$$

$$= \frac{x^2}{4} - \frac{x}{4}\sin 2x - \frac{1}{8}\cos 2x + C$$

7. $\int x \cos^2 x\,dx$ **on R.**

Sol. $\int x \cos^2 x\,dx = \frac{1}{2}\int x(1+\cos 2x)dx$

$$= \frac{1}{2}\left[\int x\,dx + \int x\cos 2x\,dx\right]$$

$$= \frac{1}{2}\left[\frac{x^2}{2} + \left\{x\frac{\sin 2x}{2} - \frac{1}{2}\int \sin 2x\,dx\right\}\right]$$

$$= \frac{x^2}{4} + \frac{x}{4}\sin 2x - \frac{1}{4}\int \sin 2x\,dx$$

$$= \frac{x^2}{4} + \frac{x}{4}\sin 2x + \frac{1}{8}\cos 2x + C$$

8. $\int \cos\sqrt{x}\,dx$ **on R.**

Sol. $x = t^2 \Rightarrow dx = 2t\,dt$

$$I = 2\int t\cdot\cos t\,dt = 2(t\sin t - \int \sin t\,dt)$$

$$= 2(t\sin t + \cos t) + C$$

$$= 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$$

9. $\int x\sec^2 2x\,dx$ **on** $I \subset R \setminus \left\{(2n\pi+1)\frac{\pi}{4} : n \in Z\right\}$

Sol. $\int x\sec^2 2x\,dx = x\frac{\tan 2x}{2} - \frac{1}{2}\int \tan 2x\,dx$

$$= x\frac{\tan 2x}{2} - \frac{1}{2}\cdot\frac{1}{2}\log|\sec 2x| + C$$

$$= x\frac{\tan 2x}{2} - \frac{1}{4}\log|\sec 2x| + C$$

10. $\int x\cot^2 x\,dx$ **on** $I \subset R \setminus \{n\pi : n \in Z\}$.

Sol. $\int x\cot^2 x\,dx = \int x(\csc^2 x - 1)dx$

$$= \int x\csc^2 x\,dx - \int x\,dx$$

$$= x(-\cot x) + \int \cot x\,dx - \frac{x^2}{2}$$

$$= -x\cot x + \log|\sin x| - \frac{x^2}{2} + C$$

11. $\int e^x(\tan x + \sec^2 x)dx$ **on** $I \subset R \setminus \left\{(2n+1)\frac{\pi}{2} : n \in Z\right\}$

Sol. $f(x) = \tan x \Rightarrow f'(x) = \sec^2 x\,dx$

$$I = \int e^x\left[f(x) + f'(x)\right]dx = e^x f(x) + C \quad = e^x\tan x + C$$

12. $\int e^x\left(\frac{1 + x\log x}{x}\right)dx$ **on (0, ∞).**

Sol. $\int e^x\left(\frac{1 + x\log x}{x}\right)dx = \int e^x\left(\log x + \frac{1}{x}\right)dx = e^x\log x + C$

III. Evaluate the following integrals.

1. $\int x\tan^{-1} x\,dx,\ x \in R$

Sol. $\int x\tan^{-1} x\,dx = (\tan^{-1} x)\frac{x^2}{2} - \frac{1}{2}\int x^2\cdot\frac{1}{1+x^2}dx$

$$= \frac{x^2(\tan^{-1} x)}{2} - \frac{1}{2}\int\left(1 - \frac{1}{1+x^2}\right)dx$$

$$= \frac{x^2(\tan^{-1} x)}{2} - \frac{1}{2}(x - \tan^{-1} x) + C$$

$$= \frac{x^2(\tan^{-1}x)}{2} - \frac{x}{2} + \frac{\tan^{-1}x}{2} + C$$

$$= \frac{(x^2+1)}{2}\tan^{-1}x - \frac{x}{2} + C$$

2. $\int x^2 \tan^{-1}x\,dx,\ x \in R$.

Sol. $\int x^2 \tan^{-1}x\,dx = (\tan^{-1}x)\frac{x^3}{3} - \frac{1}{3}\int x^3 \frac{1}{1+x^2}dx$

$$= \frac{x^3(\tan^{-1}x)}{3} - \frac{1}{3}\int \frac{x(x^2+1)-x}{1+x^2}dx$$

$$= \frac{x^3(\tan^{-1}x)}{3} - \frac{1}{3}\int x dx + \frac{1}{3}\int \frac{x dx}{1+x^2}$$

$$= \frac{x^3(\tan^{-1}x)}{3} - \frac{x^2}{6} + \frac{1}{6}\log|1+x^2| + C$$

3. $\int \frac{\tan^{-1}x}{x^2}dx, x \in I \subset R \setminus \{0\}$

Sol. $\int \frac{\tan^{-1}x}{x^2}dx = \int \tan^{-1}x \frac{1}{x^2} = (\tan^{-1}x)\left(-\frac{1}{x}\right) + \int \frac{1}{x}\frac{1}{1+x^2}dx$

$$= -\frac{\tan^{-1}x}{x} + \frac{1}{2}\int \frac{2x\,dx}{x^2(1+x^2)}$$

$$= -\frac{\tan^{-1}x}{x} + \frac{1}{2}\int\left(\frac{1}{x^2} - \frac{1}{1+x^2}\right)(2x\,dx)$$

$$= \frac{\tan^{-1}x}{x} + \int\frac{dx}{x} - \frac{1}{2}\int\frac{2x\,dx}{1+x^2}$$

$$= -\frac{\tan^{-1}x}{x} + \log|x| - \frac{1}{2}\log|1+x^2| + C$$

4. $\int x\cos^{-1}x\,dx, x \in (-1,1)$

Sol. $\int x\cos^{-1}x$

$$= \cos^{-1}\int x\,dx - \int\left[\frac{d}{dx}[\cos^{-1}x]\int x\,dx\right]dx$$

$$= \frac{x^2}{2}\cos^{-1}x - \int \frac{-1}{\sqrt{1-x^2}}\frac{x^2}{2}dx$$

$$= \frac{x^2}{2}\cos^{-1}x + \frac{1}{2}\int \frac{x^2}{1-x^2}dx$$

$$= \frac{x^2}{2}\cos^{-1}x - \frac{1}{2}\int \frac{1-x^2-1}{\sqrt{1-x^2}}dx$$

$$= \frac{x^2}{2}\cos^{-1}x - \frac{1}{2}\int \sqrt{1-x^2}dx + \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}dx$$

$$= \frac{x^2}{2}\cos^{-1}x - \frac{1}{2}\left[\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right] + \frac{1}{2}\sin^{-1}x + C$$

$$= \frac{x^2}{2}\cos^{-1}x - \frac{1}{4}x\sqrt{1-x^2} + \frac{1}{4}\sin^{-1}x + C$$

5. $\int x^2 \sin^{-1}x\,dx, x \in (-1,1)$

Sol. $\int x^2 \sin^{-1}x\,dx$

$$= (\sin^{-1}x)\frac{x^3}{3} - \frac{1}{3}\int x^3\left(\frac{1}{\sqrt{1-x^2}}\right)dx$$

$$= \frac{x^3}{3}\sin^{-1}x - \frac{1}{3}\int \frac{x[1-(1-x^2)]}{\sqrt{1-x^2}}dx$$

$$= \frac{x^3}{3}\sin^{-1}x - \frac{1}{3}\int \frac{xdx}{\sqrt{1-x^2}} + \frac{1}{3}\int x\sqrt{1-x^2}dx$$

$$= \frac{x^3}{3}\sin^{-1}x + \frac{1}{3}\sqrt{1-x^2} + \frac{1}{3}\frac{(1-x^2)^{3/2}}{(3/2)(-2)} + C$$

$$= \frac{x^3}{3}\sin^{-1}x + \frac{\sqrt{1-x^2}}{3} - \frac{1}{9}(1-x^2)^{3/2} + C$$

6. $\int x\log(1+x)dx, x \in (-1,\infty)$

Sol. $\int x\log(1+x)dx$

$$= \log(1+x)\left(\frac{x^2}{2}\right) - \frac{1}{2}\int \frac{x^2}{1+x}dx$$

$$= \frac{x^2}{2}\log(1+x) - \frac{1}{2}\int \frac{1-(1-x^2)}{1+x}dx$$

$$= \frac{x^2}{2}\log(1+x) - \frac{1}{2}\int\frac{dx}{1+x} + \frac{1}{2}\int(1-x)dx$$

$$= \frac{x^2}{2}\log(1+x) - \frac{1}{2}\log(1+x) + \frac{1}{2}\left(x - \frac{x^2}{2}\right) + C$$

$$= \frac{(x^2-1)}{2}\log(1+x) + \frac{x}{2} - \frac{x^2}{4} + C$$

7. $\int \sin\sqrt{x}\,dx$ **on (0, ∞).**

Sol. **put** $x = t^2 \Rightarrow dx = 2t\,dt$

$$= 2\left[t(-\cos t) + \int\cos t\,dt\right]$$

$$= -2t\cos t + 2\sin t$$

$$= -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C$$

8. $\int e^{ax}\sin(bx+c)dx$ **, (a, b, c ∈ R, b ≠ 0) on R.**

Sol.

Let $I = \int e^{ax}\sin(bx+c)dx$

$$= e^{ax}\left(-\frac{\cos(bx+c)}{b}\right) + \frac{1}{b}\int\cos(bx+c)e^{ax}a\,dx$$

$$= -\frac{e^{ax}\cdot\cos(bx+c)}{b} + \frac{a}{b}\int e^{ax}\cos(bx+c)dx$$

$$= -\frac{e^{ax}\cdot\cos(bx+c)}{b} + \frac{a}{b}\left(e^{ax}\cdot\sin\frac{bx+c}{b}\right) - \frac{1}{b}\int\sin(bx+c)e^{ax}\cdot a\cdot dx$$

$$= -\frac{e^{ax}\cdot\cos(bx+c)}{b} + \frac{a}{b^2}e^{ax}\sin(bx+c) - \frac{a^2}{b^2}I$$

$$\left(1+\frac{a^2}{b^2}\right)I = -\frac{e^{ax}}{b}\cos(bx+c) + \frac{a}{b^2}e^{ax}\sin(bx+c)$$

$$\frac{a^2+b^2}{b^2}I = \frac{e^{ax}}{b^2}[a\sin(bx+c) - b(\cos(bx+c)]$$

$$\therefore I = \frac{e^{ax}}{a^2+b^2}[a\sin(bx+c) - b(\cos(bx+c)] + C_1$$

9. $\int a^x \cos 2x\,dx$ **on R(a > 0 and a ≠ 1).**

Sol. $\int a^x \cos 2x\,dx$

$$= a^x \frac{\sin 2x}{2} - \frac{1}{2}\int \sin 2x \cdot a^x \log a\,dx$$

$$= \frac{a^x \cdot \sin 2x}{2} + \frac{\log a}{2}\int a^x(-\sin 2x)dx$$

$$= \frac{a^x \sin 2x}{2} + \frac{\log a}{2}\left(a^x \cdot \cos\frac{2x}{2} - \frac{1}{2}\int \cos 2x \cdot a^x \log a\,dx\right)$$

$$= \frac{a^x \sin 2x}{2} + \frac{a^x \log a \cos 2x}{4} - \frac{(\log a)^2}{4} I$$

$$\left(1 + \frac{(\log a)^2}{4}\right) I = \frac{a^x[2\sin 2x + (\log a)\cos 2x]}{4}$$

$$\frac{4 + (\log a)^2}{4} I = \frac{a^x[2\sin 2x + (\cos 2x)\log a]}{4}$$

$$\therefore I = \frac{2 \cdot a^x \cdot \sin 2x + (a^x \cdot \log a)\cos 2x}{(\log a)^2 + 4} + c$$

10. $\int \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)dx$ **on** $I \subset R \setminus \left\{-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\}$**.**

Sol. Put x = tan t ⇒ dx = sec² t dt

Then $\int \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)dx$

$$= \int \tan^{-1}\left(\frac{3\tan t - \tan^3 t}{1 - 3\tan^2 t}\right)\sec^2 t\,dt$$

$$= \int \tan^{-1}(\tan 3t)\sec^2 t\,dt = 3\int t \sec^2 t\,dt$$

$$= 3\left[t\int \sec^2 t\,dt - \int\left\{\frac{d}{dt}(t)\int \sec^2 t\,dt\right\}dt\right]$$

$$= 3[t(\tan t) - \int (1)\tan t\,dt]$$

$$= 3(t\tan t - \log|\sec t|) + C$$

$$= 3\left(x \cdot \tan^{-1} x - \log\sqrt{1 + x^2}\right) + C$$

$$= 3x\left[\tan^{-1}x - \frac{3}{2}\log(1+x^2)\right] + C$$

$$= 3x\tan^{-1}(x) - \frac{3}{2}\log(1+x^2) + C$$

11. $\int \sinh^{-1} x\,dx$ **on R.**

Sol. $\int \sinh^{-1} x\,dx = \int 1.\sinh^{-1} x\,dx$

$$= x.\sinh^{-1} x - \int \frac{1}{\sqrt{1+x^2}}.xdx$$

$$= x.\sinh^{-1} x - \frac{1}{2}\int \frac{2x}{\sqrt{1+x^2}}dx$$

$$= x.\sinh^{-1} x - \frac{1}{2}2.\sqrt{1+x^2} + c$$

$$= x.\sinh^{-1} x - \sqrt{1+x^2} + c$$

12. $\int \cosh^{-1} xdx$ **on [1, ∞].**

Sol. $\int \cosh^{-1} xdx = \int 1.\cosh^{-1} xdx$

Apply integration by parts.

Ans: $x\cosh^{-1} x - \sqrt{x^2-1} + C$

13. $\int \tanh^{-1} x\,dx$ **on (–1, 1).**

Sol. $\int \tanh^{-1} x\,dx = \int 1.\tanh^{-1} x\,dx$

$$= \int 1.\tanh^{-1} x\,dx$$

$$= x.\tanh^{-1} x - \int \frac{1}{1-x^2}xdx$$

$$= x.\tanh^{-1} x + \frac{1}{2}\int \frac{-2x}{1-x^2}dx$$

$$= x.\tanh^{-1} x + \frac{1}{2}\log\left(1-x^2\right) + c$$

INTEGRATION OF SOME SPECIAL TYPES OF FUNCTIONS

Type I :

If the integral is of the form $\int \frac{1}{ax^2+bx+c}dx$, where ax^2+bx+c is irreducible quadratic factor then reduce it in the form of $\int \frac{1}{X^2 \pm A^2}dx$ or $\int \frac{1}{A^2 \pm X^2}dx$ and then evaluate.

Type II : If the integral is of the form $\int \frac{1}{\sqrt{ax^2+bx+c}}dx$. Then reduce it in the form $\int \frac{1}{\sqrt{X^2 \pm A^2}}dx$ or $\int \frac{1}{\sqrt{A^2 \pm X^2}}dx$ and then evaluate.

Type III : If the integral is of the form $\int \sqrt{ax^2+bx+c}\,dx$. Then reduce it in the from of $\int \sqrt{A^2 \pm X^2}dx$ or $\int \sqrt{X^2 \pm A^2}dx$ and then evaluate

Type IV :

If the integral is of the form $\int \frac{px+q}{ax^2+bx+c}dx$ then take

$px + q = A\frac{d}{dx}(ax^2+bx+c)+B$.

Type V : If the integral is of the form $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$. Then take

$px + q = A\frac{d}{dx}(ax^2+bx+c)+B$.

Type VI : If the integral is of the form $\int (px+q)\sqrt{ax^2+bx+c}\,dx$. Then take

$px + q = A\frac{d}{dx}(ax^2+bx+c)+B$.

Type VII : If the integral is of the form $\int \frac{1}{(px+q)\sqrt{ax^2+bx+c}}dx$. Then to evaluate this put $px + q = \frac{1}{t}$.

Type VIII : If the integral is of the form $\int \frac{1}{(ax^2+b)\sqrt{cx^2+d}}dx$. Then to evaluate this, put $x = \frac{1}{t}$.

Type IX : If the integral is of the form $\int \frac{px+q}{\sqrt{ax+b}}dx$ or $\int \frac{\sqrt{ax+b}}{px+q}dx$ or $\int (px+q)\sqrt{ax+b}\,dx$ or $\int \frac{1}{(px+q)\sqrt{ax+b}}dx$. Then Put ax + b = t^2. Then $dx = \frac{1}{a}2t\,dt$.

INTEGRATION OF FUNCTIONS WHICH ARE RATIONAL IN sin x AND cos x.

Type X :

If the integral is of the form $\int \frac{dx}{a^2\cos^2 x + b^2 \sin^2 x}$ or $\int \frac{dx}{a\cos^2 x + b\sin x\cos x + c\sin^2 x}$ then multiply both numerator and denominator with $\sec^2 x$ and take tan x = t.

Type XI : If the integral is of the form $\int \frac{dx}{a+b\cos x}$ or $\int \frac{dx}{a+b\sin x}$ or $\int \frac{dx}{a\cos x + b\sin x + c}$, take $\tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}dx = dt \Rightarrow (1+\tan^2 x/2)dx = 2dt \Rightarrow dx = \frac{2dt}{1+t^2}$. $\sin x = \frac{2\tan x/2}{1+\tan^2 x/2} = \frac{2t}{1+t^2}$, $\cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2} = \frac{1-t^2}{1+t^2}$.

Type XII : If the integral is of the form $\int \frac{a\cos x + b\sin x}{c\cos x + d\sin x}dx$, take a cos x + b sin x = A $\frac{d}{dx}$(c cos x + d sin x)+B(c cos x + d sin x).

By equating the coefficients of cos x, sin x we get the values of A and B. Then the given integral becomes A log | c cos x + d sin x | + Bx + k.

Type XIII : If the integral is of the form $\int \frac{a\cos x + b\sin x + c}{c\cos x + d\sin x + d}dx$, take

a cos x + b sin x +c = A $\frac{d}{dx}$(c cos x + d sin x) +B(c cos x + d sin x) +C.

By equating the coefficients of cos x, sin x we get the values of A and B. Then the given integral becomes A log | c cos x + d sin x | + Bx + k.