Chapter 8

INTEGRATION

TOPICS:

1. **Integration as reverse process of differentiation.**
2. **standard forms and properties of integrals.**
3. **Integration by the method of substitution.**
4. **Integration by parts.**
5. **Integration of some standard functions.**
6. **Special types of integration.**
7. **Integration of rational functions by using partial fractions.**
8. **Reduction formulae.**

INDEFINITE INTEGRATION

DEFINITION

If f(x) and g(x) are two functions such that $f'(x) = g(x)$ then f(x) is called antiaderivative or primitive of g(x) with respect to x.

Note

If f(x) is an antiderivative of g(x) then f(x) + c is also an ant derivative of g(x) for all $c \in R$.

DEFINITION

If F(x) is an antiderivative of f(x) then F(x) + c, $c \in R$ is called indefinite integral of f(x) with respect to x. It is denoted by $\int f(x)\,dx$. The real number c is called constant of integration.

Note

1. The integral of a function need not exists. If a function f(x) has integral then f(x) is called an integrable function.

2. The process of finding the integral of a function is known as Integration.

3.The integration is the reverse process of differentiation.

COROLLARY

If f(x), g(x) are two integrable functions then $\int (f \pm g)(x)\,dx = \int f(x)dx \pm \int g(x)dx$

COROLLARY

If $f_1(x)$, $f_2(x)$, …,$f_n(x)$ are integrable functions then $\int (f_1 + f_2 + ... + f_n)(x)dx =$

$\int f_1(x)dx + \int f_2(x)dx + ... + \int f_n(x)dx$.

COROLLARY

If f(x), g(x) are two integrable functions and k, l are two real numbers then

$\int (kf + lg)(x)dx = k\int f(x)dx + l\int g(x)dx$.

{ STANDARD FORMS }

1. If $K \in R$, $\int K dx = Kx + C$.

2. If $n \neq -1$, $\int x^n dx = \frac{x^{n+1}}{n+1} + C$

3. $\int \frac{1}{x} dx = \log |x| + C$

4. $\int \frac{1}{\sqrt{x}} dx = 2\sqrt{x} + C.$

5. $\int e^x dx = e^x + C.$

6. $\int a^x dx = \frac{a^x}{\log a} + C$ (for $a > 0, a \neq 1$)

7. $\int X^{1/2} dx = \frac{2}{3} x\sqrt{x} + C$

8. $\int \sin x \, dx = -\text{Cos } x + C$

9. $\int \cos x \, dx = \text{Sin } x + C.$

10. $\int \tan x \, dx = \log |\sec x| + C$ (or) $-\log |\text{Cos } x| + C$

11. $\int \cot x \, dx = \log |\text{Sin } x| + C$ (or) $-\log |\text{Cosec } x| + C.$

12. $\int \sec x \, dx = \log |\text{Sec } x + \text{Tan } x| + C$ (or) $= \log \left| \text{Tan}\left(\frac{\pi}{4} + \frac{x}{2}\right) \right| + C.$

13. $\int \text{cosec } x \, dx = \log |\text{Cosec } x - \text{Cot } x| + C$

$= \log \left| \text{Tan} \frac{x}{2} \right| + C$

$= -\log |\text{Cosec } x + \cot x| + C.$

14. $\int \sec x \text{ Tan } x \, dx = \text{Sec } x + C$

15. $\int \text{cosec } x \cot x \, dx = -\text{Cosec } x + C$

16. $\int \sec^2 x \, dx = \text{Tan } x + C$

17. $\int \text{cosec}^2 x \, dx = -\text{Cot } x + C.$

18. $\int \sinh x \, dx = \text{Cosh } x + C.$

19. $\int \cosh x \, dx = \text{Sinh } x + C$

20. $\int \tanh x \, dx = \log |\text{Cosh } x| + C$

21. $\int \coth x \, dx = \log |\text{Sinh } x| + C$

22. $\int \text{sech } x \, dx = 2\text{Tan}^{-1}(e^x) + C$

23. $\int \text{cosech } x \, dx = \log \left| \tanh \frac{x}{2} \right| + C$

24. $\int \text{sech}^2 x \, dx = \text{Tanh } x + C$

25. $\int \text{cosech}^2 x \, dx = -\text{Coth } x + C$

26. $\int \text{sech } x \tanh x \, dx = -\text{Sech } x + C$

27. $\int \text{cosech } x \coth x \, dx = -\text{Cosech } x + C$

28. $\int \frac{dx}{\sqrt{1-x^2}} = \text{Sin}^{-1} x + C \text{ (or) } -\text{Cos}^{-1} x + C.$

29. $\int \frac{dx}{1+x^2} = \text{Tan}^{-1} x + C \text{ (or) } -\text{Cot}^{-1} x + C$

30. $\int \frac{dx}{x\sqrt{1-x^2}} = \text{Sec}^{-1} x + C \text{ (or) } - \text{Cosec}^{-1} x + C$

31. $\int \frac{1}{\sqrt{1+x^2}} dx = \text{Sinh}^{-1} x + C$

32. $\int \frac{1}{\sqrt{x^2-1}} dx = \text{Cosh}^{-1} x + C.$

EXERCISE - 8(A)

I. Evaluate the following integrals.

1. $\int(x^3-2x^2+3)dx$

Sol. $\int(x^3-2x^2+3)dx=\int x^3dx-\int 2x^2dx+3\int dx=\frac{x^4}{4}-\frac{2}{3}x^3+3x+c$

2. $\int 2x\sqrt{x}\,dx$

Sol. $\int 2x\sqrt{x}\,dx=2\int x^{3/2}dx=\frac{2x^{5/2}}{(5/2)}+c=\frac{4}{5}x^{5/2}+c$

3. $\int\sqrt[3]{2x^2}\,dx$

Sol. $\int\sqrt[3]{2x^2}\,dx=\int 2^{1/3}\cdot x^{2/3}\,dx=2^{\frac{1}{3}}\frac{3}{5}x^{\frac{5}{3}}+c$

4. $\int\frac{x^2+3x-1}{2x}dx, x\in I\subset R\setminus\{0\}$

Sol. $\int\frac{x^2+3x-1}{2x}dx=\int\left(\frac{x^2}{2x}+\frac{3}{2}-\frac{1}{2x}\right)dx$

$=\int\frac{x}{2}dx+\frac{3}{2}\int dx-\frac{1}{2}\int\frac{1}{x}dx=\frac{x^2}{4}+\frac{3}{2}x-\frac{1}{2}\log|x|+c$

5. $\int\frac{1-\sqrt{x}}{x}dx$ **on (0, ∞)**

Sol. $\int\frac{1-\sqrt{x}}{x}dx=\int\frac{dx}{x}-\int\frac{\sqrt{x}}{x}dx$

$=\log|x|-\frac{x^{-\frac{1}{2}+1}}{(1/2)}+c=\log|x|-2\sqrt{x}+c$

6. $\int\left(1+\frac{2}{x}-\frac{3}{x^2}\right)$ **dx on I ⊂ R \ {0}.**

Sol. $\int\left(1+\frac{2}{x}-\frac{3}{x^2}\right)dx=\int dx+2\int\frac{dx}{x}-3\int x^{-2}dx=x+2\log|x|+\frac{3}{x}+c$

7. $\int\left(x+\frac{4}{1+x^2}\right)dx$ **on R.**

Sol. $\int\left(x+\frac{4}{1+x^2}\right)dx=\int xdx+4\int\frac{1}{1+x^2}dx$

$$= \frac{x^2}{2} + 4\tan^{-1} x + c$$

8. $\int\left(e^x - \frac{1}{x} + \frac{2}{\sqrt{x^2-1}}\right)dx$ **.**

Sol. $\int\left(e^x - \frac{1}{x} + \frac{2}{\sqrt{x^2-1}}\right)dx$

$$= \int e^x dx - \int\frac{1}{x}dx + 2\int\frac{1}{\sqrt{x^2-1}}dx$$

$$= e^x - \log|x| + 2\log|x + \sqrt{x^2-1}| + c$$

9. $\int\left(\frac{1}{1-x^2} + \frac{1}{1+x^2}\right)dx$

Sol. $\int\left(\frac{1}{1-x^2} + \frac{1}{1+x^2}\right)dx = \int\frac{1}{1-x^2}dx + \int\frac{1}{1+x^2}dx$

$$= \tanh^{-1} x + \tan^{-1} x + c$$

10. $\int\left(\frac{1}{\sqrt{1-x^2}} + \frac{2}{\sqrt{1+x^2}}\right)dx$ **on (–1, 1).**

Sol. $\int\left(\frac{1}{\sqrt{1-x^2}} + \frac{2}{\sqrt{1+x^2}}\right)dx = \int\frac{1}{\sqrt{1-x^2}}dx + 2\int\frac{1}{\sqrt{1+x^2}}dx$

$$= \sin^{-1} x + 2\sinh^{-1} x + c$$

II. Evaluate the following integrals.

1. $\int(1-x^2)^3 dx$

Sol. $\int(1-x^2)^3 dx = \int(1 - 3x^2 + 3x^4 - x^6)dx = x - x^3 + \frac{3}{5}x^5 - \frac{x^7}{7} + c$

2. $\int\left(\frac{3}{\sqrt{x}} - \frac{2}{x} + \frac{1}{3x^2}\right)dx$

Sol. $\int\left(\frac{3}{\sqrt{x}} - \frac{2}{x} + \frac{1}{3x^2}\right)dx = \quad = 3\int\frac{dx}{\sqrt{x}} - 2\int\frac{dx}{x} + \frac{1}{3}\int x^{-2}dx$

$$= 3(2\sqrt{x}) - 2\log|x| - \frac{1}{3x} + c \quad = 6\sqrt{x} - 2\log|x| - \frac{1}{3x} + c$$

3. $\int\left(\frac{\sqrt{x}+1}{x}\right)^2 dx$

Sol. $\int\left(\frac{\sqrt{x}+1}{x}\right)^2 dx = \int\frac{x+1+2\sqrt{x}}{x^2}dx$

$= \int\frac{x}{x^2}dx + \int\frac{dx}{x^2} + 2\int\frac{\sqrt{x}}{x^2}dx$

$= \int\frac{dx}{x} + \int\frac{dx}{x^2} + 2\int x^{-3/2}dx$

$= \log|x| - \frac{1}{x} + \frac{2x^{-1/2}}{(-1/2)} + c$

$= \log|x| - \frac{1}{x} - \frac{4}{\sqrt{x}} + c$

4. $\int\frac{(3x+1)^2}{2x}dx$

Sol. $\int\frac{(3x+1)^2}{2x}dx = \int\frac{9x^2+6x+1}{2x}dx$

$= \frac{9}{2}\int x dx + 3\int dx + \frac{1}{2}\int\frac{1}{x}dx$

$= \frac{9}{2}\cdot\frac{x^2}{2} + 3x + \frac{1}{2}\log|x| + c = \frac{9}{4}x^2 + 3x + \frac{1}{2}\log|x| + c$

5. $\int\left(\frac{2x-1}{3\sqrt{x}}\right)^2 dx$

Sol. $\int\left(\frac{2x-1}{3\sqrt{x}}\right)^2 dx = \int\frac{4x^2-4x+1}{9x}dx$

$= \frac{4}{9}\int x\,dx - \frac{4}{9}\int dx + \frac{1}{9}\int\frac{dx}{x} = \frac{4}{9}\frac{x^2}{2} - \frac{4}{9}x + \frac{1}{9}\log|x| + c$

$= \frac{4}{18}x^2 - \frac{4}{9}x + \frac{1}{9}\log|x| + c$

6. $\int\left(\frac{1}{\sqrt{x}} + \frac{2}{\sqrt{x^2-1}} - \frac{3}{2x^2}\right)dx$ **on (1, ∞)**

Sol. $\int\left(\frac{1}{\sqrt{x}} + \frac{2}{\sqrt{x^2-1}} - \frac{3}{2x^2}\right)dx = \int\frac{1}{\sqrt{x}}dx + 2\int\frac{1}{\sqrt{x^2-1}}dx + \frac{3}{2}\int\frac{1}{x^2}dx$

$= 2\sqrt{x} + 2\cosh^{-1}x + \frac{3}{2x} + c$

7. $\int(\sec^2 x - \cos x + x^2)dx,$

Sol. $\int(\sec^2 x - \cos x + x^2)dx = \int\sec^2 x dx - \int\cos x\,dx + \int x^2 dx = \tan x - \sin x + \frac{x^3}{3} + c$

8. $\int\left(\sec x \tan x + \frac{3}{x} - 4\right)dx$

Sol. $\int\left(\sec x \tan x + \frac{3}{x} - 4\right)dx$

$= \int\sec x \tan x\,dx + 3\int\frac{dx}{x} - 4\int dx = \sec x + 3\log|x| - 4x + c$

9. $\int\left(\sqrt{x} - \frac{2}{1-x^2}\right)dx$ **on (0, 1).**

Sol. $\int\left(\sqrt{x} - \frac{2}{1-x^2}\right)dx = \int\sqrt{x}\,dx - 2\int\frac{dx}{1-x^2}$

$= \frac{x^{3/2}}{(3/2)} - 2\tanh^{-1}x + c = \frac{2}{3}x\sqrt{x} - 2\tanh^{-1}x + c$

10. $\int\left(x^3 - \cos x + \frac{4}{\sqrt{x^2+1}}\right)dx, x \in R$

Sol. $\int\left(x^3 - \cos x + \frac{4}{\sqrt{x^2+1}}\right)dx$

$= \int x^3 dx - \int\cos x\,dx + 4\int\frac{1}{\sqrt{x^2+1}}dx = \frac{x^4}{4} - \sin x + 4\sinh^{-1}x + c$

11. $\int\left(\cosh x + \frac{1}{\sqrt{x^2+1}}\right)dx, x \in R$

Sol. $\int\left(\cosh x + \frac{1}{\sqrt{x^2+1}}\right)dx$

$= \int\cosh x dx + \int\frac{dx}{\sqrt{x^2+1}}$

$= \sinh x + \sinh^{-1}x + c$

12. $\int\left(\sinh x + \frac{1}{(x^2-1)^{1/2}}\right)dx,$

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Sol. $\int\left(\sinh x + \frac{1}{(x^2-1)^{1/2}}\right)dx$

$$= \int \sinh x dx + \int \frac{dx}{\sqrt{x^2 - 1}}$$

$$= \cosh x + \log(x + \sqrt{x^2 - 1}) + c$$

13. $\int \frac{(a^x - b^x)^2}{a^x b^x} dx$

Sol. $\int \frac{(a^x - b^x)^2}{a^x b^x} dx = \int \frac{a^{2x} + b^{2x} - 2a^x b^x}{a^x \cdot b^x} dx$

$$= \int \frac{a^{2x}}{a^x \cdot b^x} dx + \int \frac{b^{2x}}{a^x \cdot b^x} dx - 2\int \frac{a^x b^x}{a^x \cdot b^x} dx = \int \left(\frac{a}{b}\right)^x dx + \int \left(\frac{b}{x}\right)^2 dx - 2\int dx$$

$$= \frac{(a/b)^x}{\log(a/b)} + \frac{(b/a)^x}{\log(b/a)} - 2x + c = \frac{1}{(\log a - \log b)}\left[\left(\frac{a}{b}\right)^x - \left(\frac{b}{a}\right)^x\right] - 2x + c$$

14. $\int \sec^2 x \csc^2 x \, dx$ **.**

Sol. $\int \sec^2 x \csc^2 x \, dx = \int \frac{1}{\cos^2 x \sin^2 x} dx$

$$= \int \frac{\sin^2 x + \cos^2 x}{\cos^2 x \cdot \sin^2 x} dx = \int \frac{1}{\cos^2 x} dx + \int \frac{1}{\sin^2 x} dx = \int \sec^2 x \, dx + \int \csc^2 x \, dx = \tan x - \cot x + C$$

15. $\int \frac{1 + \cos^2 x}{1 - \cos 2x} dx$ **.**

Sol. $\int \frac{1 + \cos^2 x}{1 - \cos 2x} dx = \int \frac{1 + \cos^2 x}{2 \sin^2 x} dx$

$$= \frac{1}{2}\int \frac{1}{\sin^2 x} dx + \frac{1}{2}\int \cot^2 x \, dx$$

$$= \frac{1}{2}\int \operatorname{cosec}^2 x \, dx + \frac{1}{2}\int (\csc^2 x - 1) dx$$

$$= \int \csc^2 x \, dx - \frac{1}{2}\int dx = -\cot x - \frac{x}{2} + C$$

16. $\int \sqrt{1 - \cos 2x} \, dx$

Sol. $\int \sqrt{1 - \cos 2x} \, dx = \int \sqrt{2 \sin^2 x} dx$

$$= \int \sqrt{2} \sin x dx = -\sqrt{2} \cos x + C$$

17. $\int \frac{1}{\cosh x + \sinh x} dx$ **on R.**

Sol. $\int \frac{1}{\cosh x + \sinh x} dx = \int \frac{(\cosh x - \sinh x)}{(\cosh x + \sinh x)(\cosh x - \sinh x)} dx = \int \frac{\cosh x - \sinh x}{\cosh^2 x - \sinh^2 x} dx$

$= \int (\cosh x - \sinh x) dx$

$= \sinh x - \cosh x + C$

18. $\int \frac{1}{1+\cos x} dx$ **on R**

Sol. $\int \frac{1}{1+\cos x} dx = \int \frac{1-\cos x}{(1+\cos x)(1-\cos x)} dx$

$= \int \left(\frac{1-\cos x}{1-\cos^2 x} \right) dx$

$= \int \left(\frac{1}{\sin^2 x} - \frac{\cos x}{\sin^2 x} \right) dx$

$= \int \csc^2(x) dx - \int \csc x \cot x dx$

$= -\cot x + \csc x + C$

INTEGRATION BY SUBSTITUTION THEOREM

If $\int f(x)dx = g(x)$ **and a ≠ 0 then** $\int f(ax+b)dx = \frac{1}{a} g(ax+b) + c$ **.**

Proof :

Put ax + b = t.

Then $\frac{d}{dx}(ax+b) = \frac{dt}{dx} \Rightarrow a \cdot dx = dt \Rightarrow dx = \frac{1}{a} dt$

$\therefore \int f(ax+b)dx = \int f(t) \cdot \frac{1}{a} dt$

$= \frac{1}{a} \int f(t)dt = \frac{1}{a} g(t) + c = \frac{1}{a} g(ax+b) + c$

e.g. $\int (ax+b)^n dx = \frac{1}{a} \frac{(ax+b)^{n+1}}{n+1} + c, (n \neq -1)$

THEOREM

If f(x) is a differentiable function then $\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + c$ **.**

Proof :

Put $f(x) = t \Rightarrow f'(x) = \frac{dt}{dx} \Rightarrow f'(x)dx = dt$

$\therefore \int \frac{f'(x)}{f(x)}dx = \int \frac{1}{t}dt = \log|t| + c = \log|f(x)| + c$

THEOREM

$\int \tan x\,dx = \log|\sec x| + c$ **for** $x \neq (2n+1)\frac{\pi}{2}, n \in Z$ **.**

Proof :

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}dx = -\int \frac{d(\cos x)}{\cos x}dx$$

$$= -\log|\cos x| + c = \log\frac{1}{|\cos x|} + c = \log|\sec x| + c$$

THEOREM

$\int \cot x\,dx = \log|\sin x| + c$ **for x ≠ nπ, n ∈ Z.**

Proof :

$$\int \cot x\,dx = \int \frac{\cos x}{\sin x}dx = \log|\sin x| + c$$

THEOREM

$\int \sec x\,dx = \log|\sec x + \tan x| + c = \log|\tan(\pi/4 + x/2)| + c$ **for** $x \neq (2n+1)\frac{\pi}{2}, n \in Z$ **.**

Proof :

$$\int \sec x\,dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}dx = \log|\sec x + \tan x| + c$$

$$= \log\left|\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right| + c = \log\left|\frac{1+\sin x}{\cos x}\right| + c$$

$$= \log\left|\frac{1-\cos(\pi/2 + x)}{\sin(\pi/2 + x)}\right| + c$$

$$= \log\left|\frac{2\sin^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}{2\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}\right| + c$$

$$= \log\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right| + c$$

THEOREM

$\int \csc x\,dx = \log|\csc x - \cot x| + c = \log|\tan x/2| + c$ **for** $x \neq n\pi, n \in Z$.

Proof :

$$\int \csc x\,dx = \int \frac{\csc x(\csc x - \cot x)}{\csc x - \cot x}dx$$

$$= \int \frac{\csc^2 x - \csc x \cot x}{\csc x - \cot x}dx = \log|\csc x - \cot x| + c$$

$$= \log\left|\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right| + c = \log\left|\frac{1-\cos x}{\sin x}\right| + c$$

$$= \log\left|\frac{2\sin^2 x/2}{2\sin x/2\cos x/2}\right| + c = \log|\tan x/2| + c$$

THEOREM

If f(x) is differentiable function and n ≠ –1 then $\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + c$.

Proof :

Put $f(x) = t \Rightarrow f'(x)\,dx = dt$

$$\therefore \int [f(x)]^n f'(x)dx = \int t^n dt = \frac{t^{n+1}}{n+1} + c = \frac{[f(x)]^{n+1}}{n+1} + c$$

Note : $\int \frac{f'(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)} + c$

THEOREM

If $\int f(x)dx = F(x)$ and g(x) is a differentiable function then $\int (fog)(x)g'(x)dx = F[g(x)] + c$.

Proof :

$g(x) = t \Rightarrow g'(x)\,dx = dt$

$\therefore \quad \int (fog)(x)g'(x)dx = \int f[g(x)]g'(x)dx$

$$= \int f(t)dt = F(t) + c = F[g(x)] + c$$

INTEGRATION OF SOME STANDARD FUNCTIONS THEOREM

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \text{Sin}^{-1}\left(\frac{x}{a}\right) + c \quad \textbf{for } x \in \textbf{(–a, a).}$$

Proof :

Put $x = a \sin \theta$. Then $dx = a \cos \theta \, d\theta$

$$\therefore \int \frac{1}{\sqrt{a^2 - x^2}} dx = \int \frac{1}{\sqrt{a^2 - a^2 \sin^2 \theta}} a \cos \theta \, d\theta$$

$$= \int \frac{1}{a\sqrt{1 - \sin^2 \theta}} a \cos \theta d\theta = \int \frac{1}{\cos \theta} \cos \theta d\theta$$

$$= \int d\theta = \theta + c = \text{Sin}^{-1}\left(\frac{x}{a}\right) + c$$

THEOREM

$$\int \frac{1}{\sqrt{a^2 + x^2}} dx = \text{Sinh}^{-1}\left(\frac{x}{a}\right) + c \quad \textbf{for } x \in \textbf{R.}$$

Proof :

Put $x = a \sinh \theta$. Then $dx = a \cos h \, \theta \, d\theta$

$$\therefore \int \frac{1}{\sqrt{a^2 + x^2}} dx = \int \frac{1}{\sqrt{a^2 + a^2 \sinh^2 \theta}} a \cosh \theta \, d\theta$$

$$= \int \frac{a \cosh \theta}{a \cosh \theta} = \int d\theta = \theta + c = \text{Sinh}^{-1}\left(\frac{x}{a}\right) + c$$

THEOREM

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \text{Cosh}^{-1}\left(\frac{x}{a}\right) + c \quad \textbf{for } x \in \textbf{(–}\infty\textbf{, –a)} \cup \textbf{(a, }\infty\textbf{).}$$

Proof :

Put $x = a \cos h \, \theta$. Then $dx = a \sinh \theta \, d\theta$

$$\therefore \int \frac{1}{\sqrt{x^2 - a^2}} dx = \int \frac{1}{\sqrt{a^2 \cosh^2 \theta - a^2}} a \sinh \theta \, d\theta$$

$$= \int \frac{a \sinh \theta}{a \sinh \theta} d\theta = \int d\theta = \theta + c = \text{Cosh}^{-1}\left(\frac{x}{a}\right) + c$$

THEOREM

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\text{Tan}^{-1}\left(\frac{x}{a}\right)+c \text{ for } x \in R .$$

Proof :

Put $x = a \tan\theta$. Then $dx = a \sec^2\theta \, d\theta$

$$\therefore \int \frac{1}{a^2+x^2}dx = \int \frac{1}{a^2+a^2\tan^2\theta} a\sec^2\theta \, d\theta$$

$$= \int \frac{1}{a^2(1+\tan^2\theta)} a\sec^2\theta d\theta = \frac{1}{a}\int \frac{\sec^2\theta}{\sec^2\theta} d\theta$$

$$= \frac{1}{a}\int d\theta = \frac{1}{a}\theta + c = \frac{1}{a}\text{Tan}^{-1}\left(\frac{x}{a}\right)+c$$

THEOREM

$$\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+c \text{ for } x \neq \pm a$$

Proof :

$$\int \frac{1}{a^2-x^2}dx = \int \frac{1}{(a+x)(a-x)}dx$$

$$= \frac{1}{2a}\int\left(\frac{1}{a+x}+\frac{1}{a-x}\right)dx = \frac{1}{2a}\left[\log|a+x|-\log|a-x|\right]+c$$

$$= \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+c$$

THEOREM

$$\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c \text{ for } x \neq \pm a$$

Proof :

$$\int \frac{1}{x^2-a^2}dx = \int \frac{1}{(x-a)(x+a)}dx$$

$$= \frac{1}{2a}\int\left(\frac{1}{x-a}-\frac{1}{x+a}\right)dx = \frac{1}{2a}\left[\log|x-a|-\log|x+a|\right]+c$$

$$= \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c$$

THEOREM

$$\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+c$$ **for**

$x \in (-a, a)$.

Proof :

Put $x = a \sin\theta$. Then $dx = a\cos\theta\, d\theta$

$$\therefore \int\sqrt{a^2-x^2}\,dx = \int\sqrt{a^2-a^2\sin^2\theta}\,a\cos\theta\,d\theta$$

$$= \int a\sqrt{1-\sin^2\theta}\,a\cos\theta d\theta = a^2\int\cos^2\theta\,d\theta$$

$$= a^2\int\frac{1+\cos 2\theta}{2}d\theta = \frac{a^2}{2}\left[\theta+\frac{1}{2}\sin 2\theta\right]+c$$

$$= \frac{a^2}{2}\left[\theta+\frac{1}{2}2\sin\theta\cos\theta\right]+c = \frac{a^2}{2}\left[\theta+\sin\theta\sqrt{1-\sin^2\theta}\right]$$

$$= \frac{a^2}{2}\left[\mathrm{Sin}^{-1}\left(\frac{x}{a}\right)+\frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}\right]+c$$

$$= \frac{a^2}{2}\mathrm{Sin}^{-1}\left(\frac{x}{a}\right)+\frac{x}{2}\sqrt{a^2-x^2}+c$$

THEOREM

$$\int\sqrt{a^2+x^2}\,dx=\frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\mathrm{Sinh}^{-1}\left(\frac{x}{a}\right)+c$$ **for** $x \in R$ **.**

Proof :

Put $x = \sinh\theta$. Then $dx = a\cosh\theta\, d\theta$

$$\therefore \int\sqrt{a^2+x^2}\,dx = \int\sqrt{a^2+a^2\sinh^2\theta}\,a\cosh\theta d\theta$$

$$= \int a\sqrt{1+\sinh^2\theta}\,a\cosh\theta d\theta = a^2\int\cosh^2\theta d\theta$$

$$= a^2\int\frac{1+\cosh 2\theta}{2}d\theta = \frac{a^2}{2}\left[\theta+\frac{1}{2}\sinh 2\theta\right]+c$$

$$= \frac{a^2}{2}\left[\theta+\frac{1}{2}2\sinh\theta\cosh\theta\right]+c$$

$$= \frac{a^2}{2}\left[\theta+\sinh\theta\sqrt{1+\sinh^2\theta}\right]+c$$

$$= \frac{a^2}{2}\left[\text{Sinh}^{-1}\left(\frac{x}{a}\right) + \frac{x}{a}\sqrt{1+\frac{x^2}{a^2}}\right] + c$$

$$= \frac{a^2}{2}\text{Sinh}^{-1}\left(\frac{x}{a}\right) + \frac{x}{a}\sqrt{a^2+x^2} + c$$

THEOREM

$$\int\sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\text{Cosh}^{-1}\left(\frac{x}{a}\right) + c \text{ for } x \in [a, \infty).$$

Proof :

Put $x = a\cosh\theta$. Then $dx = a\sinh\theta\, d\theta$

$$\therefore \int\sqrt{x^2-a^2}\,dx = \int\sqrt{a^2\cosh^2\theta - a^2}\, a\sinh\theta\, d\theta$$

$$= \int a\sqrt{\cosh^2\theta - 1}\, a\sinh\theta\, d\theta = a^2\int\sinh^2\theta\, d\theta$$

$$= a^2\int\frac{\cosh 2\theta - 1}{2}\,d\theta = \frac{a^2}{2}\left[\frac{1}{2}\sinh 2\theta - \theta\right] + c$$

$$= \frac{a^2}{2}\left[\frac{1}{2}2\sinh\theta\cosh\theta - \theta\right] + c$$

$$= \frac{a^2}{c}\left[\cosh\theta\sqrt{\cosh^2\theta - 1} - \theta\right] + c$$

$$= \frac{a^2}{2}\left[\frac{x}{a}\sqrt{\frac{x^2}{a^2}-1} - \text{Cosh}^{-1}\left(\frac{x}{a}\right)\right] + c$$

$$= \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\text{Cosh}^{-1}\left(\frac{x}{a}\right) + c$$