SUCCESSIVE DIFFERENTIATION -2

EXERCISE -- 7 (B)

I.

1. Find $D^3\left(\sin x.\log x\right)$ **when** $x=\frac{\pi}{2}$

Sol: $y=\sin x.\log x$

$D^3\sin x.\log x=3_{c_0}D^3\sin x.\log x+3_{c_1}D^2\sin x.D\log x+3_{c_2}D\sin x.D^2\log x+3_{c_3}\sin x.D^3\log x.$

$=1.\sin\left(x+3\frac{\pi}{2}\right).\log x-3\sin x.\frac{1}{x}+3\cos x.\frac{-1}{x^2}+1.\sin x.\frac{2}{x^3}$

$=-\cos x.\log x-3\sin x.\frac{1}{x}-3\cos x.\frac{1}{x^2}+\sin x.\frac{2}{x^3}$

$\left(D^3\sin x.\log x\right)_{at x=\frac{\pi}{2}}=0-3.1.\frac{2}{\pi}-0+1.\frac{16}{\pi^3}=\frac{16}{\pi^3}-\frac{6}{\pi}$

2. Find the sum of the middle terms is Leibnitz expansion of $D^3\left(e^x\cos x\right)$

Sol: $D^3\left(e^x\cos x\right)=3c_0\left(e^x\right)_3.\cos x+3c_1\left(e^x\right)_2.D\cos x+3c_2\left(e^x\right)_1D^2cosx+3c_3e^xD^3cosx=0$

Middle terms are 2nd and 3rd terms.

Sum of the middle terms $=3c_1\left(e^x\right)_2.D\cos x+3c_2\left(e^x\right)_1D^2cosx$

$=3\,e^x\left(-\sin x\right)+3e^x\left(-\cos x\right)$

$=-3e^x\left(\sin x+\cos x\right)$

II.

1. Find the nth derivatives of the following using Leibnitz theorem

i) $e^{2x}.\cos^2 x$ **ii)** $e^{2x}.\sin x.\cos 5x$

iii) $x^3.e^{ax}$ **iv)** $x^2\cos 2x$

Sol: (i) let $y=e^{2x}\cos^2 x=\frac{e^{2x}}{2}\left(1+\cos 2x\right)$

$=\frac{1}{2}\left(e^{2x}+e^{2x}.\cos 2x\right)$

Differentiate n times w.r.t. x,

$y_n=\frac{d^n}{dx^n}\frac{1}{2}\left(e^{2x}+e^{2x}.\cos 2x\right)$

$y_n=\frac{1}{2}\left(2^n.e^{2x}+\left(\sqrt{4+4}\right)^n.e^{2x}\cos\left(2x+n\tan^{-1}1\right)\right)$

$=e^{2x}\left(2^{n-1}+2^{3n/2}.\cos\left(2x+\frac{n\pi}{4}\right)\right)$

ii) $y = e^{2x}.\cos 5x.\sin x$

$= \frac{e^{2x}}{2}(2\cos 5x.\sin x) = \frac{e^{2x}}{2}(\sin 6x - \sin 4x) = \frac{1}{2}\left(e^{2x}\sin 6x - e^{2x}.\sin 4x\right)$

Differentiate n times w.r.t. x,

$y_n = \frac{d^n}{dx^n}\left(\frac{1}{2}\left(e^{2x}\sin 6x - e^{2x}.\sin 4x\right)\right)$

$y_n = \frac{1}{2}\left(\sqrt{4+16}\right)^n e^{2x}\sin\left(6x + n\tan^{-1}3\right) - \left(\sqrt{4+16}\right)^n e^{2x}\sin\left(4x + n\tan^{-1}2\right)$

$= \frac{e^{2x}}{2}(40^{n/2}.\sin\left(6x + n\tan^{-1}3\right) - 20^{n/2}\sin\left(4x + n\tan^{-1}2\right)$

iii) $y = x^3.e^{ax}$

$\Rightarrow y_n = D^n x^3.e^{ax}$

$= nc_o D^n e^{ax}x^3 + nc_1 D^{n-1}e^{ax}.Dx^3 + nc_2 D^{n-2}e^{ax}.D^2x^3 + nc_3 D^{n-3}e^{ax}.D^3x^3 + 0$

$= a^n e^{ax}x^3 + 3nx^2.a^{n-1}.e^{ax} + \frac{6n(n-1)}{2}.x.a^{n-2}.e^{ax}x^3 + 3nx^2.a^{n-1}.e^{ax} +$

$\frac{6n(n-1)}{2}x.a^{n-2}.e^{ax} + \frac{n(n-1)(n-2)}{6}6\ a^{n-3}e^{ax}$

$= e^{ax}\left[a^n x^3 + 3nx^2 a^{n-1} + 3n(n-1)x.a^{n-2} + n(n-1)(n-2)a^{n-3}\right]$

iv) $y = x^2\cos 2x$

$\Rightarrow y_n = D^n x^2.\cos 2x$

$= nc_o D^n \cos 2x.x^2 + nc_1 D^{n-1}\cos 2x.Dx^2 + nc_2 D^{n-2}\cos 2x.D^2x^2 + 0$

$= x^2.2^n\cos\left(2x + n\frac{\pi}{2}\right) + 2nx\cos\left(2x + \overline{n-1}\frac{\pi}{2}\right)2^{n-1} + \frac{n(n-1)}{2}2.\cos\left(2x + \left(\overline{n-2}\right)\frac{\pi}{2}\right)2^{n-2}$

$= 2^{n-1}\left[2x^2\cos\left(2x + \frac{n\pi}{2}\right)\right] + \frac{n(n-1)}{2}\cos\left(2x + (n-2)\frac{\pi}{2}\right)$

v) $e^x.\log 2x$

Sol:

Let $y = e^x.\log 2x$

$y_n = \left[\left(e^x\right)_n.\log 2x\right] + nc_1.\left(e^x\right)_{n-1}.D\log 2x + nc_2.\left(e^x\right)_{n-2}D^2\log 2x + 0 + ... + nc\ .e^x D^n\log 2x$

$\Rightarrow y_n = \left[\left(e^x\right)_n.\log 2x\right] + nc_1.\left(e^x\right)_{n-1}.\frac{1}{2x}.2 + nc_2.\left(e^x\right)_{n-2}\left(\frac{-1}{x^2}\right) + ... + nc_n.e^x\frac{(-1)^{n-1}.(n-1)!}{x^n}$

$= e^x\log 2x + ne^x.\frac{1}{x} + \frac{n(n-1)}{2}.e^x\left(\frac{-1}{x^2}\right) + ... + e^x(-1)^{n-1}.\frac{(n-1)!}{x^n}$

$= e^x\left[\log 2x + \frac{n}{x} + \frac{n(n-1)}{2}\left(\frac{-1}{x^2}\right) + ... + (-1)^{n-1}.\frac{(n-1)!}{x^n}\right]$

2. Find the n^th derivative of $x^2.\cos^2(2x)$ for $n>2$.

Sol: Let $y = x^2.\cos^2(2x) = x^2\dfrac{1+\cos 4x}{2} = \dfrac{1}{2}\left(x^2 + x^2\cos 4x\right)$

Differentiate n times using leitbnitz's theorem,

$$y_n = \frac{1}{2}\left(D^n x^2 + D^n x^2.\cos 4x\right)$$

$$y_n = \frac{1}{2}\left[0 + (\cos 4x)_n.x^2 + {}^nC_1(\cos 4x)_{n-1}D(x^2) + {}^nC_2(\cos 4x)_{n-2}D^2(x^2) + 0\right]$$

$$\Rightarrow y_n = \frac{1}{2}\left[0 + (\cos 4x)_n.x^2 + {}^nC_1(\cos 4x)_{n-1}(2x) + {}^nC_2(\cos 4x)_{n-2}(2)\right]$$

$$\frac{1}{2}\left[x^2.4^n\cos\left(4x+\frac{n\pi}{2}\right) + 2nx.4^{n-1}\cos\left(4x+\frac{(n-1)\pi}{2}\right)\right] + \frac{2n(n-1)}{2}\left[4^{n-2}\cos\left(4x+\frac{(n-2)\pi}{2}\right)\right]$$

$$= 4^{n-2}\left[8x^2\cos\left(4x+\frac{n\pi}{2}\right) + 4nx\cos\left(4x+\frac{(n-1)\pi}{2}\right) + \frac{n(n-1)}{2}\cos\left(4x+\frac{(n-2)\pi}{2}\right)\right]$$

3. Find the n^th derivative of $e^{2x+3}.\sin(2x)$.

Sol: let $u = \sin 2x, v = e^{2x+3}$

Ans: $\displaystyle\sum_{r=0}^{n} {}^nC_r.2^n.\sin\left(2x+\frac{n-r}{2}\pi\right)e^{2x+3}$.

4. If f(x) = $\tan^{-1}(x)$ then show that

$$(1+x^2)f^{n+2}(x) + 2x(n+1)f^{n+1}(x) + n(n+1)f^{(n)}(x) = 0 \forall n \geq 1.$$

Sol: Let $y = \tan^{-1}x$

Then $y_1 = \dfrac{1}{1+x^2} \Rightarrow y_1(1+x^2) = 1$

Diff . w.r.t. x,

$$y_2(1+x^2) + 2xy_1 = 0$$

$$\Rightarrow (1+x^2)y_2 + 2xy_1 = 0$$

Now diff. n times using Leibnitz's theorem,

$$D^n\left[(1+x^2)y_2\right] + 2.D^n[xy_1] = 0$$

$$D^n(y_2)(1+x^2) + {}^nC_1.D^{n-1}(y_2)D(1+x^2) + {}^nC_2.D^{n-2}(y_2).D^2(1+x^2) + 2.\left[D^n(y_1)x + {}^nC_1.D^{n-1}(y_1).Dx + 0\right] = 0$$

$$\Rightarrow D^n(y_2)(1+x^2) + {}^nC_1.D^{n-1}(y_2)$$

$$2x + {}^nC_2.D^{n-2}(y_2).2 + 2.\left[D^n(y_1)x + {}^nC_1.D^{n-1}(y_1)(1) + 0\right] = 0$$

$$\Rightarrow (1+x^2)y_{n+2} + {}^nC_1.y_{n+1}(2x) + {}^nC_2.y_n(2) + 2\left[y_{n+1}x + {}^nC_1.y_n.(1)\right] = 0$$

$$\Rightarrow \left(1+x^2\right) y_{n+2} + 2xn.y_{n+1} + \frac{2n(n-1)}{2} y_n + 2xy_{n+1} + 2ny_n = 0$$

$$\Rightarrow \left(1+x^2\right) y_{n+2} + 2x(n+1) y_{n+1} + \left[n(n-1)+2n\right] y_n = 0$$

$$\Rightarrow \left(1+x^2\right) y_{n+2} + 2x(n+1) y_{n+1} + \left(n^2+n\right) y_n = 0$$

$$\Rightarrow \left(1+x^2\right) y_{n+2} + 2x(n+1) y_{n+1} + (n+1) y_n = 0$$

$$\Rightarrow \left(1+x^2\right) f^{n+2}(x) + 2x(n+1) f^{(n+1)}(x) + n(n+1) f^{n}(x) = 0$$

5. If $y = x^n \log x$ then show that $xy_{n+1} = \underline{\lfloor n}$.

Sol:

$y = x^n \log x$ _____(1)

Diff. w.r.t. x

$$y_1 = x^n \left(\frac{1}{x}\right) + n.x^{n-1} \log x$$

Multiplying with x, we get

$$\Rightarrow xy_1 = x^n + nx^n \log x$$

$$\Rightarrow xy_1 = x^n + ny$$

usisng Leibnitz theorem Diff. n times w.r.t. x ,

$$D^n(xy_1) = D^n x^n + nD^n y$$

$$\Rightarrow nc_o D^n y_1.x + nc_1 D^{n-1} y_1.Dx + 0 = n! + ny_n$$

$$\Rightarrow xy_{n+1} + ny_n = n! + ny_n$$

$$\Rightarrow xy_{n+1} = n!$$

III.

1. If y=$\frac{\log x}{x}(x>0)$ then show that $y_n = \frac{(-1)^n .n!}{x^{n+1}}\left[\log x - 1 - \frac{1}{2} - \frac{1}{3} ..... \frac{1}{n}\right]$

Sol: $y = \frac{\log x}{x} = \frac{1}{x}.\log x$

Diff. n times,

$y_n = D^n \frac{1}{x}.\log x$, let u=$\frac{1}{x}$ and v= logx

$$y_n = \left(\frac{1}{x}\right)_n \log x + nc_1 \left(\frac{1}{x}\right)_{n-1} \frac{1}{x} + nc_2 \left(\frac{1}{x}\right)_{n-2} \left(\frac{1}{x^2}\right) ..... + \frac{1}{x}(\log x)_n$$

$$= \frac{(-1)^n n!}{x^{n+1}}.\log x + \frac{n(-1)^{(n-1)}.(n-1)!}{x^n}.\frac{1}{x} + \frac{n(n-1)}{2}\frac{(-1)^{n-2}(n-2)!}{x^{n-1}}.\frac{1}{x^2} ..... + \frac{1}{x}\frac{(-1)^{n-1}(n-1)!}{x^n}$$

$$= \frac{(-1)^n n!}{x^{n+1}}\left(\log x - 1 - \frac{1}{2} .... - \frac{1}{n}\right).$$

2. **If y=cos** $|m\log x|, x>0$ **, then show that** $x^2y_2+xy_1+m^2y=0$ **and hence deduce that** $x^2y_{n+2}+(2n+1)xy_{n+1}+(m^2+n^2)y_n=0$

Sol: y = cos (m log x)

$y_1=-\sin(m\log x).\frac{m}{x}$

$xy_1=-m.\sin(m\log x)$

Differentiating w.r.to x

$x.y_2+y_1.1=-m\cos(m\log x).\frac{m}{x}$

$x^2y_2+xy_1=-m^2\cos(m\log x)=-m^2y \quad x^2y_2+xy_1+m^2y=0$

Differentiating n times w.r.to x, using Leibnitz's theorem

$(x^2y_2)_n+(xy_1)_n+m^2y_n=0$

$\Rightarrow (y_2)_n.x^2+nC_1(y_2)_{n-1}(x^2)_1+nC_2.(y_2)_{n-2}(x^2)_2+0+\left[(y_1)_n x+nC_1((y_1)_{n-1}.Dx)+m^2(y)_n=0\right.$

$\Rightarrow (y_2)_n.x^2+nC_1(y_2)_{n-1}(2x)+nC_2.(y_2)_{n-2.}2+\left[(y_1)_n x+nC_1(y_{1n-1}1)+m^2y_n=0\right.$

$\Rightarrow x^2.y_{n+2}+2n.xy_{n+1}+\frac{n(n-1)}{2}2.y_n+(n^2-n+n+m^2)y_n=0$

$\Rightarrow x^2y_{n+2}+(2n+1)xy_{n+1}+(m^2+n^2)y_n=0$

3. **If** $y=\frac{\sinh^{-1}x}{\sqrt{1+x^2}}$ **then show that** $(1+x^2)y_2+3xy_1+y=0$ **and hence deduce the** $(1+x^2)y_{n+2}+(2n+3)xy_{n+1}+(n+1)^2y_n=0$

Sol: $y=\frac{\sinh^{-1}x}{\sqrt{1-x^2}}$ -----(1)

$\Rightarrow y\sqrt{1-x^2}=\sinh^{-1}x$

Diff. w.r.t x,

$\Rightarrow \sqrt{1+x^2}.y_1+y.\frac{1}{2\sqrt{1+x^2}}.2x=\frac{1}{\sqrt{1+x^2}}$

$\Rightarrow (1+x^2)y_1+xy=1$------(2)

Diff. w.r.t.x,

$\Rightarrow (1+x^2)y_2+2xy_1+xy_1+y=0$

$\Rightarrow (1+x^2)y_2+3xy_1+y=0$ ------(3)

Differentiating n times w.r.to x

$\left[(1+x^2)y_2\right]_n+3(xy_1)_n y_n=0$

$$\Rightarrow (1+x^2)(y_2)_n + nC_1(y_2)_{n-1}(1+x^2)_1 + nC_1(y_2)_{n-1}(1+x^2)_2 + 0 + 3\left[(y_1)_n.x + nC_1(y_1)_{n-1}.Dx\right] + (y)_n = 0$$

$$\Rightarrow (1+x^2)(y_2)_n + nC_1(y_2)_{n-1}2x + nC_1(y_2)_{n-1}.2 + 3\left[(y_1)_n.x + nC_1(y_1)_{n-1}.1\right] + y_n = 0$$

$$\Rightarrow (1+x^2)y_{n+2} + 2nxy_{n+1} + \frac{n(n-1)}{2}y_n.2 + 3xy_{n+1} + 3.n.y_n + y_n = 0$$

$$\Rightarrow (1+x^2)y_{n+2} + (2n+3)xy_{n+1} + (n+1)^2 y_n = 0.$$

4. **If** $\cos^{-1}\left(\frac{y}{b}\right) = n.\log\left(\frac{x}{n}\right), x > 0$ **, then show that** $x^2y_2 + xy_1 + n^2y = 0$ **, and hence deduce that** $x^2y_{n+2} + (2n+1)xy_{n+1} + 2n^2y_n = 0$ **.**

Sol . Given

$\cos^{-1}\frac{y}{b} = n\log\frac{x}{n} \Rightarrow \cos^{-1}\frac{y}{b} = n(\log x - \log n)$

Differentiating w.r to x ,

$$\frac{1}{\sqrt{1-\frac{y^2}{b^2}}}\frac{1}{b}.y_1 = \frac{n}{x} \Rightarrow -\frac{y_1}{\sqrt{b^2-y^2}} = \frac{n}{x}$$

$\Rightarrow x^2.y_1^2 = n^2(b^2-y^2)$

Differentiating w.r.to x,

$x^2(2y_1y_2) + y_1^2.2x = n^2(0-2yy_1)$

$\Rightarrow x^2y_2 + xy_1 = -n^2y \quad \Rightarrow x^2y_2 + xy_1 + n^2y = 0$

Differentiating n times w.r to x using leibnitz's theorem

$(x^2y_2)_n + (xy_1)_n + n^2y_n = 0$

$\Rightarrow (y_2)_n.x^2 + nC_1(y_2)_{n-1}(x^2)_1 + nC_2(y_2)_{n-2}.(x^2)_2 + 0 + \left[(y_1)_n.x + nC_1(y_1)_{n-1}.Dx\right] + n^2(y)_n = 0$

$\Rightarrow (y_2)_n.x^2 + nC_1(y_2)_{n-1}(2x) + nC_2(y_2)_{n-2}.2 + \left[(y_1)_n.x + nC_1(y_1)_{n-1}.1\right] + n\ y_n = 0$

$\Rightarrow x^2.y_{n+2} + 2nxy_{n+1} + \frac{n(n-1)}{2}y_n.2 + xy_{n+1} + ny_n + n^2.y_n = 0$

$\Rightarrow x^2y_{n+2} + (2n+1)xy_{n+1} + 2n^2y_n = 0$

5. **If** $y = e^{\sin^{-1}x}$ **, then show that** $(1-x^2)y_2 - xy_1 - y = 0$ **and hence deduce that**

$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+1)y_n = 0$

Sol: $y = e^{\sin^{-1}x}$

Diff w.r.t. x,

$$y_1 = \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}} = \frac{y}{\sqrt{1-x^2}}$$

$\Rightarrow \sqrt{1-x^2}\,y_1 = y$

Differentiating w.r to x

$\sqrt{1-x^2}.y_2 + y_1 \frac{1}{2\sqrt{1-x^2}}(-2x) = y_1$

$\left(1-x^2\right)y_2 - xy_1 = y_1\ \sqrt{1-x^2} = y$

$\Rightarrow \left(1-x^2\right)y_2 - xy_1 - y = 0$

Differentiating n times w.r.to x using Leibnitz's theorem

$\left[\left(1-x^2\right)y_2\right]_n - \left(xy_1\right)_n - y_n = 0$

$\Rightarrow (y_2)_n\left(1-x^2\right) + nC_1.(y_2)_{n-1}\left(1-x^2\right)_1 + nC_2(y_2)_{n-2}\left(1-x^2\right)_2 + 0 - \left[(y_1)_n.x + nc_1(y_1)_{n-1}.Dx\right] - y_n = 0$

$\Rightarrow (y_2)_n\left(1-x^2\right) + nC_1.(y_2)_{n-1}(-2x) + nC_2(y_2)_{n-2}(-2) - \left[(y_1)_n.x + nc_1(y_1)_{n-1}.1\right] - y_n = 0$

$\Rightarrow \left(1-x^2\right)y_{n+2} - 2nxy_{n+1} - \frac{n(n-1)}{2}.2y_n - \left[xy_{n+1} + ny_n\right] - y_n = 0$

$\Rightarrow \left(1-x^2\right)y_{n+2} - 2nx.y_{n+1} - n(n-1)y_n - xy_{n+1} - ny_n - y_n = 0$

$\Rightarrow \left(1-x^2\right)y_{n+2} - (2n+1)xy_{n+1} - \left(n^2 - n + n + 1\right)y_n = 0$

$\left(1-x^2\right)y_{n+2} - (2n+1)xy_{n+1} - \left(n^2+1\right)y_n = 0$

6. If $y = e^x \log x, x > 0$, then show that $xy_2 - (2x-1)y_1 + (x-1)y = 0$ and hence deduce that $xy_{n+2} - (2x-1-n)y_{n+1} + (x-1-2n)y_n + ny_{n-1} = 0$

Sol: $y = e^x \log x$ -----(1)

Differentiating w.r.t. x,

$\Rightarrow y_1 = e^x \frac{1}{x} + \log x.e^x = \frac{ex}{x} + y$

$\Rightarrow xy_1 = xy + e^x$

$\Rightarrow xy_1 - xy = e^x$ -----(2)

Differentiating w.r t. x,

$\Rightarrow xy_2 + y_1 - \left(xy_1 + y\right) = e^x$

$\Rightarrow xy_2 + (1-x)y_1 - y == xy_1 - xy$

$\Rightarrow xy_2 + (1-2x)y_1 + (x-1)y = 0$

$xy_2 - (2x-1)y_1 + (x-1)y = 0$

using Leibnitz's Theorem

Differentiate n times w.r t. x, $\left(xy_2\right)_n - \left[(2x-1)y_1\right]_n + \left[(x-1)y\right]_n = 0$

$\Rightarrow (y_2)_n.x + nc_1(y_2)_{n-1}.Dx + 0 - \left[(y_1)_n(2x-1) + nC_1(y_1)_{n-1}D(2x-1) + 0\right] + \left[(x-1)y_n + nC_1.(y_1)_{n-1}D(x-1) + 0\right] = 0$

$\Rightarrow (y_2)_n.x + nc_1(y_2)_{n-1}.1 - \left[(y_1)_n(2x-1) + nC_1(y_1)_{n-1}2\right] + \left[(x-1)y_n + nC_1.(y_1)_{n-1}1\right] = 0$

$\Rightarrow x.y_{n+2} n.y_{n+1} - (2x-1)y_{n+1} - 2n.y_n + (x-1)y_n + n.y_{n-1} = 0$

$\Rightarrow x.y_{n+2} - (2x-1-n)y_{n+1} (x-1-2n)y_n + n.y_{n-1} = 0$

7. If x + y = 1, then prove that $\frac{d^n}{dx^n}(x^n y^n) = n!(y^n - (nc_1)^2 xy^{n-1} + (nC_2)^2 y^{n-2}.x^2$

$\ldots\ldots\ldots + (-1)^n x^n)$

Sol: Given x + y = 1

$\Rightarrow y = 1 \Rightarrow \frac{dy}{dx} = -1$

$\Rightarrow Dy^n = ny^{n-1} y_1 = ny^{n-1}(-1) = -ny^{n-1}$

$\Rightarrow D^2 y^n = Dny^{n-1}.y_1 = Dny^{n-1}(-1) = -n(n-1)y^{n-2} y_1$

$= (-1)^2 n(n-1)y^{n-2}$

$-\,-\,-\,-\,-$

$\Rightarrow D^n y^n = (-1)^n .n!$

Now $\frac{d^n}{dx^n}(x^n.y^n) = (x^n)_n y^n + nC_1.(x^n)_{n-1} Dy^n + nC_2.(x^n)_{n-2} D^2 y^n + \ldots + x^n.D^n y^n]$

$\Rightarrow$

$\frac{d^n}{dx^n}(x^n.y^n) = y^n.(x^n)_n + nC_1.(x^n)_{n-1}(-n.y^{n-1}) + nC_2.(x^n)_{n-2} n(n-1)n.y^{n-1}(-1)^2 + \ldots + x^n.n!.(-1)^n]$

$= y^n(n!) + nC_1.n!.x.n.y^{n-1}(-1) + nC_2 \frac{n!}{2}.x^2 n(n-1)y^{n-2} + \ldots + x^n.n!(-1)^n$

$= y^n(n!) + nC_1.n!.x.nc_1.y^{n-1}(-1) + nC_2.n!.x^2 nc_2 y^{n-2} + \ldots + x^n.n!(-1)^n$

$= n!\left[y^n - (nC_1)^2.xy^{n-1} + (nC_2)^2.y^{n-2}.x^2 \ldots\ldots + (-1)^n x^n\right]$

8. If $y = e^{-x^2}$, **then prove that** $y_{n+2} + 2xy_{n+1} + 2(x+1)y_n = 0$

Sol: $y = e^{-x^2}$

$\Rightarrow y_1 = e^{-x^2}(-2x) = -2xy$

$\Rightarrow y_1 + 2xy = 0$

$\Rightarrow y_2 + 2(xy_1 + y) = 0$

$\therefore y_2 + 2xy_1 + 2y = 0$

using Leibnitz's Theorem Differentiate n times w.r t. x,

$(y_2)_n + 2(xy_1)_n + 2y_n = 0$

$\Rightarrow y_{n+2} + 2\left[(y_1)_n .x + nC_1.(y_1)_{n-1}.1\right] + 2y_n = 0$

$\Rightarrow y_{n+2} + 2xy_{n+1} + 2n.y_n + 2y_n = 0$

$\therefore\ y_{n+2}+2xy_{n+1}+2(n+1)y_n=0$

9. If y $=(x^2-1)^n$, then prove that $(x^2-1)y_{n+2}+2xy_{n+1}-n(n+1)y_n=0$

Sol: $y=(x^2-1)^n$ ------(1)

$\Rightarrow y_1=n(x^2-1)^{n-1}2x$

$\Rightarrow (x^2-1)y_1=2nx(x^2-1)^n=2nxy$

$\Rightarrow (x^2-1)y_1=2nxy$ -----(2)

Differentiating w.r.to x

$(x^2-1)y_2+2xy_1=2n(xy_1+y)$

$=2nxy_1+2ny$

$(x^2-1)y_2+2(1-n)xy_1-2ny=0$ ----(3)

using Leibnitz's Theorem

Differentiate n times w.r t. x, $((x^2-1)y_2)_n+2(1-n)(xy_1)_n-2ny_n=0$

$(x^2-1)y_{n+2}+nc_1y_{n+1}D(x^2-1)+nc_2y_n.D^2(x^2-1)+0+2(1-n)((y_1)_n x+nc_1.y_n.x+0)-2ny_n=0$

$$(x^2-1)y_{n+2}+nC_1y_{n+1}2x+\frac{n(n-1)}{2}y_n.2+2(1-n)(y_{n+1}x+nc_1.y_n)-2ny_n=0$$

$\Rightarrow (x^2-1)y_{n+2}+2nxy_{n+1}+n(n-1)y_n+2(1-n)(xy_{n+1}+xy_n)-2ny_n=0$

$\Rightarrow (x^2-1)y_{n+2}+2x.y_{n+1}-n(n+1)y_n=0$

10. If $y=e^{m\sin^{-1}x}$, then prove that $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2+m^2)y_n=0$

Sol: $y=e^{m\sin^{-1}x}$ ------(1)

$$\Rightarrow y_1=e^{m\sin^{-1}x}.\frac{m}{\sqrt{1-x^2}}$$

$\Rightarrow\ y_1\sqrt{1-x^2}=my$ -----(2)

Diff wrt x,

$$\sqrt{1-x^2}.y_2+y_1.\frac{1}{2\sqrt{1-x^2}}(-2x)=my_1$$

$\Rightarrow (1-x^2)y_2-xy_1=m.y_1\sqrt{1-x^2}$

$=m(my)=m^2y\ (\because y_1\sqrt{1-x^2}=my)$

$(1-x^2)y_2-xy_1-m^2y=0$

using Leibnitz's Theorem

Differentiate n times w.r t. x, $\left[\left(1-x^2\right)y_2\right]_n-\left(xy_1\right)_n-m^2y_n=0$

$\Rightarrow(y_2)_n\left(1-x^2\right)+nC_1.(y_n)_{n-1}D\left(1-x^2\right)+nC_2(y_2)_n\left(1-x^2\right)+nC_1.(y_2)_{n-1}D^2\left(1-x^2\right)+0-\left[(y_1)_n.x+{}^nC_1(y_1)_{n-1}.Dx+0\right]-m^2y=0$

$\Rightarrow\left(y_2\right)_n\left(1-x^2\right)+nC_1.\left(y_n\right)_{n-1}(-2x)+nC_2\left(y_2\right)_{n-2}(-2)-\left[\left(y_1\right)_n^2.x+{}^nC_1\left(y_n\right)_{n-1}\right]-m\ y\ =0$

$\Rightarrow\left(1-x^2\right)y_{n+2}-2nxy_{n+1}-\dfrac{n(n-1)}{2}y_n.2-xy_{n+1}-ny_n-m^2y_n=0$

$\Rightarrow\left(1-x^2\right)_{n+2}-(2n+1)xy_{n+1}-\left(n^2-n+n+m^2\right)y_n=0$

$\Rightarrow\left(1-x^2\right)y_{n+2}-(2n+1)xy_{n+1}-\left(n^2+m^2\right)y_n=0$.

PROBLEMS FOR PRACTICE

1. **Find the 4th derivative of** $\dfrac{1}{2x+3}$
2. **Find the 5th derivative of sin6x**
3. **Find the 3rd derivative of** $e^x\cos x$**.**
4. **Find nth derivative of** $\cos^2 x$
5. **Find the nth derivative of $e^{4x}\sin^2x$**

6. **Find the nth derivative of log $(4-x^2)$, $|x|<2$**

Sol: LET $y=\log\left(4-x^2\right)=\log(2-x)+(2+x)$

$=\log(2-x)+\log(2+x)$

Diff. n times w.r.t.x,

$y_n=D^n\log(2-x)+D^n\log(2+x)$

$y_n=\dfrac{(-1)^{n-1}(n-1)!(-1)^n}{(2-x)^n}+\dfrac{(-1)^{n-1}(n-1)!1^n}{(2+x)^n}$

$=(-1)^{n-1}.(n-1)!\left(\dfrac{(-1)^n}{(2-x)^n}+\dfrac{1}{(2+x)^n}\right)$

$(-1)^{n-1}.(n-1)!\left(\dfrac{1}{(x-2)^n}+\dfrac{1}{(x+2)^n}\right)$

7. **Find the nth derivative of** $\dfrac{1}{x^2-a^2}$

8. Find the n^th derivative of f(x) = $\frac{1}{6x^2-7x+2} \forall x \in RR-\left\{\frac{1}{2},\frac{2}{3}\right\}$

Sol: $y=\frac{1}{6x^2-7x+2}=\frac{1}{(2x-1)(3x-2)}$

$=\frac{3}{3x-2}-\frac{2}{2x-1}$

Diff . ntimes w.r.t. x,

$\Rightarrow y_n = D^n\left(\frac{3}{3x-2}-\frac{2}{2x-1}\right)$

$y_n=\frac{3(-1)^n n!3^n}{(3x-2)^{n+1}}-\frac{2(-1)^n n!2^n}{(2x-1)^{n+1}}$

$=(-1)^n n!\left(\frac{3^{n+1}}{(3x-2)^{n+1}}-\frac{2^{n+1}}{(2x-1)^{n+1}}\right)$

9. Suppose f(x) = ax + $\frac{b}{x}$ $\forall x \neq 0$**, then show that** $x^2\ f''(x)+xf'(x)-f(x)=0$

Sol: $f(x)=ax+\frac{b}{x}$

$\Rightarrow f'(x)=a-\frac{b}{x^2}$

$\Rightarrow f''(x)=\frac{2b}{x^3}$

L.H.S $= x^2 f''.(x)+x f'(x)-f(x)$

$=x^2\frac{2b}{x^3}+x\left(a-\frac{b}{x^2}\right)-\left(ax-\frac{b}{x}\right)$

$=\frac{2b}{x}+ax-\frac{b}{x}-ax-\frac{b}{x}=0$ =R.H.S

10. Show that f(x) = x+ cot x satisfies $(\sin^2 x) f''(x)+2x=2f(x)$

11. Find the n^th derivative of f(x) = log(8x^3+36x^2+54x+27)for all x >- $\frac{3}{2}$

Sol: $8x^3+36x^2+54x+27=(2x+3)^3$

$f(x)=\log(2x+3)^3=3\log(2x+3)$

Diff n times,

$\Rightarrow f^{(n)}(x)=3D^n(\log(2x+3))$

$f^n(x)=\frac{3(-1)^{n-1}(n-1)!2^n}{(2x+3)^n}$ *for* $x>-\frac{3}{2}$ and $n \in N$.

12. Find the nth derivative of f(x) = sin 7x cos x $\forall\ x \in 1$

Ans: $y_n = \frac{1}{2}\left(8^n \sin\left(8x + \frac{n\pi}{2}\right) + 6^n.\sin\left(6x + \frac{n\pi}{2}\right)\right)$

13. Let f(x) = $ax^2 + \frac{3b}{x} \forall\, x \neq 0$ **, then show that** $x^2 f''(x) = 2f(x)$

Sol: $f(x) = ax^2 + \frac{3b}{x}$

$\Rightarrow f'(x) = 2ax - \frac{3b}{x^2}$

$\Rightarrow f''(x) = 2a + \frac{6b}{x^3} = \frac{2}{x^2}\left[ax^2 + \frac{3b}{x}\right]$

LHS= $x^2 f''(x) = 2\left(ax^2 + \frac{3b}{x}\right) = 2f(x)$ =RHS

14. Find the nth derivative of f(x) $= e^{3x} \sin^2\ 4x \forall x \in \theta$

Ans : $\frac{(-1)^n\, n!}{(1+x^2)^{\frac{n+1}{2}}} \sin(n+1)\theta$.

15. Find the nth derivative of $f(\theta) = \cos^5\theta.\sin^7\theta (\theta \in R)$

Sol: Let x= $\cos\theta + i\sin\theta$

$\Rightarrow \frac{1}{x} = (\cos\theta + i\sin\theta)^{-1} = \cos\theta - i\sin\theta \Rightarrow x + \frac{1}{x} = 2\cos\theta,$ AND $x - \frac{1}{x} = 2i\sin\theta$

By De Moivres theorem

$x^n + \frac{1}{x^n} = 2\cos n\theta, x^n - \frac{1}{x^n} = 2i\sin n\theta$

$2^5\cos^5\theta = \left(x + \frac{1}{x}\right)^5, 2^7 i^7 \sin^7\theta = \left(x - \frac{1}{x}\right)^7$

$2^{12} i^7 \cos^5\theta.\sin^7\theta = \left(x + \frac{1}{x}\right)^5 \left(x - \frac{1}{x}\right)^7$

$= \left(x^2 - \frac{1}{x^2}\right)^5 \left(x - \frac{1}{x}\right)^2$

$= \left(x^{10} - 5x^6 + 10x^2 - \frac{10}{x^2} + \frac{5}{x^6} + \frac{1}{x^{10}}\right)\left(x^2 - 2 + \frac{1}{x^2}\right)$

$= \left(x^{12} - \frac{1}{x^{12}}\right) - 2\left(x^{10} - \frac{1}{x^{10}}\right) - 4\left(x^8 - \frac{1}{x^8}\right) + 10\left(x^6 - \frac{1}{x^6}\right) + 5\left(x^4 - \frac{1}{x^4}\right) - 20\left(x^2 - \frac{1}{x^2}\right)$

$\therefore -2^{12} i\cos^5\theta\sin^7\theta = 2i(\sin 12\theta - 2\sin 10\theta)$

$-2^{11}\cos^5\theta\sin^7\theta = \sin 12\theta - 2\sin 10\theta - 4.\sin 8\theta + 10\sin 6\theta + 5\sin 4\theta - 20.\sin 2\theta)$

$$D^n\left(\cos^5\theta\sin^7\theta\right) = -\frac{1}{2^{11}}\left\{12^n\sin\left(12\theta+\frac{n\pi}{2}\right)\right\} - 10^n 2\sin\left(10\ \theta+\frac{n\pi}{2}\right)$$

$$-8^n 4\sin\left(8\theta+\frac{n\pi}{2}\right)+6^n.10\sin\left(\frac{n\pi}{2}+6\theta\right)+4^n 5\sin\left(\frac{n\pi}{2}+4\theta\right)-2^n.20\sin\left(\frac{n\pi}{2}+2\theta\right)\Bigg\}$$

16. Prove that the nth derivative of f(x) = $\tan^{-1}x\left(x\in R\right)$ **is** $\frac{(-1)^{n-1}(n-1)!}{\left(x^2+1\right)^{n/2}}\sin\theta$ **where**

$\theta\in(-\pi,\pi)$ **is the angle satisfying** $\cos\theta = \frac{x}{\sqrt{x^2+1}}$ **and** $\theta = \frac{1}{\sqrt{x^2+1}}$

17. If u and v are functions of x having 4th order derivatives. Write the formula for $D^4(uv)$**.**

18. Find $D^3\left(\sin x\log x\right), x>0$**.**

19. Find $D^3\left(e^x.\cos x\right)$**, at x = 0**

20. Find D^4 (x.logx) at x = e (x > 0)

21. Find the nth derivative of x.e^x.

22. Find $D^n\left(x^2.\sin 3x\right)$

23. If $y=\sin^{-1}x\forall x\in(-1,1)$ **then show that** $\left(1-x^2\right)y_2 - xy_1 = 0$ and hence deduce that

$\left(1-x^2\right)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$

24. If y = a cos (logx) + b sin (logx) (x >0), then prove that

$x^2.y_{n+2} + (2n+1)xy_{n+1} + \left(n^2+1\right)y_n = 0$

Sol: $y = a\cos(\log x) + b\sin(\log x)$

$\Rightarrow y_1 = -a\sin(\log x).\frac{1}{x} + b\cos(\log x).\frac{1}{x}$

$\Rightarrow xy_1 = -a\sin(\log x) + b\cos(\log x)$

Differentiating w.r.to x

$x.y_2 + y_1.1 = -a\cos(\log x).\frac{1}{x} - b\sin(\log x).\frac{1}{x}$

$x^2y_2 + xy_1 = -\left[a\cos(\log x) + b\sin\log x\right] = -y$

$\Rightarrow x^2y_2 + xy_1 + y = 0$

Differentiating n times w.r.to x

$\left(x^2y_2\right)_n + \left(xy_1\right)_n + y_n = 0$

Differentiating n times w.r.to x

$\left(x^2y_2\right)_n + \left(xy_1\right)_n + y_n = 0$

$(y_2)_n.x^2 + nC_1(y_2)_{n-1}.Dx^2 + nC_2(y_2)_{n-2}.D^2x^2 + 0 + \left[(y_1)_n.x + nC_1.(y_1)_{n-1}.Dx + 0\right] + y_n = 0$

$$(y_2)_n.x^2 + nC_1(y_2)_{n-1}.2x + nC_2(y_2)_{n-2}.2 + \left[(y_1)_n.x + nC_1.(y_1)_{n-1}.1\right] + y_n = 0$$

$$x^2.y_{n+2} + 2nx.y_{n+1} + \frac{n(n-1)}{2}.y_n.2 + xy_{n+1} + n.y_n + y_n = 0$$

$$x^2.y_{n+2} + (2n+1)xy_{n+1} + (n^2 - n + n + 1)y_n = 0$$

$$x^2.y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0.$$

25. If $f(x) = x^{n-1}.\log x$ and g(x) $= x^n.\log x$ for $x > 0$, then prove that

i) $f^n(x) = \frac{(n-1)!}{x}$

ii) $g^{n+1}(x) = \frac{n!}{x}$

26. If log y = $\tan^{-1} x (y > 0)$**, then show that** $(1+x^2)y_{n+2} + (2nx + 2x - 1)y_{n+1} + n(n+1)y_n = 0$

Sol: $\log y = \tan^{-1} x$

Diff. wrt. X,

$$\Rightarrow \frac{1}{y}.y_1 = \frac{1}{1+x^2} \Rightarrow (1+x^2)y_1 = y$$

Differentiating w.r.to x

$$\Rightarrow (1+x^2)y_2 + 2xy_1 = y_1$$

$$\Rightarrow (1+x^2)y_2 + (2x-1)y_1 = 0$$

Differentiating n times w.r.to x

$$\left((1+x^2)y_2\right)_n + \left((2x-1)y_1\right)_n = 0$$

$$(y_2)_n.(1+x^2) + nc_1(y_2)_{n-1}.2x + nc_2(y_2)_{n-2}2 + \left((y_1)_n.(2x-1)\right) + nc_1(y_1)_{n-1}2) = 0$$

$$(1+x^2)y_{n+2} + 2nxy_{n+1} + n\frac{(n-1)}{2}.2.y_n + n(n-1+2)y_n = 0$$

$$(1+x^2)y_{n+2} + (2nx + 2x - 1)y_{n+1} + n(n+1)y_n = 0$$

27. If y = $\sin(m\sin^{-1}x); n \in (-1,1)$**, then show that** $(y_{n+2})_0 = (x^2 - m^2)(y_n)_0$**. i.e.,**

$$\left(\frac{d^{n+2}y}{dx^{n+2}}\right)_{at\ x=0} = (n^2 - m^2)\left(\frac{d^n y}{dx^n}\right)_{x=0}$$

Sol: $y = \sin(m\sin^{-1}x)$--------(1)

Diff wrt x,

$$y_1 = \cos(m\sin^{-1}x).\frac{m}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\,y_1 = m\cos(m\sin^{-1}x)$$

Squaring on both sides,

$\left(1-x^2\right)y_1{}^2 = m^2.\cos^2\left(m\sin^{-1}x\right)] \Rightarrow \left(1-x^2\right)y_1{}^2 = m^2\left(1-y^2\right)$ -----(2)

Differentiating w.r.to x

$\left(1-x^2\right).2y_1y_2 + y_1^2\left(-2x\right) = -2m^2yy_1$

Dividing with $2y_1$, we have

$\left(1-x^2\right)y_2 - xy_1 = -m^2y$

$\left(1-x^2\right)y_2 - xy_1 + m^2y = 0$

Differentiating n times w.r to x

$\left(\left(1-x^2\right)y_2\right)_n - \left(xy_1\right)_n + m^2y_n = 0$

$(y_2)_n\left(1-x^2\right)+nc_1.(y_2)_{n-1}D\left(1-x^2\right)+nc_2.(y_2)_{n-2}D^2\left(1-x^2\right)+0-\left[(y_1)_n.x+nc_1.(y_1)_{n-1}Dx\right]+m^2y_n=0$

$\Rightarrow (y_2)_n\left(1-x^2\right)+nc_1.(y_2)_{n-1}(-2x)+nc_2.(y_2)_{n-2}(-2)-\left[(y_1)_n.x+nc_1.(y_1)_{n-1}1\right]+m^2y_n=0$

$\Rightarrow \left(1-x^2\right)y_{n+2} - 2nx.y_{n+1} - \frac{n(n-1)}{2}y_n.2 - xy_n - xy_n + m^2y_n = 0$

$\Rightarrow \left(1-x^2\right)y_{n+2} - (2n+1)xy_{n+1} + \left(m^2-n^2\right)y_n = 0$

At x = 0,

we get $\left(y_{n+2}\right)_0 = -\left(m^2-n^2\right)\left(y_n\right)_0 = \left(n^2-m^2\right)\left(y_n\right)_0$.

28. Find the nth derivative of f(x) = $\frac{1}{2x+5}x\log(4x+6)\left(x > -\frac{3}{2}\right)$

29. If x = cost and y = sin pt, where p is a constant and t is not a multiple of π, then prove that $\left(1-x^2\right)y_{n+2} - (2n+1)\,xy_{n+1} - \left(x^2-p^2\right)y_n = e\forall n \in (-1,1)$

Sol:

Given x = cos t, y = sin pt

$\frac{dx}{dt} = -\sin t$ and $\frac{dy}{dt} = p\cos pt$

$y_1 = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dt}{dx}} = \frac{-p\cos pt}{\sin t}$

$y_1^2 = \frac{p^2.\cos^2 pt}{\sin^2 t} = \frac{p^2\left(1-y^2\right)}{1-x^2}$

$\left(1-x^2\right)y_1^2 = p^2\left(1-y^2\right)$ ------(1)

Differentiating w.r.to x

$\left(1-x^2\right).2y_1y_2 + y_1^2\left(-2x\right) = p^2\left(-2yy_1\right)$

$\left(1-x^2\right)y_2 - xy_1 = -p^2y$

$\left(1-x^2\right)y_2 - xy_1 + p^2y = 0$ -----(2)

Using Leibnitz's theorem,

Differentiate n times w.r.to x

$$\left(\left(1-x^2\right)y_2\right)_n-\left(xy_1\right)_n+p^2y_n=0$$

$$\Rightarrow \left(y_2\right)_n\left(1-x^2\right)+{}^nC_1.\left(y_2\right)_{n-1}D\left(1-x^2\right)+nc_2\left(y_2\right)_{n-2}D^2\left(1-x^2\right)+0-\left[\left(y_1\right)_n^n.x+\ C_1\left(y_1\right)_{x-1}.Dx+0\right]+p^2y_n=0$$

$$\left(y_2\right)_n\left(1-x^2\right)+{}^nC_1.\left(y_2\right)_{n-1}\left(-2x\right)+\frac{n\left(n-1\right)}{2}\left(y_2\right)_{n-2}\left(-2\right)-\left[\left(y_1^2\right)_n^n.x+\ C_1\left(y_1\right)_{x-1}.1\right]+p\ y_n=0$$

$$\Rightarrow \left(1-x^2\right)y_{n+2}-\left(2n+1\right)xy_{n+1}.\left(n^2-n+n-p^2\right)y_n=0$$

$$\Rightarrow \left(1-x^2\right)y_{n+2}-\left(2n+1\right)xy_{n+1}-\left(n^2-p^2\right)y_n=0$$

$$\Rightarrow x^2\left(2y_1y_2\right)+y_1^2.2x=n^2\left(-2yy_1\right)$$

$$\Rightarrow x^2y_2+xy_1+n^2y=0$$