CHAPTER 6

POLAR COORDINATES

TOPICS:

1.Relation Between Polar Coordinates And Cartesian Coordinates

2.Distance Between Two Points

3.Area Of The Triangle

4.Equation Of Straight Line- Various Forms

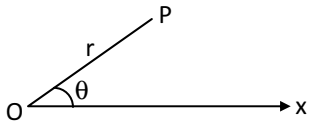
5.Equation Of The Circle.

6.Equation Of The Conic.

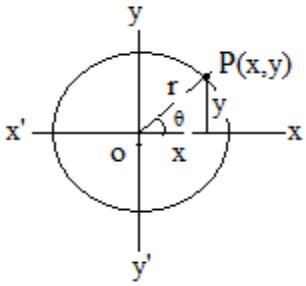
POLAR COORDINATES

POLAR COORDINATE SYSTEM

Let O be a fixed point in a plane and $\vec{O}x$ be a fixed ray in the plane. With respect to these two we can determine the position of any point in the plane. The fixed point O is called pole and the fixed ray $\vec{O}x$ is called initial line or polar axis. Let P be a point in the plane such that OP = r. If $\angle Pox = \theta$ then r, θ are called polar coordinates of P. The point P is denoted by (r, θ). The non-negative real number r is called radial distance, the vector $\vec{r} = \overrightarrow{OP}$ is called radius vector and the angle θ is called vectorial angle of the point P.



RELATION BETWEEN POLAR AND CARTESIAN COORDINATES



Let P(x, y) be a point in the Cartesian coordinate plane. Take the origin O as pole and the positive direction of x-axis as polar axis (initial line). Let (r, θ) be the polar coordinates of P. Then $r = OP = \sqrt{x^2 + y^2}$.

$\cos\theta = \frac{x}{r}, \sin\theta = \frac{y}{r}$. Thus $x = r\cos\theta, y = r\sin\theta$.

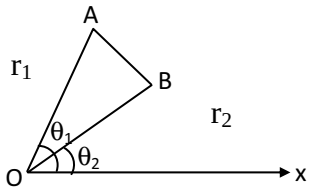
Note . Conversion of Cartesian coordinates into polar coordinates is $r = \sqrt{x^2 + y^2}$, $\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$.

DISTANCE BETWEEN TWO POINTS

The distance between points (r_1, θ_1), (r_2, θ_2) is $\sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}$.

Proof :

Let $A(r_1, \theta_1)$, $B(r_2, \theta_2)$ be the polar coordinates of two points w.r.t to O and OX.



Then $OA = r_1$, $OB = r_2$, $\angle AOx = \theta_1$, $\angle BOx = \theta_2$.

From ΔOAB,

$$AB^2 = OA^2 + OB^2 - 2OA \cdot OB\cos\angle AOB$$

$$= r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)$$

$$\therefore AB = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}.$$

AREA OF THE TRIANGLE

The area of the triangle formed by the points (r_1, θ_1), (r_2, θ_2), (r_3, θ_3) is $\frac{1}{2}\left|\Sigma r_1r_2\sin(\theta_1 - \theta_2)\right|$.

EXERCISE – 6(A)

1. Taking the origin as the pole and the positive x-axis as initial ray, convert the following polar equations into Cartesian forms.

i) $r^2 \cos 2\theta = a^2$

Sol. The Relations between polar and cartesian coordinates are $x=r\cos\theta$, $y = r\sin\theta$

Given equation is $r^2(\cos^2\theta - \sin^2\theta) = a^2$

$\Rightarrow (r\cos\theta)^2 - (r\sin\theta)^2 = a^2$

$\Rightarrow x^2 - y^2 = a^2$

ii) $\theta = \tan^{-1}(m)$

Sol.

The Relations between polar and cartesian coordinates are $x=r\cos\theta$, $y = r\sin\theta$

Given $\tan\theta = m$

$$\Rightarrow \frac{\sin\theta}{\cos\theta} = m$$

$$\Rightarrow \frac{r\sin\theta}{r\cos\theta} = m \Rightarrow \frac{y}{x} = m$$

(The Relations between polar and cartesian coordinates are $x=r\cos\theta$, $y = r\sin\theta$)

i.e. $y = mx$

iii) $r = a\sin^2\frac{\theta}{2}$

Sol. The Relations between polar and cartesian coordinates are $x=r\cos\theta$, $y = r\sin\theta$

Given equation

$$r = a\frac{(1-\cos\theta)}{2} \Rightarrow 2r = a(1-\cos\theta) = a\left(1-\frac{x}{r}\right)$$

$$2r = a\frac{(r-x)}{r} \Rightarrow 2r^2 = ar - ax$$

$$2r^2 + ax = ar \Rightarrow 2(x^2+y^2) + ax = ar$$

$$[2(x^2+y^2)+ax]^2 = a^2r^2$$

$$4(x^2+y^2)^2 + a^2x^2 + 4ax(x^2+y^2) = a^2(x^2+y^2)$$

$$4(x^2+y^2)^2 + 4ax(x^2+y^2) + a^2x^2 = a^2x^2 + a^2y^2$$

$$4(x^2+y^2)(x^2+y^2+ax) = a^2y^2$$

2. Taking origin as the pole and the positive x-axis as initial ray. Convert the following Cartesian equation into polar forms.

Sol. i) $x^2 + y^2 = a^2$

The Relations between polar and cartesian coordinates are $x=r\cos\theta$, $y = r\sin\theta$

$r^2 \cos^2\theta + r^2 \sin^2\theta = a^2$

$r^2(\cos^2\theta + \sin^2\theta) = a^2$

i.e. $r^2 = a^2$

ii) $y^2 = 4ax$

The Relations between polar and cartesian coordinates are $x=r\cos\theta$, $y = r\sin\theta$

$(r\sin\theta)^2 = 4a(r\cos\theta)$

$r^2 \sin^2\theta = 4ar\cos\theta$

$$r = 4a\frac{\cos\theta}{\sin^2\theta}$$

$$\Rightarrow r = 4a\frac{\cos\theta}{\sin\theta}\frac{1}{\sin\theta} = 4a\cot\theta\csc\theta$$

iii) $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

The Relations between polar and cartesian coordinates are $x=r\cos\theta$, $y = r\sin\theta$

$$r^2\frac{\cos^2\theta}{a^2}+r^2\frac{\sin^2\theta}{b^2}=1$$

$$r^2\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)=1$$

iv) $x^2 - y^2 = a^2$

$r^2 \cos^2\theta - r^2 \sin^2\theta = a^2$

$r^2(\cos^2\theta - \sin^2\theta) = a^2$

$r^2 \cos 2\theta = a^2$

3. Find the distance between the following pairs of points with polar coordinates.

Sol. i) Given points $P\left(2,\frac{\pi}{6}\right), Q\left(4,\frac{2\pi}{3}\right)$

Distanct between the points is $PQ = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)}$

$$= = \sqrt{4 + 16 - 2\times2\times4\cos(120^\circ - 30^\circ)} \quad = \sqrt{20 - 16\cos 90^\circ = 20 - 16(0)}$$

$PQ = \sqrt{20} = 2\sqrt{5}$ units

ii) $P\left(3,\frac{\pi}{4}\right), Q\left(4,\frac{7\pi}{12}\right)$

Ans: $\sqrt{13}$ units

iii) Given points $P\left(a,\frac{\pi}{2}\right), Q\left(4a,\frac{\pi}{6}\right)$

$PQ = \sqrt{13}\,a$ units

II.

1. Show that the points with polar coordinates (0, 0), (3, π/2) and (3, π/6) form an equilateral triangle.

Sol. Vertices of the triangle are A(0, 0), B(3, π/2) and C(3, π/6) .

$AB = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)}$

$= \sqrt{0 + 9 - 2\times0\times3\cos(0 - p/2)} = \sqrt{9} = 3$

$BC = = \sqrt{9 + 9 - 2\times3\times3\cos(90° - 30°)}$

$= = \sqrt{18 - 18\times\frac{1}{2}} = \sqrt{18 - 9} = 3$

$CA = = \sqrt{9 + 0 - 2\times3\times0\cos(p/6)} = \sqrt{9} = 3$

$\Rightarrow AB = BC = CA$

ABC is an equilateral triangle.

2. Find the area of the triangle formed by the following points with polar coordinates.

i) $(a,0), \left(2a,\theta+\frac{\pi}{3}\right), \left(3a,\theta+\frac{2\pi}{3}\right)$

ii) $\left(-5,-\frac{2\pi}{3}\right), \left(4,\frac{5\pi}{6}\right), (0,0)$

Sol. i) vertices of the triangle are $A(a,0), B\left(2a,\theta+\frac{\pi}{3}\right), C\left(3a,\theta+\frac{2\pi}{3}\right)$

Area of the trangle is $\frac{1}{2}\left|\Sigma r_1r_2\sin(\theta_1-\theta_2)\right|$

$= \frac{1}{2}\left[2a^2\sin\left(\theta+\frac{\pi}{3}-\theta\right) + 6a^2\sin\left(\theta+\frac{2\pi}{3}-\theta-\frac{\pi}{3}\right) + 3a^2\sin\left(\theta-\frac{2\pi}{3}-\theta\right)\right]$

$= \frac{1}{2}[2a^2\sin(60°) + 6a^2\sin(60°) + 3a^2\sin(120°)]$

$$= \frac{1}{2}\left[2a^2 \cdot \left(\frac{-\sqrt{3}}{2} \right) + 6a^2 \left(-\frac{\sqrt{3}}{2} \right) + 3a^2 - \frac{\sqrt{3}}{2} \right]$$

$$= \frac{\sqrt{3}}{4}\left| 2a^2 + 6a^2 - 3a^2 \right|$$

$$= \frac{\sqrt{3}}{4}(5a^2) = \frac{5\sqrt{3}}{4}a^2 \text{ sq.units}$$

ii) $A\left(-5, -\frac{2\pi}{3}\right), B\left(4, \frac{5\pi}{6}\right), C(0,0)$ **are the vertices of the triangle.**

Ans: 10 sq.units.

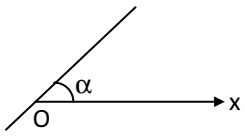
POLAR EQUATION OF A STRAIGHT LINE

The polar equation of a line passing through pole and making an angle α with the initial line is $\theta = \alpha$.

Proof :

If P(r, θ) is a point in the line then ∠POX= θ and hence θ = α.

Conversely if P(r, θ) is a point such that θ = α then P lies in the line.



∴ The equation to the locus of a point P in the line is θ = α.

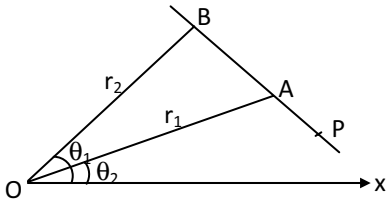
∴ The polar equation of the line is θ = α.

THEOREM

The polar equation of a line passing through the points $(r_1, \theta_1), (r_2, \theta_2)$ is

$$\frac{\sin(\theta_2 - \theta_1)}{r} + \frac{\sin(\theta - \theta_2)}{r_1} + \frac{\sin(\theta_1 - \theta)}{r_2} = 0.$$

Proof : Let $A(r_1, \theta_1)$, $B(r_2, \theta_2)$ and $P(r, \theta)$.



Then P lies in the line $\overleftrightarrow{AB}$ ⇔ Area of ΔPAB = 0

$$\frac{1}{2}\left| r_1 r_2 \sin(\theta - \theta_1) + (r_1 r_2)\sin(\theta_1 - \theta_2) + (r_2 r)\sin(\theta_2 - \theta) \right| = 0$$

$$\Leftrightarrow r\left[r_1 \sin(\theta - \theta_1) - r_2 \sin(\theta - \theta_2) \right] = r_1 r_2 \sin(\theta_2 - \theta_1)$$

$\therefore$ The polar equation of line i.e. the locus of P is $r\left[r_1 \sin(\theta-\theta_1) - r_2 \sin(\theta-\theta_2)\right] = r_1 r_2 \sin(\theta_2-\theta_1)$.

$\Rightarrow \dfrac{\sin(\theta_2-\theta_1)}{r} + \dfrac{\sin(\theta-\theta_2)}{r_1} + \dfrac{\sin(\theta_1-\theta)}{r_2} = 0$

Note . The polar equation of a line passing through the points (r_1, θ_1), (r_2, θ_2) is

$r\left[r_1 \sin(\theta-\theta_1) - r_2 \sin(\theta-\theta_2)\right] = r_1 r_2 \sin(\theta_2-\theta_1)$.

COROLLARTY

The polar equation of a line passing through the pole and the point (r_1, θ_1) is $\theta = \theta_1$.

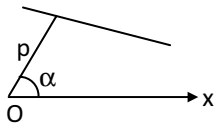
THEOREM

The polar equation of a line which is at a distance of p from the pole and whose normal makes an angle α with the initial line is $r \cos(\theta - \alpha) = p$.

Proof : Let O be the pole of $\vec{O}x$ be the initial line.

$\therefore$ ON = p, $\angle NOx = \alpha$.

P(r, θ) is a point on the line.



$\Leftrightarrow \cos\angle NOP = \dfrac{ON}{OP}$

$\Leftrightarrow \cos(\theta-\alpha) = \dfrac{p}{r} \Leftrightarrow r\cos(\theta-\alpha) = p$.

Note : If p = 0 then $r\cos(\theta-\alpha) = p \Rightarrow r\cos(\theta-\alpha) = 0 \Rightarrow \theta - \alpha = \pi/2 \Rightarrow \theta = \pi/2 + \alpha \Rightarrow \theta = \theta_1$, which is a line passing through pole.

Note . $r\cos(\theta-\alpha) = p$

$\Rightarrow r(\cos\theta\cos\alpha + \sin\theta\sin\alpha) = p$

$\Rightarrow (r\cos\theta)\cos\alpha + (r\sin\theta)\sin\alpha = p$

$\Rightarrow x\cos\alpha + y\sin\alpha = p$ (normal form in Cartesian system).

NOTE

The equation of a straight line which is at a distance of p units from pole and perpendicular to the initial ray is $r\cos\theta = p$.

NOTE

The equation of a straight line which is at a distance p units from the pole and parallel to the initial ray is $r\sin\theta = p$.

THEOREM

The polar form of the line ax + by + c = 0 is a cos θ + b sin θ = k/r where k = –c.

THEOREM

The polar equation of a line parallel to the line a cos θ + b sin θ = k/r is $a\cos\theta + b\sin\theta = \frac{k_1}{r}$**.**

THEOREM

The polar equation of a line perpendicular to the line $a\cos\theta + b\sin\theta = \frac{k}{r}$ **is**

$$a\cos\left(\frac{\pi}{2}+\theta\right)+b\sin\left(\frac{\pi}{2}+\theta\right)=\frac{k_1}{r}.$$

NOTE

The polar equation of a line perpendicular to the line $r\cos(\theta-\alpha)=p$ is $r\sin(\theta-\alpha)=p_1$.

EXERCISE --- 6(B)

1. Find the polar equation of the line joining the points (5, π/2) and (–5, π/6).

Sol. Given points are A(5, π/2) and B(–5, π/6)

Equation of the line joining $A(r_1, \theta_1)$, $B(r_2, \theta_2)$ is

$$\frac{\sin(\theta_2-\theta_1)}{r}+\frac{\sin(\theta-\theta_2)}{r_1}+\frac{\sin(\theta_1-\theta)}{r_2}=0$$

$$\frac{\sin\left(\frac{\pi}{6}-\frac{\pi}{2}\right)}{r}+\frac{\sin\left(\theta-\frac{\pi}{6}\right)}{5}+\frac{\sin\left(\frac{\pi}{2}-\theta\right)}{-5}=0$$

$$-\frac{\sqrt{3}}{2r}+\frac{\sin\left(\theta-\frac{\pi}{6}\right)}{5}-\frac{\cos\theta}{5}=0\Rightarrow\frac{1}{5}\left(\sin\left(\theta-\frac{\pi}{6}\right)-\cos\theta\right)=\frac{\sqrt{3}}{2r}$$

$$\Rightarrow\sin\left(\theta-\frac{\pi}{6}\right)-\cos\theta=\frac{5\sqrt{3}}{2r}$$

2. Find the polar coordinates of the points of intersection of the lines $\frac{2}{r}=\sin\left(\theta+\frac{\pi}{3}\right)+\sin\theta$ and $\frac{4}{r}=3\sin\theta-\sqrt{3}\cos\theta$.

Sol. Given lines are

$$\frac{2}{r}=\sin\left(\theta+\frac{\pi}{3}\right)+\sin\theta \quad \ldots(1)$$

$$\frac{4}{r}=3\sin\theta-\sqrt{3}\cos\theta \quad \ldots(2)$$

Eliminating 'r' , we get

$$2\left(\sin\left(\theta+\frac{\pi}{3}\right)+\sin\theta\right)=3\sin\theta-\sqrt{3}\cos\theta$$

$$2\left(\left(\sin\theta\cdot\cos\frac{\pi}{3}+\cos\theta\cdot\sin\frac{\pi}{3}\right)+\sin\theta\right)=3\sin\theta-\sqrt{3}\cos\theta$$

$$2\sin\theta\frac{1}{2}+2\cos\theta\frac{\sqrt{3}}{2}+2\sin\theta=3\sin\theta-\sqrt{3}\cos\theta\ 3\sin\theta+\sqrt{3}\cos\theta=3\sin\theta-\sqrt{3}\cos\theta$$

$$2\sqrt{3}\cos\theta=0\Rightarrow\cos\theta=0\Rightarrow\theta=\frac{\pi}{2}$$

Substituting in (1)

$$\frac{4}{r}=3\sin\frac{\pi}{2}-4\cos\frac{\pi}{2}=3\cdot1-4\cdot0=3\Rightarrow r=\frac{4}{3}$$

Point of intersection is $P\left(\frac{4}{3},\frac{\pi}{2}\right)$.

3. Find the polar equation of a straight line with intercepts 'a' and 'b' on the rays θ = 0 and θ = π/2 respectively.

Sol. Given line is passing through the points

A(a, 0) and B(b, π/2)

Equation of the line is

$$\frac{\sin(\theta_2-\theta_1)}{r}+\frac{\sin(\theta-\theta_2)}{r_1}+\frac{\sin(\theta_1-\theta)}{r_2}=0$$

$$\frac{\sin\frac{\pi}{2}}{r}+\frac{\sin\left(\theta-\frac{\pi}{2}\right)}{a}+\frac{\sin(0-\theta)}{b}=0$$

$$\frac{1}{r}-\frac{\cos\theta}{a}-\frac{\sin\theta}{b}=0\Rightarrow\frac{1}{r}=\frac{\cos\theta}{a}+\frac{\sin\theta}{b}$$

$$\frac{1}{r}=\frac{b\cos\theta+a\sin\theta}{ab}$$

$r(b\cos\theta + a\sin\theta) = ab.$

II.

1. Find the polar equations of the lines passing through (2, π/4).

(i) parallel to, (ii) perpendicular to the straight line $\frac{7}{r}=4\cos\theta+3\cos\theta$**.**

Sol. i) Given line is $4\cos\theta+3\sin\theta=\frac{7}{r}$

Equation of the line parallel to this line is $4\cos\theta+3\sin\theta=\frac{k}{r}$

This line is passing through $A\left(2,\frac{\pi}{4}\right)$

$$\Rightarrow 4\cos 45^\circ+3\sin 45^\circ=\frac{k}{2}$$

$$\Rightarrow k=2\left(4\frac{1}{\sqrt{2}}+3\frac{1}{\sqrt{2}}\right)=2\cdot\frac{7}{\sqrt{2}}=7\sqrt{2}$$

Equation of the parallel line is

$$4\cos\theta+3\sin\theta=\frac{7\sqrt{2}}{r}$$

ii) Equation of the lint perpendicular to this line is

$$4\cos\left(\frac{\pi}{2}+\theta\right)+3\sin\left(\frac{\pi}{2}+\theta\right)=\frac{k'}{r}\Rightarrow -4\sin\theta+3\cos\theta=\frac{k'}{r}$$

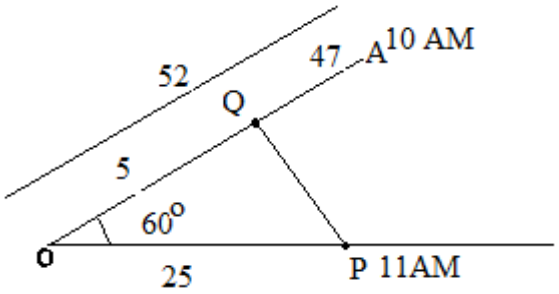
This line is passing through $A\left(2,\frac{\pi}{4}\right)$

$$\Rightarrow -4\sin 45^\circ+3\cos 45^\circ=\frac{k}{2}\Rightarrow\frac{k}{2}=-4\frac{1}{\sqrt{2}}+3\frac{1}{\sqrt{2}}=-\frac{1}{\sqrt{2}}\Rightarrow k=-\frac{2}{\sqrt{2}}=-\sqrt{2}$$

Equation of the perpendicular line is $-4\sin\theta+3\cos\theta=-\frac{\sqrt{2}}{r}\Rightarrow 4\sin\theta-3\cos\theta=\frac{\sqrt{2}}{r}$

2. Two straight roads intersect at angle 60°. A car is moving away from the crossing at the rate of 25 kmph at 10 A.M. Another car is 52 km away from the crossing on the other and is moving towards the crossing at the rate of 47 kmph at 10 A.M. Then find the distance between the cars at 11 A.M.

Sol. Let P and Q be the positions of the cars at 11 A.M. and 0 be the pole.



Let O be the intersection of the roads.

Let A be the position of the second car at 10AM

OA = 52km

Let o be the position of the 1st car at 10 am.

Velocity of the car is 25 kmph and that of the second is 47 kmph. Let P,Q be the positions of two cars at 11 am respectively.

OP = 25, OQ = 5, $\angle POQ = 60°$

from $\Delta OPQ, PQ^2 = r_1^2 + r_2^2 - 2r_1r_2 \cos\angle POQ$

$= 625 + 25 - 2 \cdot 25 \cdot 5 \cos 60°$

$= 650 - 2 \cdot 125 \cdot \frac{1}{2} = 525$

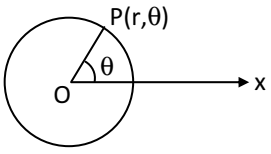
$PQ = \sqrt{525} = 5\sqrt{21}$ km.

POLAR EQUATION OF THE CIRCLE.

The polar equation of the circle of radius 'a' and having centre at the pole is r = a.

Proof :

Let O be the pole and $\vec{O}x$ be the initial line. If a point P(r, θ) lies in the circle then r = OP = radius = a. Conversely if P(r, θ) is a point such that r = a then P lies in the circle.



∴ The polar equation of the circle is r = a.

THEOREM

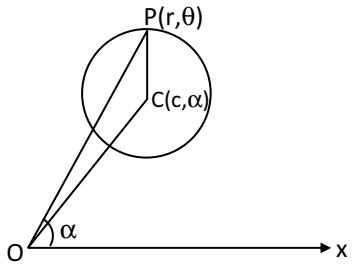
The polar equation of the circle of radius 'a' and the centre at

(c, α) is $r^2 - 2cr\cos(\theta - \alpha) = a^2 - c^2$.

Proof :

Let O be the pole and $\vec{O}x$ be the initial line.

Let P(r, θ) be any point on the circle.



Centre of the circle C = (c, α) and radius of the circle = r.

$\therefore$ OC = c, CP = r, $\angle POx = \theta$, $\angle Cox = \alpha$.

From ΔOPC, we get :

$$CP^2 = OP^2 + OC^2 - 2OP \cdot OC \cdot \cos\angle POC$$

$$\Rightarrow a^2 = r^2 + c^2 - 2rc\cos(\theta - \alpha)$$

$$\Rightarrow r^2 - 2rc\cos(\theta - \alpha) = a^2 - c^2$$

$\therefore$ The locus of P is $r^2 - 2rc\cos(\theta - \alpha) = a^2 - c^2$.

$\therefore$ The polar equation of the circle is :

$$r^2 - 2rc\cos(\theta - \alpha) = a^2 - c^2.$$

NOTE

The polar equation of the circle of radius 'a' and passing through the pole is $r = 2a\cos(\theta - \alpha)$ where α is the vectorial angle of the centre.

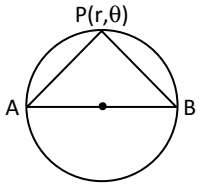
NOTE

The polar equation of the circle of radius 'a', passing through the pole and its diameter as the initial line is $r = 2a\cos\theta$.

NOTE

The polar equation of the circle of radius 'a' and touching the initial line at the pole is $r = 2a\sin\theta$.

POLAR EQUATION OF A CIRCLE FOR WHICH (R_1, θ_1) AND (R_2, θ_2) ARE EXTREMITIES OF DIAMETER.



Given $A(r_1, \theta_1)$, $B(r_2, \theta_2)$ are the ends of the diameter. LET $P(r, \theta)$ be any point on the circle.
AB is the diameter $\Rightarrow \angle APB = 90°$

$AP^2 + PB^2 = AB^2$ …(1)

$AP^2 = r^2 + r_1^2 - 2rr_1 \cos(\theta - \theta_1)$

$PB^2 = r^2 + r_2^2 - 2rr_2 \cos(\theta - \theta_1)$

$AB^2 = r^2 + r_2^2 - 2r_1r_2 \cos(\theta_1 - \theta_2)$

Substituting in (1)

$r^2 + r_1^2 - 2rr_1 \cos(\theta - \theta_1) + r^2 + r_2^2 - 2rr_2 \cos(\theta - \theta_2)$

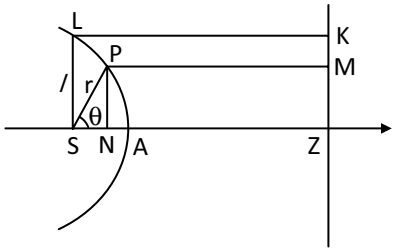
$= r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_1 - \theta_2)$

Equation of the required circle is : $r^2 - r\left[\cos(\theta - \theta_1) + \cos(\theta - \theta_2)\right] + r_1r_2 \cos(\theta_1 - \theta_2) = \theta$.

POLAR EQUATION OF THE CONIC.

The polar equation of a conic in the standard form is $\frac{\ell}{r} = 1 + e\cos\theta$**.**

Proof : Let S be the focus and Z be the projection of S on the directrix.



Let S be the pole and SZ be the initial line.

Let SL = ℓ be the semi latus rectum and e be the eccentricity of the conic.

Let K be the projection of L on the directrix.

L lies on the conic = $\frac{LS}{LK} = e$

$\Rightarrow LS = eLK \Rightarrow \ell = eSZ \Rightarrow SZ = \ell / e$

Let $P(r, \theta)$ be a point on the conic.

Let M be the projection of P on the directrix and N be the projection of P on SZ.

P lies on the conic $\Rightarrow \frac{SP}{PM} = e \Rightarrow PS = ePM$

$\Rightarrow r = eNZ = e(SZ - SN) = e(SZ - SP\cos\theta)$

$= e\left(\frac{\ell}{e} - r\cos\theta\right)$

$= \ell - er\cos\theta$

$\Rightarrow r + re\cos\theta = \ell \Rightarrow r(1+\cos\theta) = \ell \Rightarrow \frac{\ell}{r} = 1 + e\cos\theta$

$\therefore$ The equation of the conic is $\frac{\ell}{r} = 1 + e\cos\theta$.

Note : If we take ZS as the initial line then the polar equation of the conic becomes $\frac{\ell}{r} = 1 - e\cos\theta$.

Note : The polar equation of a parabola is $\frac{\ell}{r} = 1 + \cos\theta \Rightarrow \frac{\ell}{r} = 2\cos^2\frac{\theta}{2}$.

Note : The polar equation of a parabola having latus rectum 4a is $\frac{2a}{r} = 2\cos^2\frac{\theta}{2} \Rightarrow a = r\cos^2\frac{\theta}{2}$.

Note :

The equation of the directrix of the conic $\frac{\ell}{r} = 1 + e\cos\theta$ is $\frac{\ell}{r} = e\cos\theta$.

Proof : Equation of the conic is $\frac{\ell}{r} = 1 + e\cos\theta$

Let Z be the foot of the perpendicular from S on the directrix $\Rightarrow$ SZ = ℓ / e.

Let Q(r,θ) be any point on the directrix of the conic

$\Rightarrow$ SZ = SQ cos θ $\Rightarrow \frac{\ell}{e} = r\cos\theta \Rightarrow \frac{\ell}{r} = e\cos\theta$, which is the equation of the directrix of the conic.

EXERCISE – 6(C)

1. Find the centre and radius of the circle $r^2 - 2r(3\cos\theta + 4\sin\theta) = 39$.

Sol.

Equation of the circle is $r^2 - 2r(3\cos\theta + 4\sin\theta) = 39$

$\sqrt{3^2 + 4^2} = 5$

$\Rightarrow r^2 - 2r.5\left(\frac{3}{5}\cos\theta + \frac{4}{5}\sin\theta\right) = 39$

Let $\cos\alpha = 3/5$, $\sin\alpha = 4/5$

$\Rightarrow r^2 - 10r(\cos\theta\cos\alpha + \sin\theta\sin\alpha) = 39$

$r^2 - 10r\cos(\theta - \alpha) = 39$ where $\tan\alpha = \frac{4}{3}$

Comparing with $r^2 - 2cr\cos(\theta - \alpha) = a^2 - c^2$

we get :

$c = 5$, $\alpha = \tan^{-1}(4/3)$

$a^2 - c^2 = 39 \Rightarrow a^2 = 39 + c^2 = 25+39 = 64$

$a^2 = 64 \Rightarrow a = 8$

centre is $c\left(5, \tan^{-1}\frac{4}{3}\right)$, radius $a = 8$.

2. Find the polar equation of the circle whose end points of the diameter are $\left(\sqrt{2}, \frac{\pi}{4}\right)$ and $\left(\sqrt{2}, \frac{3\pi}{4}\right)$.

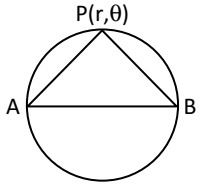
Sol.

Given $A\left(\sqrt{2}, \frac{\pi}{4}\right)$, $B\left(\sqrt{2}, \frac{3\pi}{4}\right)$ are the ends of the diameter of the circle.

Let $P(r, \theta)$ be any point on the circle.

AB is a diameter of the circle $\Rightarrow \angle APB = 90°$

$\therefore AP^2 + PB^2 = AB^2$ -----(1)



$$AP^2 = r^2 + 2 - 2r\sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right)$$

$$= r^2 + 2 - 2\sqrt{2}r\cos\left(\theta - \frac{\pi}{4}\right)$$

$$BP^2 = r^2 + 2 - 2r\cdot\sqrt{2}\cos\left(\theta - \frac{3\pi}{4}\right)$$

$$AB^2 = 2 + 2 - 2\sqrt{2}\cdot\sqrt{2}\cos\left(\frac{3\pi}{4} - \frac{\pi}{4}\right)$$

$= 4 - 4\cos 90° = 4 - 0 = 4$

From (1),

$$r^2 + 2 - 2\sqrt{2}r\cos\left(\theta - \frac{\pi}{4}\right) + r^2 + 2 - 2\sqrt{2}r\cos\left(\theta - \frac{3\pi}{4}\right) = 4$$

$$\Rightarrow 2r^2 - 2\sqrt{2}r\left[\cos\left(\theta - \frac{\pi}{4}\right) + \cos\left(\theta - \frac{3\pi}{4}\right)\right] = 0$$

$$\Rightarrow 2r^2 - 2\sqrt{2}r\left\{\cos\left(\theta - \frac{\pi}{4}\right) + \cos\left[-\pi + \left(\frac{\pi}{4} + \theta\right)\right]\right\} = 0$$

$$\Rightarrow 2r^2 - 2\sqrt{2}r\left[\cos\left(\theta - \frac{\pi}{4}\right) - \cos\left(\theta + \frac{\pi}{4}\right)\right] = 0$$

$$\Rightarrow 2r^2 - 2\sqrt{2}r\left[2\sin\theta\cdot\sin\frac{\pi}{4}\right] = 0$$

$$\Rightarrow 2r^2 - 2\sqrt{2}r\left(2\cdot\sin\theta\cdot\frac{1}{\sqrt{2}}\right) = 0$$

$$\Rightarrow 2r^2 - 4r\sin\theta = 0 \Rightarrow 2r(r - 2\sin\theta) = 0$$

$r - 2\sin\theta = 0 \Rightarrow r = 2\sin\theta$

Equation of the required circle is $r = 2\sin\theta$.

3. Find the polar equation of the circle with its centre at (4, π/6) and radius 5 units.

Sol. Centre is $c\left(4, \frac{\pi}{6}\right), r = 5$

$\therefore$ $c = 4$, $\alpha = \pi/6$, $a = 5$

Equation of the circle with centre (c, α) and radius 'a' is

$$r^2 - 2cr\cos(\theta - \alpha) + c^2 = a^2$$

$$r^2 - 8r\cos\left(\theta - \frac{\pi}{6}\right) + 16 = 25$$

$$r^2 - 8r\left(\theta - \frac{\pi}{6}\right) - 9 = 0$$

II.

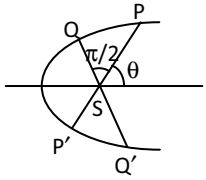
1. If PP′ and QQ′ are the perpendicular focal chords of the conic, prove that $\frac{1}{(SP)(SP')} + \frac{1}{(SQ)(SQ')} =$ constant.

Sol.

Equation of two conic is $\frac{\ell}{r} = 1 + e\cos\theta$

Given PP' and QQ' are two perpendicular focal chords.

Let P(SP,θ) where S is the focus then $\frac{\ell}{SP} = 1 + e\cos\theta \Rightarrow \frac{1}{SP} = \frac{1 + e\cos\theta}{\ell}$



Since PP' is focal chord, P′(SP′, 180° + θ)

$$\frac{\ell}{SP'} = 1 + e\cos(180° + \theta) = 1 - e\cos\theta$$

$$\frac{1}{SP'} = \frac{1 - e\cos\theta}{\ell}$$

PSP′, QSQ′ are perpendicular focal chords.

Coordinates of Q are $\left(SQ, \frac{\pi}{2} + \theta\right)$ and Q′ are $\left(SQ', \frac{3\pi}{2} + \theta\right)$.

$Q\left(SQ, \frac{\pi}{2} + \theta\right)$ is a point on the conic.

$$\frac{\ell}{SQ} = 1 + e\cos\left(\frac{\pi}{2} + \theta\right) = 1 - e\sin\theta$$

$$\frac{1}{SQ} = \frac{1 - e\sin\theta}{\ell}$$

$Q'\left(SQ', \frac{3\pi}{2} + \theta\right)$ is a point on the conic.

$$\frac{\ell}{SQ'} = 1 + e\cos\left(\frac{3\pi}{2} + \theta\right) = 1 + e\sin\theta$$

$$\frac{1}{SQ'} = \frac{1+e\sin\theta}{\ell}$$

$$\frac{1}{(SP)(SP')} + \frac{1}{(SQ)(SQ')} =$$

$$\frac{1+e\cos\theta}{\ell} \cdot \frac{1-e\cos\theta}{\cdot} + \frac{1-e\sin\theta}{\ell} \cdot \frac{1+e\sin\theta}{\ell}$$

$$= \frac{1-e^2\cos^2\theta+1-e^2\sin^2\theta}{\ell^2}$$

$= \frac{2-e^2}{\ell^2}$ which is a constant.

2. In a parabola, prove that the length of a focal chord which is inclined at π/6 to the axis is four times the length of the latus rectum.

Sol. Equation of the conic is $\frac{\ell}{r} = 1+e\cos\theta$

For a parabola e = 1

Equation of the parabola is $\frac{\ell}{r} = 1+\cos\theta$ and latusrectum is LL' = 2ℓ.

Let PP' be the given focal chord.

Let $P\left(SP, \frac{\pi}{6}\right)$ then $P'\left(SP', \frac{7\pi}{6}\right)$

P and P' are two points on the parabola, therefore

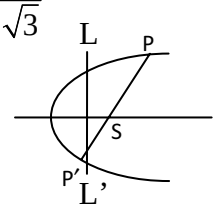
$$\frac{\ell}{SP} = 1+\cos 30° = 1+\frac{\sqrt{3}}{2} = \frac{2+\sqrt{3}}{2}$$

$$\Rightarrow SP = \frac{2\ell}{2+\sqrt{3}}$$

And $\frac{\ell}{SP'} = 1+\cos 210° = 1+\cos(180°+30°)$

$$= 1-\cos 30° = 1-\frac{\sqrt{3}}{2} = \frac{2-\sqrt{3}}{2}$$

$$SP' = \frac{2\ell}{2-\sqrt{3}}$$



$$PP' = SP + SP' = \frac{2\ell}{2+\sqrt{3}} + \frac{2\ell}{2-\sqrt{3}}$$

$$= \frac{2\ell(2-\sqrt{3}+2+\sqrt{3})}{4-3} = 8\ell = 4(2\ell)$$

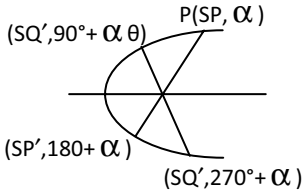
= 4 (latus rectum)

3. Prove that the perpendicular focal chords of a rectangular hyperbola are of equal length.

Sol.

Equation of the rectangular hyperbola is $\frac{\ell}{r} = 1 + \sqrt{2}\cos\theta$

($\because$ For a rectangular hyperbola $e = \sqrt{2}$)



Let PP' and QQ' be the perpendicular focal chords.

Let P(SP, α) then P′(SP′, 180° + α) , Q(SQ, 90°+α) and Q′(SQ′, 270°+α).

Since P and P' are points on the rectangular hyperbola,

$$\therefore \frac{\ell}{SP} = 1 + \sqrt{2}\cos\alpha \Rightarrow \frac{1}{SP} = \frac{1+\sqrt{2}\cos\alpha}{\ell}$$

$$SP = \frac{\ell}{1+\sqrt{2}\cos\alpha}$$

$$\frac{\ell}{SP'} = 1 + \sqrt{2}\cos(180° + \alpha) = 1 - \sqrt{2}\cos\alpha$$

$$SP' = \frac{\ell}{1-\sqrt{2}\cos\alpha}$$

$$PP' = SP + SP = \frac{\ell}{1+\sqrt{2}\cos\alpha} + \frac{\ell}{1-\sqrt{2}\cos\alpha}$$

$$= \frac{\ell(1-\sqrt{2}\cos\alpha+1+\sqrt{2}\cos\alpha)}{1-2\cos^2\alpha} = \frac{-2\ell}{\cos 2\alpha}$$

$$\therefore |PP'| = \frac{2\ell}{\cos 2\alpha} \qquad \ldots(1)$$

Q(SQ, 90°+α) is a point on the rectangular hyperbola , $\therefore$

$$\frac{\ell}{SQ} = 1 + \sqrt{2}\cos(90° + \alpha) = 1 - \sqrt{2}\sin\alpha$$

$$SQ = \frac{\ell}{1 - \sqrt{2}\sin\alpha}$$

$Q'(SQ', 270°+\alpha)$ is a point on the rectangular hyperbola , $\therefore$

$$\frac{\ell}{SQ'} = 1 + \sqrt{2}\cos(270° + \alpha)$$

$$= 1 + \sqrt{2}\sin\alpha = SQ' = \frac{\ell}{1 + \sqrt{2}\sin\alpha}$$

$$QQ' = SQ + SQ' = \frac{\ell}{1 - \sqrt{2}\sin\alpha} + \frac{\ell}{1 + \sqrt{2}\sin\alpha}$$

$$= \frac{\ell(1 + \sqrt{2}\sin\alpha + 1 - \sqrt{2}\sin\alpha)}{1 - 2\sin^2\alpha} = \frac{2\ell}{\cos 2\alpha} \quad \ldots(2)$$

From (1) and (2) we get $PP' = QQ'$.

PROBLEMS FOR PRACTICE

1. **Taking origin as the pole and the positive X-axis as the initial ray, find the Cartesian coordinates of point whose polar coordinates are $(1, -\pi/4)$.**

Ans. $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

2. **Taking origin as the pole and positive x-axis as initial ray, find the polar coordinates of the point p whose Cartesian coordinates are** $\left(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right)$**.**

Ans. Polar coordinates of p are $\left[3, -\frac{\pi}{4}\right]$

3. **Taking origin as pole and the positive x-axis as the initial ray, convert the following polar equation into Cartesian forms.**

i) $r^2 \sin 2\theta = 18$ **.Ans =** $xy = 9$ **ii)** $r = \frac{5}{\sin\theta - \cos\theta}$ **Ans =** $y - x = 5$

iii) $r = 5\cos\theta$ **Ans :** $x^2 + y^2 = 5x$ **iv)** $r\cos^2\frac{\theta}{2} = a$ **, Ans:** $y^2 + 4ax = 4a^2$

4. **Taking the origin as the pole and positive X-axis as initial ray, convert the following Cartesian equations into polar equations.**

   i) $x^2 - y^2 = 4y$ **ii)** $y = x \tan \alpha$

   iii) $x^3 = y^2(4 - x)$ **iv)** $x^2 + y^2 - 4x - 4y + 7 = 0$

Ans. i) $r \cos 2\theta = 4 \sin \theta$, ii) $\tan\theta = \tan\alpha \Rightarrow \theta = \alpha$, iii)$r \cos\theta = 4 \sin^2 \theta$,

iv)$r^2 + 7 = 4r(\cos\theta + \sin\theta)$

5. **Show that the points with polar coordinates (0, 0),** $\left(5, \frac{\pi}{18}\right), \left(5, \frac{7\pi}{18}\right)$ **form an equilateral triangle.**

6. **Find the area of the triangle formed by the points whose polar coordinates are** $\left(1, \frac{\pi}{6}\right), \left(2, \frac{\pi}{3}\right), \left(3, \frac{\pi}{2}\right)$.

7. **Find the polar equation of the straight line joining the points (3, 3π/4) and (2, π/4).**

Ans. $5r(\sin\theta - \cos\theta) = 6\sqrt{2}$

8. **Find the polar equation of the line passing through (3, π/3) and (i) parallel, (ii) perpendicular to the line** $3\cos\theta + 4\sin\theta = \frac{3}{r}$.

Ans. (i) $3\cos\theta + 4\sin\theta = \frac{3(3+4\sqrt{3})}{2r}$

ii) $4\cos\theta - 3\sin\theta = \frac{3(4-3\sqrt{3})}{2r}$

9. **Find the polar equation of a straight line passing through (4, 20°) and making an angle 140° with the initial ray.**

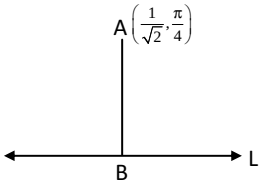
Ans. $r \cdot \cos(\theta - 50°) = 2\sqrt{3}$

10. **Find the polar coordinates of the point of intersection of the lines** $\cos\theta + \cos\left(\theta - \frac{\pi}{3}\right) = \frac{2}{r}$ **and** $\cos\theta + \cos\left(\theta + \frac{\pi}{3}\right) = \frac{2}{r}$.

Ans. P(4/3, 0°)

11. Find the foot of the perpendicular drawn from $\left(\frac{1}{\sqrt{2}}, \frac{\pi}{4}\right)$ on the line $\sin\theta - \cos\theta = \frac{1}{r}$.

Sol.



Let B be foot the perpendicular drawn from A on the line $\sin\theta - \cos\theta = \frac{1}{r}$.

$\therefore$ Equation of AB is

$$\sin\left(\frac{\pi}{2}+\theta\right) - \cos\left(\frac{\pi}{2}+\theta\right) = \frac{k}{r}$$

$$\cos\theta + \sin\theta = \frac{k}{r}$$

AB is passing through $A\left(\frac{1}{\sqrt{2}}, \frac{\pi}{4}\right)$

$$\Rightarrow \cos 45° + \sin 45° = \frac{k}{(1/\sqrt{2})}$$

$$k = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = \frac{1}{2} + \frac{1}{2} = 1$$

Equation of AB is $\cos\theta + \sin\theta = \frac{1}{r}$ …(1)

Equation of L is $\sin\theta - \cos\theta = \frac{1}{r}$ …(2)

Solving (1) and (2) :

$\cos\theta + \sin\theta = \sin\theta - \cos\theta$

$2\cos\theta = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \pi/2$

$$\therefore \frac{1}{r} = \sin\frac{\pi}{2} - \cos\frac{\pi}{2} = 1 - 0 = 1$$

$r = 1$.

The foot of the perpendicular from A on L is $B(1, \pi/2)$.

12.Find the equation of the circle centered at (8,120°) which pass through the point (4, 60°).

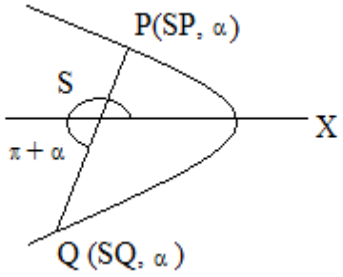
Ans. $r^2 - 16r\cos(\theta - 120°) + 16 = 0$

13. Find the centre and radius of the circle $r^2 - 8r(\sqrt{3}\cos\theta + \sin\theta) + 15 = 0$.

Ans. Centre (c, α) = $\left(8, \frac{\pi}{6}\right)$, Radius a = 7.

14. If PSQ is a chord passing through the focus S of a conic and l is the semi-latus rectum, show that $\frac{1}{SP} + \frac{1}{SQ} = \frac{2}{\ell}$.

Sol.



Let S be the focus and SX be the initial ray.

Equation of the conic is $\frac{\ell}{r} = 1 + e\cos\theta$

Let P(SP, α) be a point on the conic.

$\Rightarrow \frac{\ell}{r} = 1 + e\cos\alpha \Rightarrow \frac{\ell}{SP} = 1 + e\cos\alpha \Rightarrow \frac{1}{SP} = \frac{1 + e\cos\theta}{\ell}$ …(1)

Q(SQ, 180° + α) is a point on the conic.

$\Rightarrow \frac{\ell}{SP} = 1 + e\cos(180° + \alpha) = 1 - e\cos\alpha$

$\Rightarrow \frac{1}{SQ} = \frac{1 - e\cos\alpha}{\ell}$ …(2)

Adding (1) and (2)

$\frac{1}{SP} + \frac{1}{SQ} = \frac{1 + e\cos\alpha}{\ell} + \frac{1 - e\cos\alpha}{\ell}$

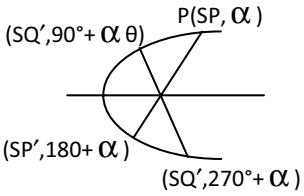
$= \frac{1 + e\cos\alpha + 1 - e\cos\alpha}{\ell} = \frac{2}{\ell}$

15. If PP′, QQ′ are perpendicular focal chords of a conic, show that $\frac{1}{PP'}+\frac{1}{QQ'}$ = a (constant).

Sol.

Let Let PP' and QQ' be the perpendicular focal chords of the conic $\frac{\ell}{r}=1+e\cos\theta$

Let P(SP, α) then P′(SP′, 180° + α) , Q(SQ, 90°+α) and Q′(SQ′, 270°+α).



$$SP=\frac{\ell}{1+e\cos\alpha}, SP'=\frac{\ell}{1-e\cos\alpha}$$

$$SQ=\frac{\ell}{1-e\sin\alpha}, SQ'=\frac{\ell}{1+e\sin\alpha}$$

$$PP = SP + SP' = \frac{\ell}{1+e\cos\alpha}+\frac{\ell}{1-e\cos\alpha}$$

$$=\frac{\ell(1-e\cos\alpha+1+e\cos\alpha)}{1-e^2\cos^2\alpha}=\frac{2\ell}{1-e^2\cos^2\alpha}$$

$$QQ = SQ + SQ' = \frac{\ell}{1-e\sin\alpha}+\frac{\ell}{1+e\sin\alpha}$$

$$=\frac{\ell(1+e\sin\alpha+1-e\sin\alpha)}{1-e^2\sin^2\alpha}=\frac{2\ell}{1-e^2\sin^2\alpha}\ \frac{1}{PP'}+\frac{1}{QQ'}=\frac{1-e^2\cos^2\alpha}{2\ell}+\frac{1-e^2\sin^2\alpha}{2\ell}$$

$$=\frac{2-e^2}{2\ell}(\text{constant})$$