# CHAPTER 5

# HYPERBOLA

**TOPICS:**

**1.Equation Of Hyperbola In Standard From-Parametric Form.**

**2.Various Forms And Certain Important Properties.**

**3.Equation Of Tangent And Normal, Pole-Polar, Conjugate Points And Lines.**

**4.Auxiliary Circle, Director Circle, Asymptotes.**

# HYPERBOLA

**DEFINITION**

A conic section is said to be a hyperbola if it's eccentricity is greater than 1.

**EQUATION OF A HYPERBOLA IN STANDARD FROM.**

**The equation of a hyperbola in the standard form is** $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$**.**

**Proof :**

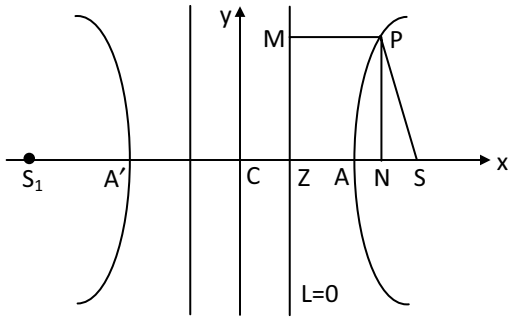
Let S be the focus, L = 0 be the directrix and e be the eccentricity of the hyperbola.

Let P be a point on the hyperbola. Let M, Z be the projections of P, S on the directrix L = 0 respectively.

Let N be the projection of P on SZ.

Let A, A′ be the points which are dividing SZ in the ratio e : 1 internally and externally respectively.

Let AA′ = 2a. Let C be the midpoint of AA′. The points A, A′ lie on the hyperbola $\Rightarrow \frac{SA}{AZ}=e$, $\frac{SA'}{A'Z}=e$.




$\therefore$ SA = eAZ, SA′ = eA′Z.

Now SA + SA′ = eAZ + eA′Z

$\Rightarrow$ CS – CA + CS + CA′ = e(AZ + A′Z)

$\Rightarrow$ 2CS = eAA′ ($\because$ CA = CA′)

$\Rightarrow$ 2CS = e2a $\Rightarrow$ CS = ae

Aslo SA′ – SA = eA′Z – eAZ

$\Rightarrow$ AA′ = e(A′Z – AZ)

$\Rightarrow$ 2a = e[CA′ + CZ – (CA – CZ)]

$\Rightarrow 2^a$ = e 2CZ ($\because$ CA = CA′) $\Rightarrow$ CZ = $\frac{a}{e}$.

Take CS, the principal axis of the hyperbola as
x-axis and Cy perpendicular to CS as y-axis. Then

C=(0,0), S = (ae, 0) .

Let P($x_1$, $y_1$) be a point on the hyperbola.

Now PM = NZ = CN – CZ = $x_1 - \frac{a}{e}$.

P lies on the hyperbola $\Rightarrow \frac{PS}{PM} = e$

$\Rightarrow PS = ePM \Rightarrow PS^2 = e^2PM^2$

$\Rightarrow (x_1 - ae)^2 + (y_1 - 0)^2 = e^2\left(x_1 - \frac{a}{e}\right)^2$

$\Rightarrow (x_1 - ae)^2 + y_1^2 = (x_1e - a)^2$

$\Rightarrow x_1^2 + a^2e^2 - 2x_1ae + y_1^2 = x_1^2e^2 + a^2 - 2x_1ae$

$\Rightarrow x_1^2(e^2 - 1) - y_1^2 = a^2(e^2 - 1)$

$\Rightarrow \frac{x_1^2}{a^2} - \frac{y_1^2}{a^2(e^2-1)} = 1 \Rightarrow \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$ where $b^2 = a^2(e^2 - 1)$

The locus of P is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

$\therefore$ The equation of the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

NATURE OF THE CURVE $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Let C be the curve represented by $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.Then

i) $(x, y) \in C \Leftrightarrow (x, -y) \in C$ and $(x, y) \in C \Leftrightarrow (-x, y) \in C$.

   Thus the curve is symmetric about both the x-axis and the y-axis. Hence the coordinate axes are two axes of the hyperbola.

ii) $(x, y) \in C \Leftrightarrow (-x, -y) \in C$.

   Thus the curve is symmetric about the origin O.

   Therefore the origin is the center of the hyperbola.

   Hence O is the midpoint of every chord of the hyperbola through O.

iii) $(x, y) \in C$ and $y = 0 \Rightarrow x^2 = a^2 \Rightarrow x = \pm a$.

Thus the curve meets x-axis (Principal axis) at two points A(a, 0), A′(–a, 0). Hence hyperbola has two vertices. The axis AA′ is called transverse axis. The length of transverse axis is AA′ = 2a.

iv) $(x, y) \in C$ and $x = 0 \Rightarrow y^2 = -b^2 \Rightarrow$ y is imaginary.

Thus the curve does not meet the y-axis. The points B(0, b), B′(0, –b) are two points on y-axis. The axis BB′ is called conjugate axis. BB′ = 2b is called the length of conjugate axis.

v) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow y = \frac{b}{a}\sqrt{x^2 - a^2} \Rightarrow$ y has no real value for $-a < x < a$.

Thus the curve does not lie between x = –a and x = a.

Further $x \to \infty \Rightarrow y \to \pm\infty$ and

$x \to -\infty \Rightarrow y \to \pm\infty$.

Thus the curve is not bounded (closed) on both the sides of the axes.

vi) The focus of the hyperbola is S(ae, 0). The image of S with respect to the conjugate axis is S′(–ae, 0). The point S′ is called second focus of the hyperbola.

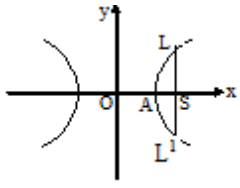
vii) The directrix of the hyperbola is x = a/e. The image of x = a/e with respect to the conjugate axis is x = –a/e. The line x = –a/e is called second directrix of the hyperbola corresponding to the second focus S′.

**THEOREM**

**The length of the latus rectum of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{2b^2}{a}$.**

**Proof**

Let LL′ be the length of the latus rectum of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.




Let f SL = l, since S= (ae,o) then L = (ae, l)

Since L is a point on the hyperbola $\Rightarrow \frac{(ae)^2}{a^2} - \frac{l^2}{b^2} = 1$

$$\Rightarrow \frac{l^2}{b^2} = e^2 - 1 \Rightarrow l^2 = b^2(e^2 - 1)$$

$$\Rightarrow l^2 = b^2 \times \frac{b^2}{a^2} \Rightarrow l = \frac{b^2}{a} \Rightarrow SL = \frac{b^2}{a}$$

$$\therefore LL' = 2SL = \frac{2b^2}{a}.$$

**THEOREM**

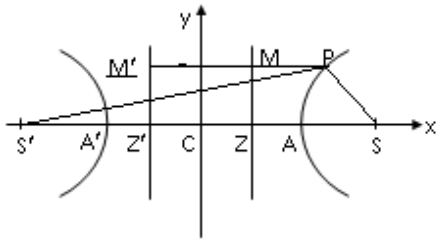
**The difference of the focal distances of any point on the hyperbola is constant i.e., if P is appoint on the hyperbola** $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ **with foci S and S′ then** $|PS' - PS| = 2a$

**Proof :**

Let e be the eccentricity and L = 0, L′ = 0 be the directrices of the hyperbola.

Let C be the centre and A, A′ be the vertices of the hyperbola.

∴ AA′ = 2a.




Foci of the hyperbola are S(ae, 0), S′(–ae, 0).

Let P($x_1$, $y_1$) be a point on the hyperbola.

Let M, M′ be the projections of P on the directrices L = 0, L′ = 0 respectively.

∴ $\frac{SP}{PM} = e, \frac{S'P}{PM'} = e$.

Let Z, Z′ be the points of intersection of transverse axis with directrices.

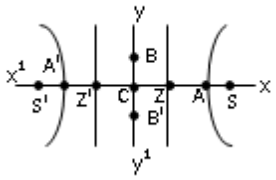
∴ $MM' = ZZ' = CZ + CZ' = 2a/e$

$$PS' - PS = ePM' - ePM = e(PM' - PM)$$

$$= e(MM') = e(2a/e) = 2a$$

Various from of the hyperbola

**1) For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$**




i) Centre = (0,0)

ii) foci ($\pm$ ae,0)

iii) Vertices ($\pm$ a,0)

iv) directrices x = $\pm \frac{a}{e}$

v) Transverse axis is x axis i.e., y = 0 and its length = 2a

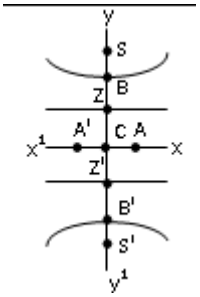
vi) Conjugate axis is y axis i.e., x = 0 and its length = 2b

vii) The length of the latus rectum = $\frac{2b^2}{a}$

viii) Equations of the latus rectum are x = $\pm$ ae

ix) Eccentricity e = $\sqrt{\frac{a^2+b^2}{a^2}}$

**2) For the hyperbola $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$**




i) Centre = (0,0)

ii) foci (0, $\pm$ be)

iii) Vertices $(0, \pm b)$

iv) Equation of the directrices $y = \pm \frac{b}{e}$

v) Equation to the transverse axis $x = 0$ and its length $= 2b$

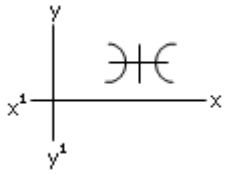
vi) Equation to the conjugate axis $y = 0$ and its length is $2a$.

vii) The length of the latus rectum $= \frac{2a^2}{b}$

viii) Equations of the latus rectum are $y = \pm be$

ix) Eccentricity $e = \sqrt{\frac{a^2+b^2}{b^2}}$

**3) The equation of the hyperbola with centre at (h, k) and its transverse axis parallel to x-axis is?**




$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

i) Centre = (h,k)

ii) foci = $(h \pm a, k)$

iii) Vertices $(h \pm ae, k)$

iv) Equation of the directrices $x = h \pm \frac{a}{e}$

v) Equation to transverse axis is $y-k = 0$ and its length $= 2a$

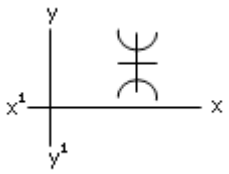
vi) Equation to Conjugate axis is $x-h = 0$ and its length $= 2b$

vii) The length of the latus rectum $= \frac{2b^2}{a}$

viii) Equations of the latus rectum are $x = h \pm ae$

ix) Eccentricity e = $\sqrt{\frac{a^2+b^2}{a^2}}$

**4) The equation of the hyperbola with centre at (h,k) and its transverse axis parallel to y-axis is ?**




$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$

i) Centre = (h,k)

ii) foci = (h,k $\pm$ b)

iii) Vertices (h,k$\pm$ be)

iv) Equation of the directrices y = k $\pm \frac{b}{e}$

v) Equation to transverse axis is x-h = 0 and its length = 2b

vi) Equation to Conjugate axis is y-k = 0 and its length = 2a

vii) The length of the latus rectum = $\frac{2a^2}{b}$

viii) Equations of the latus rectum are y = k$\pm$ be

ix) Eccentricity e = $\sqrt{\frac{a^2+b^2}{b^2}}$

**5) If P is a point on the hyperbola with foci S and $S^1$ then $|PS^1 - PS|$ = 2a**

**Notation :** We use the following notation in this chapter.

$$S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1,\ S_1 \equiv \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1,$$

$$S_{11} \equiv S(x_1, y_1) = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1,\ S_{12} \equiv \frac{x_1x_2}{a^2} - \frac{y_1y_2}{b^2} - 1.$$

**Note :**

Let $P(x_1, y_1)$ be a point and $S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$ be a hyperbola. Then

i) P lies on the hyperbola $S = 0 \Leftrightarrow S_{11} = 0$

ii) P lies inside the hyperbola $S = 0 \Leftrightarrow S_{11} > 0$

iii) P lies outside the hyperbola $S = 0 \Leftrightarrow S_{11} < 0$

**THEOREM**

**The equation of the chord joining the two points $A(x_1, y_1)$, $B(x_2, y_2)$ on the hyperbola $S = 0$ is $S_1 + S_2 = S_{12}$.**

**THEOREM**

The equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $P(x_1, y_1)$ is $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$.

**THEOREM**

The condition that the line $y = mx + c$ may be a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$c^2 = a^2m^2 - b^2$.

**Note :**

The equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ may be taken as $y = mx \pm \sqrt{a^2m^2 - b^2}$.

The point of contact is $\left( \frac{-a^2m}{c}, \frac{-b^2}{c} \right)$ where $c^2 = a^2m^2 - b^2$.

**THEOREM**

Two tangents can be drawn to a hyperbola from an external point.

**Note :**

If $m_1$, $m_2$ are the slopes of the tangents through P, then $m_1$, $m_2$ become the roots of $(x_1^2 - a^2)m^2 - 2x_1y_1m + (y_1^2 + b^2) = 0$.

Hence $m_1 + m_2 = \frac{2x_1y_1}{x_1^2 - a^2}$, $m_1m_2 = \frac{y_1^2 + b^2}{x_1^2 - a^2}$.

**THEOREM**

**The point of intersection of two perpendicular tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ lies on the circle $x^2 + y^2 = a^2 – b^2$.**

**Proof**

Equation of any tangent to the hyperbola is : $y = mx \pm \sqrt{a^2m^2 - b^2}$

Suppose $P(x_1, y_1)$ is the point of intersection of tangents.

P lies on the tangent $\Rightarrow$

$y_1 = mx_1 \pm \sqrt{a^2m^2 - b^2} \Rightarrow y_1 - mx_1 = \pm\sqrt{a^2m^2 - b^2}$

$\Rightarrow (y_1 - mx_1)^2 = a^2m^2 - b^2$

$\Rightarrow y_1^2 + m^2x_1^2 - 2mx_1y_1 - a^2m^2 + b^2 = 0$

$\Rightarrow m^2(x_1^2 - a^2) - 2mx_1y_1 + (y_1^2 + b^2) = 0$

This is a quadratic in m giving the values for m say $m_1$ and $m_2$.

The tangents are perpendicular :

$\Rightarrow m_1m_2 = -1 \Rightarrow \frac{y_1^2 + b^2}{x_1^2 - a^2} = -1$

$\Rightarrow y_1^2 + b^2 = -x_1^2 + a^2 \Rightarrow x_1^2 + y_1^2 = a^2 - b^2$

$P(x_1, y_1)$ lies on the circle $x^2 + y^2 = a^2 – b^2$.

**DEFINITION**

The point of intersection of perpendicular tangents to a hyperbola lies on a circle, concentric with the hyperbola. This circle is called director circle of the hyperbola.

**DEFINITION**

The feet of the perpendiculars drawn from the foci to any tangent to the hyperbola lies on a circle, concentric with the hyperbola. This circle is called auxiliary circle of the hyperbola.

**COROLLARTY**

The equation to the auxiliary circle of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $x^2 + y^2 = a^2$.

**THEOREM**

The equation to the chord of contact of $P(x_1, y_1)$ with respect to the hyperbola $S = 0$ is $S_1 = 0$.

**THEOREM**

**The equation of the polar of the point $P(x_1, y_1)$ with respect to the hyperbola $S = 0$ is $S_1 = 0$.**

**THEOREM**

The pole of the line $lx + my + n = 0$ $(n \neq 0)$ with respect to the hyperbola $S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$ is $\left(\frac{-a^2 l}{n}, \frac{b^2 m}{n}\right)$.

**Note**

The condition for the points $P(x_1, y_1)$, $Q(x_2, y_2)$ to be conjugate with respect to the hyperbola $S = 0$ is $S_{12} = 0$.

**MIDPOINT OF A CHORD THEOREM**

The equation of the chord of the hyperbola $S = 0$ having $P(x_1, y_1)$ as it's midpoint is $S_1 = S_{11}$.

**PAIR OF TANGENTS THEOREM**

The equation to the pair of tangents to the hyperbola $S = 0$ from $P(x_1, y_1)$ is $S_1^2 = S_{11}S$.

**ASYMPTOTES**

**DEFINITION**

The tangents of a hyperbola which touch the hyperbola at infinity are called asymptotes of the hyperbola.

**Note :**

1. The equation to the pair of asymptotes of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$.
2. The equation to the pair of asymptotes and the hyperbola differ by a constant.
3. Asymptotes of a hyperbola passes through the centre of the hyperbola.
4. Asymptotes are equally inclined to the axes of the hyperbola.
5. Any straight line parallel to an asymptote of a hyperbola intersects the hyperbola at only one point.

**THEOREM**

**The angle between the asymptotes of the hyperbola $S = 0$ is $2\tan^{-1}(b/a)$.**

**Proof:**

The equations to the asymptotes are $y = \pm\frac{b}{a}x$.

If $\theta$ is an angle between the asymptotes, then

$$\tan\theta = \frac{\frac{b}{a} - \left(-\frac{b}{a}\right)}{1 + \left(\frac{b}{a}\right)\left(-\frac{b}{a}\right)} = \frac{2\left(\frac{b}{a}\right)}{1 - \frac{b^2}{a^2}} = \frac{2\tan\alpha}{1 - \tan^2\alpha} = \tan 2\alpha \text{ where } \tan\alpha = \frac{b}{a}.$$

$$\therefore \theta = 2\alpha = 2\text{Tan}^{-1}\frac{b}{a}.$$

**PARAMETRIC EQUATIONS**

A point (x, y) on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ represented as x = a sec θ, y = b tan θ in a single parameter θ. These equations x = a sec θ, y= b tanθ are called parametric equations of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The point (a sec θ, b tan θ) is simply denoted by θ.

**Note :** A point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ can also be represented by (a cosh θ, b sinh θ). The equations x = a cosh θ, y = sinh θ are also called parametric equations of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

**THEOREM**

The equation of the chord joining two points α and β on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is :

$$\frac{x}{a}\cos\frac{\alpha - \beta}{2} - \frac{y}{b}\sin\frac{\alpha + \beta}{2} = \cos\frac{\alpha + \beta}{2}.$$

**THEOREM**

The equation of the tangent at P(θ) on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$.

**THEOREM**

The equation of the normal at P(θ) on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$.

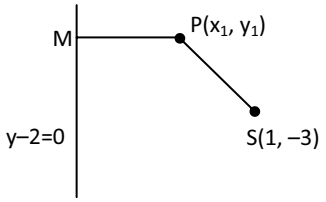
## EXERCISE -- 5(A)

**1. One focus of a hyperbola is located at the point (1, –3) and the corresponding directrix is the line y = 2. Find the equation of the hyperbola if its eccentricity is 3/2.**

**Sol.**

Focus S(1, –3) and directrix L= y – 2 = 0.

Eccentricity e = 3/2.




let $P(x_1, y_1)$ be any point on the hyperbola.

Let PM be the perpendicular from P to the directrix .

then $SP = e \cdot PM \Rightarrow SP^2 = e^2 \cdot PM^2$

$$\Rightarrow (x_1-1)^2+(y_1+3)^2 = \frac{9}{4}\left|\frac{y_1-2}{\sqrt{1+0}}\right|^2$$

$$\Rightarrow x_1^2+1-2x_1+y_1^2+9+6y_1 = \frac{9}{4}(y_1-2)^2$$

$$\Rightarrow 4x_1^2+4y_1^2-8x_1+24y_1+40 = 9(y_1^2+4-4y_1) = 9y_1^2-36y_1+36$$

$$\Rightarrow 4x_1^2-5y_1^2-8x_1+60y_1+4=0$$

Locus of $P(x_1, y_1)$ is $4x^2 – 5y^2 – 8x + 60y + 4 = 0$.

**2. If the lines 3x – 4y = 12 and 3x + 4y = 12 meets on a hyperbola S = 0 then find the eccentricity of the hyperbola S = 0.**

**Sol.**

Given lines $3x – 4y = 12$ , $3x + 4y = 12$

The combined equation of the lines is $(3x – 4y)(3x + 4y) = 144$

$$\Rightarrow 9x^2 – 16y^2 = 144$$

$$\Rightarrow \frac{x^2}{\frac{144}{9}}-\frac{y^2}{\frac{144}{16}}=1 \Rightarrow \frac{x^2}{16}-\frac{y^2}{9}=1$$

$$\Rightarrow a^2=16, b^2=9$$

Eccentricity e = $\sqrt{\frac{a^2+b^2}{a^2}}$ $= \sqrt{\frac{16+9}{16}} = \sqrt{\frac{25}{16}} = \frac{5}{4}$

**3. Find the equations of the hyperbola whose foci are (±5, 0) , the transverse axis is of length 8.?**

**Sol.** Foci are (±5, 0),

$\therefore$ SS'=2ae = 10. $\Rightarrow$ ae = 5.

Length of transverse axis is 2a = 8 $\Rightarrow$ a = 4

$\therefore$ e = 5/4

$b^2 = a^2(e^2 - 1) = 16\left(\frac{25}{9} - 1\right) = 9$

Equation of the hyperbola is $\frac{x^2}{16} - \frac{y^2}{9} = 1$

$\Rightarrow 9x^2 - 16y^2 = 144.$

**4. Find the equation of the hyperbola, whose asymptotes are the straight lines x + 2y +3 = 0, 3x + 4y + 5 = 0 and which passes through the point (1, –1).**

**Sol.** Combined equation of the asymptotes is

(x + 2y + 3)(3x + 4y + 5) = 0

$\therefore$ Equation of the hyperbola can be taken as

(x + 2y + 3)(3x + 4y + 5) + k = 0

Given the hyperbola is passing through p(1, –1)

$\Rightarrow$ (1 – 2 + 3)(3 – 4 + 5) + k = 0

$\Rightarrow 8 + k = 0 \Rightarrow k = -8$

Equation of the hyperbola is

(x + 2y + 3)(3x + 4y + 5) – 8 = 0

$\Rightarrow 3x^2 + 6xy + 9x + 4xy + 8y^2 + 12y + 5x + 10y + 15 - 8 = 0$

$\Rightarrow 3x^2 + 10xy + 8y^2 + 14x + 22y + 7 = 0.$

**5. If 3x – 4y + k = 0 is a tangent to $x^2 - 4y^2 = 5$, find value of k.**

**Sol.** Equation of the hyperbola $x^2 - 4y^2 = 5$

$\frac{x^2}{5} - \frac{y^2}{(5/4)} = 1 \Rightarrow a^2 = 5, b^2 = \frac{5}{4}$

Equation of the line is 3x – 4y + k = 0

$\Rightarrow 4y = 3x + k \Rightarrow y = \frac{3}{4}x + \frac{k}{4}$ ----(1)

$\Rightarrow m = \frac{3}{4}, c = \frac{k}{4}$

If (1) is a tangent to the hyperbola then

$c^2 = a^2m^2 - b^2$

$\Rightarrow \frac{k^2}{16} = 5 \cdot \frac{9}{16} - \frac{5}{4}$

$\Rightarrow k^2 = 45 - 20 = 25 \Rightarrow k = \pm 5$

**6. Find the product of lengths of the perpendiculars from any point on the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ to its asymptotes.**

**Sol.** Equation of the hyperbola is $\frac{x^2}{16} - \frac{y^2}{9} = 1$

$a^2 = 16, b^2 = 9$

Product of the perpendiculars from any point on the hyperbola to its asymptotes

$= \frac{a^2b^2}{a^2 + b^2} = \frac{16 \times 9}{16 + 9} = \frac{144}{25}$

**7. If the eccentricity of a hyperbola is 5/4, then find the eccentricity of its conjugate hyperbola.**

**Sol.** Eccentricity e = 5/4

If e and $e_1$ are the eccentricity of a hyperbola and its conjugate hyperbola, then

$\frac{1}{e^2} + \frac{1}{e_1^2} = 1$

$\Rightarrow \quad \frac{16}{25} + \frac{1}{e_1^2} = 1$

$\Rightarrow \quad \frac{1}{e_1^2} = 1 - \frac{16}{25} = \frac{9}{25} \Rightarrow e_1^2 = \frac{25}{9} \Rightarrow e_1 = \frac{5}{3}.$

**II.**

**1. Find the centre, foci, eccentricity, equation of the directrices, length of the lacus rectum of the following hyperbola.**

**i) $16y^2 - 9x^2 = 144$**

**Sol.**

Equation of the hyperbola is $16y^2 - 9x^2 = 144$

$\frac{y^2}{9} - \frac{x^2}{16} = 1 \Rightarrow a^2 = 16, b^2 = 9$

Centre c(0, 0)

Eccentricity $\sqrt{\frac{a^2 + b^2}{b^2}} = \sqrt{\frac{16+9}{9}} = \frac{5}{3}$

Foci are $(0, \pm be) = (0, \pm 5)$

Equation of the directrices are $y = \pm\frac{b}{e} \Rightarrow y = \pm 3 \cdot \frac{3}{5} \Rightarrow 5y = \pm 9$

Length of the latus rectum= $2 \cdot \frac{a^2}{b} = 2 \cdot \frac{16}{3} = \frac{32}{3}$

**ii) $x^2 - 4y^2 = 4$**

**Sol.** Equation of the hyperbola is $\frac{x^2}{4} - \frac{y^2}{1} = 1$

$a^2 = 4, b^2 = 1$

$\Rightarrow$ Centre c(0, 0)

$\Rightarrow$ Eccentricity $= \sqrt{\frac{a^2 + b^2}{a^2}} = \sqrt{\frac{4+1}{4}} = \frac{\sqrt{5}}{2}$

$\Rightarrow$ Foci are $(\pm ae, 0) = (\pm\sqrt{5}, 0)$

$\Rightarrow$ Equations of directrices are

$x = \pm\frac{a}{e} = \pm 2 \cdot \frac{2}{\sqrt{5}} \Rightarrow \sqrt{5}x = \pm 4 \Rightarrow \sqrt{5}x \pm 4 = 0$

Length of the latus rectum = $\frac{2b^2}{a} = \frac{2 \cdot 1}{2} = 1$

**iii) $5x^2 - 4y^2 + 20x + 8y = 4$**

**Sol.** Given equation is $5x^2 - 4y^2 + 20x + 8y = 4$

$\Rightarrow 5(x^2 + 4x) - 4(y^2 - 2y) = 4$

$\Rightarrow 5((x + 2)^2 - 4) - 4((y - 1)^2 - 1) = 4$

$\Rightarrow 5(x+2)^2 - 4(y-1)^2 = 20$

$\Rightarrow \frac{(x+2)^2}{4} - \frac{(y-1)^2}{5} = 1$

$\Rightarrow a^2 = 4, b^2 = 5 \Rightarrow a < b$

Centre C(–2, +1)=(h, k).

Eccentricity = $\sqrt{\frac{a^2+b^2}{a^2}} = \sqrt{\frac{4+5}{4}} = \frac{3}{2}$

$ae = 3$

Foci are $(h \pm ae, k)$

$= (-2 \pm 3, 1) = (-5, 1)$ and $(1, 1)$

Equations of directrices are

$x - h = \pm \frac{a}{e} \Rightarrow x + 2 = \pm 2 \cdot \frac{2}{3}$

$3x + 6 = \pm 4 \Rightarrow 3x + 10 = 0$ or $3x + 2 = 0$

Length of the latus rectum = $\frac{2b^2}{a} = \frac{2 \cdot 5}{2} = 5$

**iv) $9x^2 - 16y^2 + 72x - 32y - 16 = 0$**

**Sol.** Equation of the hyperbola is :

$9x^2 - 16y^2 + 72x - 32y - 16 = 0$

$\Rightarrow 9(x^2 + 8x) - 16(y^2 + 2y) = 16$

$\Rightarrow 9(x^2 + 8x + 16) - 16(y^2 + 2y + 1)$

$= 16 + 144 - 16$

$\Rightarrow 9(x + 4)^2 - 16(y + 1)^2 = 144$

$\Rightarrow \frac{(x+4)^2}{16} - \frac{(y+1)^2}{9} = 1$

Comparing with $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

$a^2 = 16, b^2 = 9, h = -4, k = -1$

Centre (h, k) = (–4, –1)

$e = \sqrt{\frac{a^2+b^2}{a^2}} = \sqrt{\frac{16+9}{16}} = \sqrt{\frac{25}{16}} = \frac{5}{4}$

Foci = $(h \pm ae, k) = \left(-4 \pm 4 \cdot \frac{5}{4}, 1\right)$

$= (-4 \pm 5, -1) = (1, -1)$ and $(-9, -1)$

Equation of the directrices are :

$x + 4 = \pm 4 \cdot \frac{4}{5} = \pm \frac{16}{5}$

$\Rightarrow 5x + 20 = \pm 16$

Equation of the directrices are $5x + 4 = 0$ and $5x + 36 = 0$

Length of the latus rectum = $2\frac{b^2}{a} = 2 \cdot \frac{9}{4} = \frac{9}{2}$

**2. Find the equation to the hyperbola whose foci are (4, 2) and (8, 2) and eccentricity is 2.**

**Sol**. Foci are S(4, 2) and $S^1$(8, 2) and eccentricity e =2.

Centre C = the mid point of the foci= $\left(\frac{4+8}{2}, \frac{2+2}{2}\right) = (6, 2)$

And $SS^1 = 2ae = 8 - 4 = 4$

$\Rightarrow ae = 2$

$\Rightarrow e = 2 \Rightarrow a.2 = 2 \Rightarrow a = 1.$

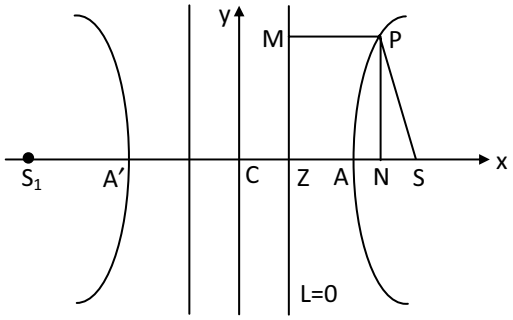
$b^2 = a^2(e^2 - 1) = 1(4 - 1) = 3$

Equation of the hyperbola is $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

$\Rightarrow \frac{(x-6)^2}{1} - \frac{(y-2)^2}{3} = 1$

**3. Find the equation of the hyperbola of given length of transverse axis is 6 whose vertex bisects the distance between the centre and the focus.**

**Sol.**



Let the hyperbola be $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Given CA = AS

$\Rightarrow a = ae - a \Rightarrow 2a = ae \Rightarrow e = 2$

Length of transverse axis is $2a = 6 \Rightarrow a = 3$

$b^2 = a^2(e^2 - 1) = 9(4 - 1) = 27$

Equation of the hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$\Rightarrow \frac{x^2}{9}-\frac{y^2}{27}=1 \Rightarrow 3x^2 - y^2 = 27$

**4. Find the equations of the tangents to the hyperbola $x^2 - 4y^2 = 4$ which are (i) parallel (ii) perpendicular to the line x + 2y = 0.**

**Sol.** Equation of the hyperbola is $x^2 - 4y^2 = 4$

$\frac{x^2}{4}-\frac{y^2}{1}=1 \Rightarrow a^2 = 4, b^2 = 1$

**i) given line is x + 2y = 0**

since tangent is parallel to x + 2y = 0, slope of the tangent is m = $-\frac{1}{2}$

$\Rightarrow c^2 = a^2m^2 - b^2 = 4\cdot\frac{1}{4}-1 = 1-1 = 0$

$\Rightarrow c = 0$

Equation of the parallel tangent is

$y = mx + c = -\frac{1}{2}x$

$\Rightarrow 2y = -x \Rightarrow x + 2y = 0$

**ii) The tangent is perpendicular to x + 2y = 0**

Slope of the tangent $m = \frac{-1}{(-1/2)} = 2$

$c^2 = a^2m^2 - b^2 = 4\cdot4 - 1 = 15$

$\Rightarrow c = \pm\sqrt{15}$

Equation of the perpendicular tangent is $y = 2x \pm \sqrt{15}$.

**5. Find the equations of tangents drawn to the hyperbola $2x^2 - 3y^2 = 6$ through (–2, 1).**

**Sol.** Equation of the hyperbola is $2x^2 - 3y^2 = 6$

$$\Rightarrow \frac{x^2}{3} - \frac{y^2}{2} = 1$$

Let m be the slope of the tangent .

The tangent id passing through p(–2, 1).

Equation of the tangent is

$y - 1 = m(x + 2) = mx + 2m$

$\Rightarrow y = mx + (2m + 1)$ …(1)

since (1) is a tangent to the hyperbola,

$c^2 = a^2m^2 - b^2$

$\Rightarrow (2m + 1)^2 = 3m^2 - 2$

$\Rightarrow 4m^2 + 4m + 1 = 3m^2 - 2$

$\Rightarrow m^2 + 4m + 3 = 0 \Rightarrow (m + 1)(m + 3) = 0$

$\Rightarrow m = -1$ or $-3$

**Case 1** $m = -1$

Equation of the tangent is

$y = -x - 1 \Rightarrow x + y + 1 = 0$

**Case 2** $m = -3$

Equation of the tangent is

$y = -3x - 5 \Rightarrow 3x + y + 5 = 0$

**III.**

**1. Tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ makes angles $\theta_1, \theta_2$ with transverse axis of a hyperbola. Show that the point of intersection of these tangents lies on the curve $2xy = k(x^2 - a^2)$ when $\tan\theta_1 + \tan\theta_2 = k$.**

**Sol.** Given hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Inclinations of the tangents are $\theta_1, \theta_2$.

$\Rightarrow$ slopes of the tangents are $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$

Equation of the tangent to the hyperbola is $y = mx \pm \sqrt{a^2m^2 - b^2}$

let $p(x_1, y_1)$ be the point of intersection of the tangents , then

$\Rightarrow y_1 = mx_1 \pm \sqrt{a^2m^2 - b^2}$

$\Rightarrow y_1 - mx_1 = \pm\sqrt{a^2m^2 - b^2}$

**s.o.b.s**

$(y_1 - mx_1)^2 = a^2m^2 - b^2$

$\Rightarrow y_1^2 + m^2x_1^2 - 2mx_1y_1 - a^2m^2 + b^2 = 0$

$m^2(x_1^2 - a^2) - 2mx_1y_1 + (y_1^2 + b^2) = 0$

Which is a quadratic equation in m. Therefore it has two roots from m , say $m_1, m_2$

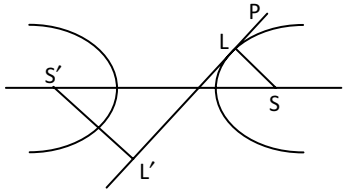
$m_1 + m_2 = \frac{2x_1y_1}{x_1^2 - a^2} \Rightarrow \tan\theta_1 + \tan\theta_2 = \frac{2x_1y_1}{x_1^2 - a^2}$

$\Rightarrow k = \frac{2x_1y_1}{x_1^2 - a^2}$ or $2x_1y_1 = k(x_1^2 - a^2)$

Therefore locus of $p(x_1, y_1)$ is $2xy = k(x^2 - a^2)$

**2. Show that the feet of the perpendiculars drawn from foci to any tangent of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ lie on the auxiliary circle of the hyperbola.**

**Sol.** Equation of the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$




Equation of the tangent to the hyperbola is

$y = mx \pm \sqrt{a^2m^2 - b^2}$

$\Rightarrow y - mx = \pm\sqrt{a^2m^2 - b^2}$ …(1)

Slope of the perpendicular line = -1/m

Equation of the line perpendicular to (1) and passing through foci (±ae,0) is

$y = -\frac{1}{m}(x \pm ae) \Rightarrow my = -(x \pm ae)$

$x + my = \pm ae$ …(2)

Squaring and adding (1) and (2)

$(y^2 - mx)^2 + (x + my)^2 = a^2m^2 - b^2 + a^2e^2$

$\Rightarrow y^2 + m^2x^2 - 2mxy + x^2 + m^2y^2 + 2mxy = a^2m^2 - a^2(e^2 - 1) + a^2e^2$

$\Rightarrow (x^2 + y^2)(1 + m^2) = a^2m^2 - a^2e^2 + a^2 + a^2e^2 = a^2(1 + m^2)$

$\Rightarrow x^2 + y^2 = a^2$ which is the auxiliary circle.

**3. Prove that the poles of normal chords of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ lie on the curve $\frac{a^6}{x^2} - \frac{b^6}{y^2} = (a^2 + b^2)^2$.**

**Sol.** Equation of the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Let $P(x_1, y_1)$ be the pole.

Equation of the polar is $S_1=0$

$\Rightarrow \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ …(1)

Equation of the normal to the hyperbola is

$$\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2 \qquad \ldots(2)$$

(1) and (2) are representing the

$$\therefore \frac{\left(\frac{x_1}{a^2}\right)}{\left(\frac{a}{\sec\theta}\right)} = \frac{\left(-\frac{y_1}{b^2}\right)}{\left(\frac{b}{\tan\theta}\right)} = \frac{1}{a^2 + b^2}$$

$$\frac{x_1 \sec\theta}{a^3} = \frac{y_1 \tan\theta}{-b^3} = \frac{1}{a^2 + b^2}$$

$$(a^2 + b^2)\sec\theta = \frac{a^3}{x_1} \qquad \ldots(i)$$

$$(a^2 + b^2)\tan\theta = -\frac{b^3}{y_1} \quad \ldots(ii)$$

**$(i)^2 - (ii)^2 \Rightarrow$**

$$(a^2 + b^2)^2(\sec^2\theta - \tan^2\theta) = \frac{a^6}{x_1^2} - \frac{b^6}{y_1^2}$$

Locus of $P(x_1, y_1)$ is $\frac{a^6}{x^2} - \frac{b^6}{y^2} = (a^2 + b^2)^2$

## PROBLEMS FOR PRACTICE

**1. Find the centre eccentricity, foci directrices and length of the latus rectum of the following hyperbola (i) $4x^2 - 9y^2 - 8x - 32 = 0$, (ii) $4(y + 3)^2 - 9(x - 2)^2 = 1$.**

**Sol**. i) $4x^2 - 9y^2 - 8x - 32 = 0$

$4(x^2 - 2x) - 9y^2 = 32$

$4(x^2 - 2x + 1) - 9y^2 = 36$

$$\frac{(x-1)^2}{9} - \frac{y^2}{4} = 1$$

Centre of the hyperbola is (1, 0)

$a^2 = 9, b^2 = 4 \Rightarrow a = 3, b = 2$

$$e = \sqrt{\frac{a^2 + b^2}{a^2}} = \sqrt{\frac{9+4}{9}} = \frac{\sqrt{13}}{3}$$

Foci are $\left(1 \pm 3 \cdot \frac{\sqrt{13}}{3}, 0\right) = (1 \pm \sqrt{13}, 0)$

Equations of directrices are $x = 1 \pm \frac{3 \cdot 3}{\sqrt{13}} \Rightarrow x = 1 \pm \frac{9}{\sqrt{13}}$

Length of the latusrectum = $\frac{2b^2}{a} = \frac{2 \cdot 4}{3} = \frac{8}{3}$

ii) The equation of the hyperbola is : $4(y + 3)^2 - 9(x - 2)^2 = 1$

$$\frac{y-(-3)^2}{1/4} - \frac{(x-2)^2}{1/9} = 1$$

Centre =(2, –3) =(h, k)

Semi transverse axis = b = 1/2

Semi conjugate axis = a = 1/3

$$e = \sqrt{\frac{a^2 + b^2}{b^2}} = \sqrt{\frac{(1/9) + (1/4)}{(1/4)}} = \sqrt{\frac{13}{9}} = \frac{\sqrt{13}}{3}$$

Foci are (h, k±be) = $\left(2, -3 \pm \frac{1}{2} \cdot \frac{\sqrt{13}}{3}\right) = \left(2, -3 \pm \frac{\sqrt{13}}{6}\right)$

Equations of the directrices are :

$$y = k \pm \frac{b}{e} = -3 \pm \frac{1}{2} \cdot \frac{3}{\sqrt{3}}$$

$$y = -3 \pm \frac{3}{2\sqrt{13}}$$

Length of latus rectum = $\frac{2a^2}{b} = \frac{2 \cdot \frac{1}{9}}{1/2} = \frac{4}{9}$.

**2. If e, $e_1$ be the eccentricity of a hyperbola and its conjugate hyperbola then $\frac{1}{e^2} + \frac{1}{e_1^2} = 1$.**

**Sol.** Equation of the hyperbola is S= $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$e = \sqrt{\frac{a^2 + b^2}{a^2}} \Rightarrow e^2 = \frac{a^2 + b^2}{a^2}$$

$$\therefore \quad \frac{1}{e^2} = \frac{a^2}{a^2 + b^2} \qquad \ldots(1)$$

Equation of the conjugate hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \Rightarrow \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

$$e_1 = \sqrt{\frac{a^2 + b^2}{b^2}} \Rightarrow e_1^2 = \frac{a^2 + b^2}{b^2} \Rightarrow \frac{1}{e_1^2} = \frac{b^2}{a^2 + b^2} \qquad \ldots(2)$$

Adding (1) and (2), $\frac{1}{e^2} + \frac{1}{e_1^2} = \frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2} = \frac{a^2 + b^2}{a^2 + b^2} = 1$

**3. i) If the line lx + my + n = 0 is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then show that $a^2l^2 - b^2m^2 = n^2$.**

**ii) If the line lx + my = 1 is a normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then show that $\frac{a^2}{l^2} - \frac{b^2}{m^2} = (a^2 + b^2)^2$.**

**Sol.** i) Equation of the hyperbola is S= $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Equation of the given tangent is lx + my + n = 0 …(1)

Equation of the tangent P(θ) is $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta - 1 = 0$ …(2)

Equations (1) and (2) are representing same lines,

$$\Rightarrow \frac{\sec\theta}{al} = \frac{\tan\theta}{-bm} = \frac{-1}{n}$$

$$\Rightarrow \sec\theta = -\frac{al}{n},\ \tan\theta = \frac{bm}{n}$$

$$\Rightarrow \sec^2\theta - \tan^2\theta = 1 \Rightarrow \frac{a^2l^2}{n^2} - \frac{b^2m^2}{n^2} = 1$$

$a^2l^2 - b^2m^2 = n^2$.

**ii) Equation of the hyperbola is**

$$S= \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Equation of the given line is $lx + my = 1$ …(1)

Equation of the normal at $P(\theta)$ is

$$\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2 \qquad \ldots(2)$$

Equations (1) and (2) are representing same lines,

$$\frac{l}{\left(\frac{a}{\sec\theta}\right)} = \frac{m}{\left(\frac{b}{\tan\theta}\right)} = \frac{1}{a^2+b^2}$$

$$\Rightarrow \frac{l\sec\theta}{a} = \frac{m\tan\theta}{b} = \frac{1}{a^2+b^2}$$

$$\Rightarrow \frac{(a^2+b^2)}{\sec^2\theta - \tan^2\theta} = \frac{a^2}{l^2} - \frac{b^2}{m^2}$$

The required condition is $\frac{a^2}{l^2} - \frac{b^2}{m^2} = (a^2+b^2)^2$

**4. Find the equations of the tangents to the hyperbola $3x^2 - 4y^2 = 12$ which are (i) parallel and (ii) perpendicular to the line $y = x - 7$.**

**Ans**. (i) $y = x \pm 1$ ; (ii) $x + y = \pm 1$

**5. Prove that the point of intersection of two perpendicular tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ lies on the circle $x^2+a^2 = a^2 - b^2$.**

**Sol.** Equation of the hyperbola is $S= \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Let $P(x_1, y_1)$ be the point of intersection of two perpendicular tangents to the hyperbola .

Equation of the tangent is

$y = mx \pm \sqrt{a^2m^2 - b^2}$

This tangent passes through $P(x_1, y_1)$

$\Rightarrow y_1 = mx_1 \pm \sqrt{a^2m^2 - b^2}$

$\Rightarrow (y_1 - mx_1)^2 = a^2m^2 - b^2$

$\Rightarrow y_1^2 + m^2x_1^2 - 2mx_1y_1 = a^2m^2 - b^2$

$\Rightarrow m^2x_1^2 - a^2m^2 - 2mx_1y_1 + y_1^2 + b^2 = 0$

$\Rightarrow m^2(x_1^2 - a^2) - 2mx_1y_1 + (y_1^2 + b^2) = 0$

which is a quadratic in m. Therefore it has two roots from m, say $m_1$, $m_2$ which are the slopes of the tangents passing through P.

The tangents are perpendicular $\Rightarrow m_1m_2 = -1$

$\frac{y_1^2 + b^2}{x_1^2 - a^2} = -1 \Rightarrow y_1^2 + b^2 = -x_1^2 + a^2$

$x_1^2 + y_1^2 = a^2 - b^2$

Locus of $P(x_1, y_1)$ is $x^2 + y^2 = a^2 - b^2$

This circle is called director circle of the hyperbola.

**6. Prove that the product of the perpendicular distance from any point on a hyperbola to its asymptotes is constant.**

**Sol**. Equation of the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Any point on the hyperbola is $P(a\sec\theta, b\tan\theta)$

Equation of the asymptotes are $\frac{x}{a} = \pm\frac{y}{b}$

$\Rightarrow \frac{x}{a} - \frac{y}{b} = 0$ and $\frac{x}{a} + \frac{y}{b} = 0$

PM = Perpendicular distance from P on $\frac{x}{a} - \frac{y}{b} = 0$

$= \frac{|\sec\theta - \tan\theta|}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}}$

PN = Perpendicular distance from P on $\frac{x}{a}+\frac{y}{b}=0=\frac{|\sec\theta+\tan\theta|}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}$

$$PM\cdot PN=\frac{|\sec\theta-\tan\theta|}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}\frac{|\sec\theta+\tan\theta|}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}$$

$$=\frac{|\sec^2\theta-\tan^2\theta|}{\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}=\frac{1}{\frac{a^2+b^2}{a^2b^2}}=\frac{a^2b^2}{a^2+b^2}\ =\text{constant}$$

**7. Show that the angle between two asymptotes of a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $2\tan^{-1}\left(\frac{b}{a}\right)$ or $2\sec^{-1}(e)$.**