CHAPTER 2

SYSTEMS OF CIRCLES

TOPICS:

1. **ANGLE BETWEEN TWO INTERSECTING CIRCLES.**
2. **ORTHOGONAL CIRCLES.**
3. **RADICAL AXIS, RADICAL CENTRE.**
4. **COAXIAL SYSTEM OF CIRCLES, SIMPLEST FORM.**
5. TYPES OF COAXIAL SYSTEMS
6. **LIMITING POINTS.**
7. **ORTHOGONAL SYSTEM.**

SYSTEM OF CIRCLES

ANGLE BETWEEN TWO CIRCLES

DEFINITION

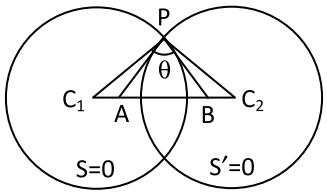
If two circles intersect at a point P, then the angle between the tangents of the two circles at P is called the angle between the circles at P.

THEOREM

If d is the distance between the centers of two intersecting circles with radii r_1, r_2 and θ is the angle between the circles then $\cos\theta = \dfrac{d^2 - r_1^2 - r_2^2}{2r_1r_2}$.

Proof :

Let C_1, C_2 be the centers of the two circles S = 0, S′ = 0 with radii r_1, r_2 respectively. Thus $C_1C_2 = d$. Let P be a point of intersection of the two circles. Let PB, PA be the tangents of the circles S = 0, S′ = 0 respectively at P.



Now $PC_1 = r_1$, $PC_2 = r_2$, $\angle APB = \theta$

Since PB is a tangent to the circle S = 0, $\angle C_1PB = \pi/2$

Since PA is a tangent to the circle S′ = 0, $\angle C_2PA = \pi/2$

Now $\angle C_1PC_2 = \angle C_1PB + \angle C_2PA - \angle APB = \pi/2 + \pi/2 - \theta = \pi - \theta$

From ΔC_1PC_2, by cosine rule,

$C_1^2C_2^2 = PC_1^2 + PC_2^2 - 2PC_1 \cdot PC_2 \cos\angle C_1PC_2 \Rightarrow$

$d^2 = r_1^2 + r_2^2 - 2r_1r_2\cos(\pi - \theta) \Rightarrow d^2 = r_1^2 + r_2^2 + 2r_1r_2\cos\theta$

$\Rightarrow 2r_1r_2\cos\theta = d^2 - r_1^2 - r_2^2 \Rightarrow \cos\theta = \dfrac{d^2 - r_1^2 - r_2^2}{2r_1r_2}$

COROLLARY

If θ is the angle between the circles $x^2 + y^2 + 2gx + 2fy + c = 0$, $x^2 + y^2 + 2g'x + 2f'y + c' = 0$ then $\cos\theta = \dfrac{c + c' - 2(gg' + ff')}{2\sqrt{g^2 + f^2 - c}\sqrt{g'^2 + f'^2 - c'}}$.

Proof :

Let C_1, C_2 be the centres and r_1, r_2 be the radii of the circles $S = 0$, $S' = 0$ respectively and $C_1C_2 = d$.

$\therefore$ $C_1 = (-g, -f)$, $C_2 = (-g', -f')$,

$r_1 = \sqrt{g^2 + f^2 - c}$, $r_2 = \sqrt{g'^2 + f'^2 - c'}$

Now $\cos\theta = \dfrac{d^2 - r_1^2 - r_2^2}{2r_1r_2} = \dfrac{(g - g')^2 + (f - f')^2 - (g^2 + f^2 - c) - (g'^2 + f'^2 - c')}{2\sqrt{g^2 + f^2 - c}\sqrt{g'^2 + f'^2 - c'}}$

$= \dfrac{g^2 + g'^2 - 2gg' + f^2 + f'^2 - 2ff' - g^2 - f^2 + c - g'^2 - f'^2 + c'}{2\sqrt{g^2 + f^2 - c}\sqrt{g'^2 + f'^2 - c'}}$

$= \dfrac{c + c' - 2(gg' + ff')}{2\sqrt{g^2 + f^2 - c}\sqrt{g'^2 + f'^2 - c'}}$

DEFINITION

Two intersecting circles are said to cut each other orthogonally if the angle between the circles is a right angle.

COROLLARY

Let d be the distance between the centers of two intersecting circles with radii r_1, r_2. The two circles cut orthogonally if $d^2 = r_1^2 + r_2^2$.

COROLLARY

The condition that the two circles $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, $S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0$ may cut each other orthogonally is $2gg' + 2ff' = c + c'$.

Proof :

Let C_1, C_2 be the centers and r_1, r_2 be the radii of the circles $S = 0$, $S' = 0$ respectively.

$\therefore$ $C_1 = (-g, -f)$, $C_2 = (-g', -f')$

$r_1 = \sqrt{g^2 + f^2 - c}$, $r_2 = \sqrt{g'^2 + f'^2 - c'}$

Let P be point of intersection of the circles.

The two circles cut orthogonally at P

$\Leftrightarrow \angle C_1PC_2 = 90^0$.

$\Rightarrow C_1C_2^2 = C_1P^2 + C_2P^2 \Leftrightarrow (g-g')^2 + (f-f')^2 = r_1^2 + r_2^2$

$\Leftrightarrow g^2 + g'^2 - 2gg' + f^2 + f'^2 - 2ff' = g^2 + f^2 - c + g'^2 + f'^2 + c'$

$\Leftrightarrow -(2gg' + 2ff') = -(c + c')$

$\Rightarrow 2gg' + 2ff' = c + c'$

NOTE:

1. The equation of the common chord of the intersecting circles s= 0 and s^1=0 is s-s^1 =0.
2. The equation of the common tangent of the touching circles s= 0 and s^1=0 is s-s^1 =0

3.If the circle s=0 and the line L =0 are intersecting then the equation of the circle passing through the points of intersection of s=0 and L=0 is S+ λ L =0.

4. The equation of the circle passing through the point of intersection of S=0 and S^1 =0 is S+ λS^1 =0.

EXERCISE – 2(a)

1. Find 'k' if the following pair of circles are orthogonal.

i) $x^2+y^2 + 2by-k = 0, x^2+y^2+2ax+8=0$

Sol. Given circles are $x^2+y^2 + 2by-k = 0, x^2+y^2+2ax+8=0$

from above equations $g_1 = 0$; $f_1 = b$; $c_1 = -k$

$g_2 = a$; $f_1 = 0$; $c_1 = 8$

since the circles are orthogonal , $2g_1g_2 + 2f_1f_2 = c_1 + c_2$

$2(0)(a) + 2(b)(0) = -k + 8$

$0 = -k + 8$

K = 8

ii) $x^2 + y^2 + 6x - 8y + 12 = 0; x^2 + y^2 - 4x + 6y + k = 0$

ANS k= -48

iii) $x^2+y^2 - 5x - 14y - 34 = 0; x^2+y^2 + 2x + 4y + k = 0$

Sol. **k = 1**

iv $x^2 + y^2 + 4x + 8 = 0; x^2 + y^2 - 196y + k = 0$

Ans: **k = −8**

2 Find the angle between the circles given by the equations

i) $x^2 + y^2 - 12x - 6y + 41 = 0;$ $x^2 + y^2 - 4x + 6y - 59 = 0$

Sol. Given circle is $x^2 + y^2 - 12x - 6y + 41 = 0$

Centre C_1 = (6,3) radius $r_1 = \{36 + 9 - 41\}^{1/2}$ =2

$x^2 + y^2 - 4x + 6y - 59 = 0$, then centre $C_2 = (-2,-3)$

radius $r_2 = \{4 + 9 - 59\}^{1/2} = \{72\}^{1/2} = 6\sqrt{2}$

$C_1C_2 = d = \sqrt{(6+2)^2 + (3+3)^2} = \sqrt{64+36} = 10$

Let θ be the angle between the circles, then $\cos\theta = \dfrac{d^2 - r_1^2 - r_1^2}{2r_1r_2}$

$= \dfrac{100 - 4 - 72}{2 \times 2\sqrt{72}} = \dfrac{24}{4 \times 6\sqrt{2}} = \dfrac{1}{\sqrt{2}}$

$\theta = 45^0$

ii) $x^2 + y^2 + 6x - 10y - 135 = 0$; $x^2 + y^2 - 4x + 14y - 116 = 0$.

Ans: $\theta = \dfrac{2\pi}{3}$

3. Show that the angle between the circles $x^2 + y^2 = a^3$ and $x^2 + y^2 = ax$ is $\dfrac{3\pi}{2}$

Sol. Equations of the circles are $S = x^2 + y^2 - a^2 = 0$,$S^1 = x^2 + y^2 - ax - ay = 0$

$C_1(0,0)$, $C_2\left(\dfrac{a}{2},\dfrac{a}{2}\right)$ $\therefore$ $C_1C_2^2 = \left(0-\dfrac{a}{2}\right)^2 + \left(0-\dfrac{a}{2}\right)^2$

$\dfrac{a^2}{4} + \dfrac{a^2}{4} = \dfrac{a^2}{2} = d^2$

$r_1 = a$, $r_2 = \sqrt{\dfrac{a^2}{4} + \dfrac{a^2}{4}} = \sqrt{\dfrac{a^2}{4}} = \dfrac{a}{\sqrt{2}}$

$\cos\theta = \dfrac{d^2 - r_1^2 - r_2^2}{2r_1r_2} = \dfrac{\frac{a^2}{2} - a^2 - \frac{a^2}{2}}{2.a\frac{a}{\sqrt{2}}} = \dfrac{-a^2}{\sqrt{2}.a^2} = \dfrac{-1}{\sqrt{2}} = \cos\dfrac{3\pi}{4}$

$\theta = \dfrac{3\pi}{4}$

4. Show that circles given by the following equations intersect each other orthogonally.

i). $x^2 + y^2 - 2x - 2y - 7 = 0$; $3(x^2 + y^2) - 8x + 29y = 0$

Sol. Given circles are

$x^2 + y^2 - 2x - 2y - 7 = 0$; $3(x^2 + y^2) - 8x + 29y = 0$

from above circles,

$g = -1$, $f = -1$, $c = -7$, $g^1 = \dfrac{-4}{3}$, $f^1 = \dfrac{29}{6}$; $c^1 = 0$

$c + c^1 = -7 + 0$

$2gg^1 + 2ff^1 = -2(-1)\left(\frac{-4}{3}\right) + 2(-1)\frac{29}{6}$

$= \frac{8}{3} - \frac{29}{3} = \frac{-21}{3} = -7$

Therefore, $2gg^1 + 2ff^1 = c + c^1$

Hence the given circles cut each other orthogonally.

Hence both the circles cut orthogonally.

ii). $x^2 + y^2 + 4x - 2y + 11 = 0$; $x^2 + y^2 - 4x + 8y + 11 = 0$.

iii) $x^2 + y^2 + 2x - 4y + 4 = 0$; $x^2 + y^2 - 3x + 4y + 1 = 0$

iv) **$x^2 + y^2 + 2lx + 4 = 0$; $x^2 + y^2 + 2my - g = 0$**

Sol. Given circles $x^2 + y^2 + 2lx + 4 = 0$; $x^2 + y^2 + 2my - g = 0$

from these equations,

$g_1 = -l$; $f_1 = 0$, $c_1 = g$, $g_2 = 0$, $f_2 = m$, $c_2 = -g$

now $2g_1g_2 + 2f_1f_2 = c_1 + c_2$

$2(-l)(0) + 2(0)(m) = g - g$

$0 = 0$

$\therefore$ Two circles are orthogonal.

II SHORT ANSWER QUESTIONS

1 Find the equation of the circle which passes through the origin and intersects the circles given by the following equations orthogonally.

i) $x^2 + y^2 - 4x + 6y + 10 = 0$, $x^2 + y^2 + 12y + 6 = 0$

Sol. Let equation of the circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$ ----(1)

Above circle is passing through (0,0),

$0+0+0+0+c = 0 \Rightarrow c=0$.

Circle (1) is orthogonal to

$x^2 + y^2 - 4x + 67 + 10 = 0$ then

$2gg^1 + 2ff^1 = c + c^1$

$2g(-2) + 2f(3) = 0 + 10$

$-4g + 6f = 10$ -----(2)

Circle (1) is orthogonal to

$x^2 + y2 + 12y + 6 = 0$

$\therefore$ $2g(0) + 2f(6) = 6 + 0$

$12f = 6$------(3)

$f = \frac{1}{2}$

from (2) and (3) ,

$-4g + 6 \frac{1}{2} = 10$

$-4g = 10 - 3$

$g = -\frac{7}{4}$

$\therefore$ Equation of circle is $x^2 + y^2 - \frac{7}{2} x + y = 0$

$2x^2 + 2y^2 - 7x + 2y = 0.$

2. Find the equation of the circle which passes through the point (0, -3) and intersects the circles given by the equations $x^2 + y^2 - 6x + 3y + 5 = 0$, $x^2 + y^2 - x - 7y = 0$ orthogonally.

Sol.

Let circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$-------(1)

(1) is orthogonal to $x^2 + y^2 - 6x + 3y + 5 = 0$

$\therefore$ $2g(-3) + 2f\left(\frac{+3}{2}\right) = c + 5$

$-6g + 3f = c + 5$---------- (2)

(1) is orthogonal to $x^2 + y^2 - x - 7y = 0$

$\therefore$ $2g\left(\frac{+1}{2}\right) + 2f\left(\frac{+7}{2}\right) = c$

$-g - 7f = c$ ----(3)

Given (1) is passing through (0, –3)

$0 + 9 - 6f + c = 0$

(3) – (2)

$5g - 10f = -5$

$g - 2f = -1$

(iii) + (iv)

$9 - g - 13f = 0 \Rightarrow g + 13f = 9$

$\underline{g - 2f = -1}$

$15f = 10$

$f = \frac{2}{3}$

$g = 2.\frac{2}{3} - 1$

$g = +\frac{1}{3}$

$9 - 6.\frac{2}{3} + c = 0$

$c = -5$

$x^2 + y^2 + \frac{4}{3}y + \frac{4}{3}x - 5 = 0$

(or) $3x^2 + 3y^2 + 2x + 4y - 15 = 0$

(3) Find the equation of the circle passing through the origin, having its centre on the line $x + y = 4$ and intersecting the circle $x^2 + y^2 - 4x + 2y + 4 = 0$ orthogonally.

Sol. **let** $S = x^2 + y^2 + 2gx + 2fy + c = 0$

$S=0$ is passing through $(0, 0)$

$0 + 0 + 2g.0 + 2f.0 + c = 0$

$x^2 + y^2 + 2gx + 2fy = 0$

Centre $(-g,-f)$ is on $x + y = 4$

$\therefore\ -g - f = 4$----------(1)

$S=0$ is orthogonal to

$x^2 + y^2 - 4x + 2y + 4 = 0$

$-4g + 2f = 4 + 0$

$f - 2g = 2$ -------(2)

Solving (1) and (2) we get

$-3g = 6$

$g = -2$

$f = -2$

Equation of circle is $x^2 + y^2 - 4x - 4y = 0$

4. Find the equation of the circle which passes through the points (2, 0), (0, 2) and orthogonal to the circle $2x^2 + 2y^2 + 5x - 6y + 4 = 0$

Sol.

Let $S = x^2 + y^2 + 2gx + 2fy + c = 0$

$S=0$ is passing through $(2, 0)$, $(0, 2)$,

$4 + 0 + 4g + c = 0$ ----(1)

and $0 + 4 + 4f + c = 0$ -----(2)

(1) –(2) $f-g = 0 \Rightarrow g = f$

$S=0$ is orthogonal to $x^2 + y^2 + \frac{5}{2}x - \frac{6}{2}y + 2 = 0$

$2g\left(\frac{5}{4}\right) + 2f\left(-\frac{3}{2}\right) = 2 + c$

$\frac{5}{2}g - 3f = 2 + c$

But $g = f$

$\frac{5}{2}$ g – 3g = 2 +c

$\Rightarrow$ – g = 4 + 2c

Putting value of g in equation (1)

–16–8c + c = –4

$c = -\frac{12}{7}$

$-g = 4 - \frac{24}{7} = +\frac{4}{7}$

Equation of the circle is

$x^2 + y^2 - \frac{8x}{7} + \frac{8y}{7} + \frac{12}{7} = 0$

$\Rightarrow 7(x^2 + y^2) - 8x - 8y - 12 = 0$

(5) Find the equation of the circle which cuts orthogonally the circle $x^2 + y^2 - 4x + 2y - 7 = 0$ and having centre at (2,3)

Sol. Given circle is

$x^2 + y^2 - 4x + 2y - 7 = 0$ -----(1)

Let the required circle be

$S = x^2 + y^2 + 2gx + 2fy + c = 0$

centre (–g, –f) = (2,3) given

g = –2, f = – 3

circles (1) and S=0 are cutting each other orthogonally.

$2g\,g^1 + 2f\,f^1 = c + c^1$

2(–2) (–2) + 2(–3) (1) = – 7 + c

8 – 6 = – 7 + c

+ 2 = – 7 + c

c = 7 + 2 = 9 $\Rightarrow$ c = 9

Hence the required circle is,

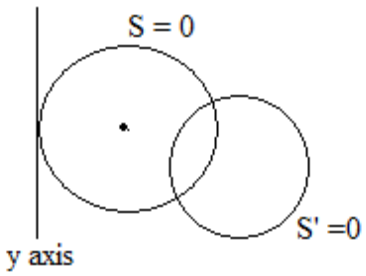
$x^2 + y^2 - 4x - 6y + 9 = 0$

III LONG ANSWER QUESTIONS

1 Find the equation of the circle which intersects the circle $x^2 + y^2 - 6x + 4y - 3 = 0$ orthogonally and passes through the point (3, 0) and touches the Y–axis.

Sol. let (h,k) be the centre of the circle.

Since the circle is touching the y axis, therefore radius is |h|



Therefore equation of the circle is

$(x - h)^2 + (y - k)^2 = h^2$

$S = x^2 - 2hx + y^2 - 2ky + k^2 = 0$

S=0 is Passing through (3,0),

$\Rightarrow 9 - 6h + k^2 = 0$ — (i)

S=0 is Orthogonal to $x^2 + y^2 - 6x + 4y - 3 = 0$

$\Rightarrow 2(-h)(-3) + 2(-k)(2) = -3 + k^2 \quad \Rightarrow 6h - 4k = -3 + k^2$

$\Rightarrow 6h - 4k + 3 - k^2 = 0$-----(2)

(1) + (2) $\Rightarrow$ $12 - 4k = 0 \Rightarrow k = 3 \Rightarrow h = 3$,

Equation of circle be

$y^2 + x^2 - 6x - 6y + 9 = 0$

(2) Find the equation of the circle which cuts the circles $x^2 + y^2 - 4x - 6y + 11 = 0$, $x^2 + y^2 - 10x - 4y + 21 = 0$ orthogonally and has the diameter along the straight line $2x + 3y = 7$.

Sol.

Let circle be $S = x^2 + y^2 + 2gx + 2fy + c = 0$

S=0 is Orthogonal to **$x^2 + y^2 - 4x - 6y + 11 = 0$, $x^2 + y^2 - 10x - 4y + 21 = 0$**

$\Rightarrow 2g(-2) + 2f(-3) = 11 + c$ —— (1)

$\Rightarrow 2g(-5) + 2f(-2) = 21 + c$ —— (2)

(1) –(2) $\Rightarrow$ $-6g + 2f = 10$ —— (3)

centre (-g,-f) is on $2x + 3y = 7$,

$\therefore -2g - 3f = 7$ —— (4)

Solving (3) and (4)

$f = -1$, $g = -2$,

sub. These values in (1), then $c = 3$

Equation of circle $x^2 + y^2 - 4x - 2y + 3 = 0$

3) If P,Q are conjugate points with respect to a circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ then prove that the circle PQ as diameter cuts the circles S = 0 orthogonally.

Sol. Equation of the circle is

$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$

Let $P = (x_1,y_1)$, $Q (x_2,y_2)$ be the conjugate points w.r.t. the circle S=0.

Since $P = (x_1,y_1)$, $Q (x_2,y_2)$ are conjugate points, $S_{12} = 0$.

$\Rightarrow x_1x_2 + y_1y_2+g(x_1+x_2)+f(y_1+y_2)+c=0$

$\Rightarrow x_1x_2 + y_1y_2 +c= - g(x_1+x_2)-f(y_1+y_2)$----(1)

Equation of the circle on PQ as diameter is

$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$

$\Rightarrow S^1 = x^2 + y^2 - (x_1+x_2)x - (y_1+y_2)y + (x_1x_2+y_1y_2) = 0.$

Given S=0 and $S^1 = 0$ are orthogonal,

$2 g_1g_2 + 2f_1f_2$

$= 2g\left[-\left(\frac{x_1 + x_2}{2}\right)\right] + 2f\left[-\left(\frac{y_1 + y_2}{2}\right)\right]$

$= -g(x_1+x_2)-f(y_1+y_2) = x_1x_2 + y_1y_2 +c$

And $C + c^1 = x_1x_2 + y_1y_2 + c$

$\therefore 2g_1 g_2 + 2f_1 f_2 = C + c^1$

Hence circles are orthogonal to each other.

4) **If the equation of two circles whose radii are a, a^1 be S = 0, S^1=0, then show that the circles $\frac{S}{a} \pm \frac{S'}{a'} = 0$ will interest orthogonally.**

5) Find the equation of the circle which intersects each of the following circles orthogonally
$x^2 + y^2 + 2x + 4y + 1 = 0$; $x^2 + y^2 - 2x + 6y - 3 = 0$; $2x^2 + 2y^2 + 6x + 8y - 3 = 0$

Sol. Let equation of circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$

this circle is orthogonal to **$x^2 + y^2 + 2x + 4y + 1 = 0$; $x^2 + y^2 - 2x + 6y - 3 = 0$;**
$x^2 + y^2 + 3x + 4y - 3/2 = 0$

$2g(1) + 2f(2) = c + 1$ - (i)

$2g\left(\frac{3}{2}\right) + 2f(2) = c - \frac{3}{2}$ - (ii)

$2g(-1) + 2f(3) = c - 3$ - (iii)

(iii) – (i)

$-5g + 2f = \frac{-3}{2}$ (or) $-10g + 4f = -3$ –(iv)

(iii) – (i)

$-4g + 2f = -4$

$f - 2g = -2$

Solving (iv) and (v) we get

f = -7, g = - 5/2, c = -34

∴ Equation of circle be

$x^2 + y^2 – 5x –14y - 34 = 0$

ii) $x^2 + y^2 + 4x + 2y + 1 = 0$;

$2x^2 + 2y^2 + 8x + 6y - 3 = 0$;

$x^2 + y^2 + 6x – 2y - 3 = 0$.

Ans. $x^2 + y^2 – 5y – 14x - 34 = 0$

6) If the Straight line 2x + 3y = 1 intersects the circle $x^2 + y^2 = 4$ at the points A and B. Find the equation of the circle having AB as diameter.

Sol. circle is S = $x^2 + y^2 = 4$

Equation of the line is L = 2x + 3y = 1

Equation of circle passing through S=0 and L=0 is $S+\lambda L =0$

$\Rightarrow (x^2 + y^2 – 4) + \lambda (2x + 3y – 1) = 0$

$\Rightarrow x^2 + y^2 + 2\lambda x + 3\lambda y – 4 - \lambda = 0$

$\Rightarrow$ Center $\left(-\lambda, \frac{-3\lambda}{2}\right)$

Centre lies on 2x + 3y – 1 = 0

$\Rightarrow 2(-\lambda) + 3\frac{-3\lambda}{2} - 1 = 0$

$\lambda = \frac{-2}{13}$

∴ Equation of circle be

$13 (x^2 + y^2) - 4 \times 13 – 2(2x + 3y - 1) = 0$

$13 (x^2 + y^2) - 4x – 6y - 50 = 0$.

7) if x + y = 3 is the equation of the chord AB of the circle $x^2 + y^2 - 2x + 4y - 8 = 0$, Find the equation of the circle having AB as diameter.

Ans: $x^2 + y^2 - 6x + 4 = 0$

8) **Find the equation of the circle passing through the intersection of the circles $x^2 + y^2 = 2ax$ and $x^2 + y^2 = 2by$ and having its center on the line $\frac{x}{a} - \frac{y}{b} = 2$.**

Sol.

Let $S = x^2 + y^2 = 2ax$

$S^1 = x^2 + y^2 = 2by$

Equation of circle passes through the point of intersected of $S = 0$ and $S^1 = 0$ can be written as $S+\lambda S^1 = 0$

$\Rightarrow x^2 + y^2 - 2ax + \lambda (x^2 + y^2 - 2by) = 0$

$\Rightarrow x^2 (1+\lambda) + y^2 (1+\lambda) + x(-2a) – (2b\lambda)y = 0$

$\Rightarrow x^2 + y^2 - \frac{2ax}{1+\lambda} - \frac{2b\lambda y}{1+\lambda} = 0$

Centre $C = \left[\frac{+a}{1+\lambda}, \frac{+b\lambda}{1+\lambda}\right]$

Centre is a point on $\frac{x}{a} - \frac{y}{b} = 2$

$\Rightarrow \quad \frac{+a}{a(1+\lambda)} - \frac{b\lambda}{(1+\lambda)b} = 2$

$\Rightarrow \quad 1 - \lambda = 2(1+\lambda)$

$\Rightarrow \quad \lambda = -1/3$

Equation of circle be

$3x^2 + 3y^2 - 6ax - x^2 - y^2 + 2by = 0$

$\Rightarrow 2x^2 + 2y^2 - 6ax + 2by = 0$

$\Rightarrow x^2 + y^2 - 3ax + by = 0.$

RADICAL AXIS

The locus of the point, for which the powers with respect to two given non concentric circles are equal is a straight line, called the radical axis of the given circles.

THEOREM

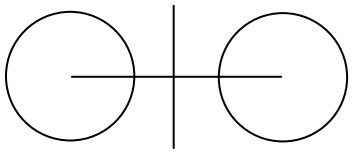
The equation of the radical axis of the circles $S= 0$, $S' = 0$ is $S - S' = 0$.

THEOREM

The radical axis of two circles is perpendicular to their line of centers.

Proof :

Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, $S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0$ be the given circles.



The equation of the radical axis is $S - S' = 0$

$\Rightarrow \quad 2(g-g')x + 2(f-f')y + (c-c') = 0$

$\Rightarrow a_1x + b_1y + c_1 = 0$ where

$a_1 = 2(g - g')$, $b_1 = 2(f - f')$, $c_1 = c - c'$

The centers of the circles are $(-g, -f)$, $(-g', -f')$

The equation to the line of centers is :

$(x + g)(f - f') = (y + f)(g - g')$

$\Rightarrow \quad (f-f')x - (g-g')y - gf' + fg' = 0$

$\Rightarrow \quad a_2x + b_2y + c_2 = 0$ where

$a_2 = f - f'$, $b_2 = -(g - g')$, $c_2 = fg' - gf'$

Now $a_1a_2 + b_1b_2 =$

$2(g - g')(f - f') - 2(f - f')(g - g') = 0.$

THEOREM

If two circles intersect, then the radical axis is their common chord.

THEOREM

If two circles touch each other, then the radical axis is their common tangent at the point of contact.

THEOREM

The radical axes of three circles, whose centers are non-collinear, taken in pairs, are concurrent.

DEFINITION

The point of concurrence of the radical axes of three circles, whose centers are non collinear, taken in pairs, is called the radical centre of the circles.

THEOREM

The center of a circle cutting two circles orthogonally lies on the radical of the two circles.

EXERCISE 2B

1. Find the equation of the radical axis of the following circles.

i) $x^2 + y^2 - 3x - 4y + 5 = 0,\ 3(x^2 + y^2) - 7x + 8y + 11 = 0$

Sol. let $S \equiv x^2 + y^2 - 3x - 4y + 5 = 0$

$$S' = x^2 + y^2 - \frac{7}{3}x + \frac{8}{3}y + \frac{11}{3} = 0$$

Radical axis is $S - S' = 0$

$$(x^2 + y^2 - 3x - 4y + 5) - \left(x^2 + y^2 - \frac{7}{3}x + \frac{8}{3}y + \frac{11}{3}\right) = 0$$

$$-\frac{2}{3}x - \frac{20}{3}y + \frac{4}{3} = 0 \Rightarrow x + 10y - 2 = 0$$

ii) $x^2 + y^2 + 2x + 4y + 1 = 0,\ x^2 + y^2 + 4x + y = 0.$

Ans; $2x - 3y + 1 = 0$

iii) $x^2 + y^2 + 4x + 6y - 7 = 0, 4(x^2 + y^2) + 8x + 12y - 9 = 0.$

Ans. $8x + 12y - 19 = 0$

iv) $x^2 + y^2 - 2x - 4y - 1 = 0,\ x^2 + y^2 - 4x - 6y + 5 = 0$

Ans. $x + y - 3 = 0$

2. Find the equation of the common chord of the following pair of circles.

i) **$x^2 + y^2 - 4x - 4y + 3 = 0,\ \ x^2 + y^2 - 5x - 6y + 4 = 0$**

Sol. $S = x^2 + y^2 - 4x - 4y + 3 = 0$

$S^1 = x^2 + y^2 - 5x - 6y + 4 = 0$

Common chord is $S - S' = 0$

$(x^2 + y^2 - 4x - 4y + 3) - (x^2 + y^2 - 5x - 6y + 4) = 0$

$x + 2y - 1 = 0$

ii) $x^2 + y^2 + 2x + 3y + 1 = 0,\ x^2 + y^2 + 4x + 3y + 2 = 0.$

Ans. $-2x - 1 = 0$

iii) $(x - a)^2 + (y - b)^2 = c^2,\ (x - b)^2 + (y - a)^2 = c^2\ (a \neq b)$

Ans. $x - y = 0.$

II.

1. Find the equation of the common tangent of the following circles at their point of contact.

i) $x^2 + y^2 + 10x - 2y + 22 = 0,\ x^2 + y^2 + 2x - 8y + 8 = 0.$

Sol. $S=x^2 + y^2 + 10x - 2y + 22 = 0$

Centre A $=(-5, 1)$, radius $r_1 = 2$

$S' = x^2 + y^2 + 2x - 8y + 8 = 0.$

Centre B $=(-1, 4)$ radius $r_2 = 3$

AB $= \sqrt{16+9} = 5$

Therefore AB $=5 = 3+2 = r_1+r_2.$

Given circles touch each other externally.

When circles touch each other ,their common tangent is $S - S' = 0$

$\therefore (x^2 + y^2 + 10x - 2y + 22) - (x^2 + y^2 + 2x - 8y + 8) = 0$

$8x + 6y +14 = 0$ (or) $4x + 3y + 7 = 0$

ii) $x^2 + y^2 - 8y - 4 = 0,\ x^2 + y^2 - 2x - 4y = 0.$

Ans. $x - 2y - 2 = 0$

2. Show that the circles $x^2 + y^2 - 8x - 2y + 8 = 0$ and $x^2 + y^2 - 2x + 6y + 6 = 0$ touch each other and find the point of contact.

Sol. $S = x^2 + y^2 - 8x - 2y + 8 = 0$

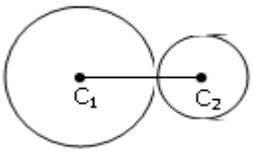
$C_1 = (4, 1)\ ;\ r_1 = \sqrt{16+1-8} = 3$

$S^1 = x^2 + y^2 - 2x + 6y + 6 = 0$

$C_2 = (1, -3),\ r_2 = \sqrt{1+9-6} = 2$

$C_1C_2 = \sqrt{(4-1)^2 + (1+3)^2} = 5$

$r_1 + r_2 = C_1 + C_2$ they touch each other externally



The point of contact divides the centre of circles in the ratio $r_1 : r_2$ internally.

Point of contact is

$$= \left(\frac{3(1)+2(4)}{3+2}, \quad \frac{3(-3)+2(1)}{3+2} \right) = (11/5, -7/5)$$

$\therefore$ Point of contact is $\left(\frac{11}{5}, \frac{-7}{5} \right)$.

3. If the two circles $x^2 + y^2 + 2gx + 2fy = 0$ and $x^2 + y^2 + 2g'x + 2f'y = 0$ touch each other then show that $f'g = fg'$.

Sol.

$S = x^2 + y^2 + 2gx + 2fy = 0$

Centre $C_1 = (-g, -f)$, radius $r_1 = \sqrt{g^2 + f^2}$

$S^1 = x^2 + y^2 + 2g'x + 2f'y = 0$

$C_2 = (-g', -f')$, $r_2 = \sqrt{g'^2 + f'^2}$

Given circles are touching circles,

$\therefore C_1C_2 = r_1 + r_2$

$\Rightarrow (C_1C_2)^2 = (r_1 + r_2)^2$

$$(g'-g)^2 + (f'-f)^2 = g^2 + f^2 + g'^2 + f'^2 + 2\sqrt{g^2+f^2}\sqrt{g'^2+f'^2}$$

$$-2(gg' + ff') = 2\{g^2g'^2 + f^2f'^2 + g^2f'^2 + f^2g'^2\}^{1/2}$$

$$\Rightarrow (gg' + ff')^2 = g^2g'^2 + f^2f'^2 + g^2f'^2 + f^2g'^2$$

$$g^2g'^2 + f^2f'^2 + 2gg'ff' = g^2g'^2 + f^2f'^2 + g^2f'^2 + f^2g'^2$$

$$\Rightarrow 2gg'ff' = g^2f'^2 + f^2g'^2$$

$$\Rightarrow g^2f'^2 + f^2g'^2 - 2gg'ff' = 0$$

$$\Rightarrow (gf' - fg')^2 = 0 \Rightarrow gf' = fg'$$

4. Find the radical centre of the following circles

i) $x^2 + y^2 - 4x - 6y + 5 = 0, x^2 + y^2 - 2x - 4y - 1 = 0, x^2 + y^2 - 6x - 2y = 0$

Sol. Given circles

$S = x^2 + y^2 - 4x - 6y + 5 = 0$

$S' = x^2 + y^2 - 2x - 4y - 1 = 0$

$S'' = x^2 + y^2 - 6x - 2y = 0$

Radical axis 0f $S = 0$ AND $S' = 0$ is $S - S' = 0$

$\Rightarrow -2x - 2y + 6 = 0$

$\Rightarrow$ x + y – 3 = 0 …(1)

R.A. of S'=0 and S" =0 is S' –S" =0

$\Rightarrow$ 4x – 2y – 1 = 0 …(2)

Solving (1) and (2),

x = 7/6, $y = \frac{11}{6}$

Radical centre is (7/6, 11/6) .

ii) $x^2 + y^2 + 4x – 7 = 0, 2x^2 + 2y^2 + 3x + 5y – 9 = 0, x^2 + y^2 + y = 0.$

Ans. P(2, 1)

III.

1. Show that the common chord of the circles $x^2 + y^2 – 6x – 4y + 9 = 0$ and $x^2 + y^2 – 8x – 6y + 23 = 0$ is the diameter of the second circle also and find its length.

Sol. S =$x^2 + y^2 – 6x – 4y + 9 = 0$, S' = $x^2 + y^2 – 8x – 6y + 23 = 0$

Common chord is S-S" =0

$\Rightarrow$ $(x^2 + y^2 – 6x – 4y + 9) – (x^2 + y^2 – 8x – 6y + 23) = 0$

$\Rightarrow$ 2x + 2y – 14 = 0

$\Rightarrow$ x + y – 7 = 0 …(i)

Centre of circle (4, 3)

Substituting (4, 3) in x + y – 7 =0 , we get

4+3-7 =0 $\Rightarrow$ 0=0.

(i) is a diameter of S' =0.

Radius is $\sqrt{4^2 + 3^2 – 23} = \sqrt{2} \Rightarrow$ diameter =$2\sqrt{2}$

2. Find the equation and the length of the common chord of the following circles.

i) $x^2 + y^2 + 2x + 2y + 1 = 0, x^2 + y^2 + 4x + 3y + 2 = 0$

sol.

S = $x^2 + y^2 + 2x + 2y + 1 = 0$,

S'=$x^2 + y^2 + 4x + 3y + 2 = 0$

Equation of common chord is S – S′ = 0

$(x^2 + y^2 + 2x + 2y + 1) – (x^2 + y^2 + 4x + 3y + 2) = 0$

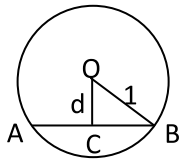
$–2x – y – 1 = 0 \Rightarrow 2x + y + 1 = 0$

Centre of S =0 is (–1, –1)

Radius = $\sqrt{1 + 1 – 1} = 1$

Length of ⊥ from centre (–1,–1) to the chord is

$$d = \left|\frac{2(-1)+(-1)+1}{\sqrt{2^2+1^2}}\right| = \frac{2}{\sqrt{5}}$$



AB=2BC

$$\text{length of the chord} = 2\sqrt{r^2-d^2} = 2\sqrt{1-\frac{4}{5}} = \frac{2}{\sqrt{5}}$$

ii) $x^2 + y^2 - 5x - 6y + 4 = 0, x^2 + y^2 - 2x - 2 = 0$

Ans: $2\sqrt{\frac{14}{5}}$

3. Prove that the radical axis of the circles $x^2 + y^2 + 2gx + 2fy + c = 0$ and $x^2 + y^2 + 2g'x + 2f'y + c' = 0$ is the diameter of the later circle (or) former bisects the circumference of later) if $2g'(g - g') + 2f'(f - f') = c - c'$.

Sol. $S = x^2 + y^2 + 2gx + 2fy + c = 0$

$S^1 = x^2 + y^2 + 2g'x + 2f'y + c' = 0$

Radical axis is $S - S' = 0$

$(x^2 + y^2 + 2gx + 2fy + c) - (x^2 + y^2 + 2g'x + 2f'y + c') = 0$

$2(g - g')x + 2(f - f')y + c - c' = 0$ …(i)

Centre of second circle is $(-g', -f')$

Radius = $\sqrt{g'^2 + f'^2 - c'}$

Now $(-g', -f')$ should lie on (i)

$\therefore -2g(g - g') - 2f'(f - f') + c - c' = 0$

or $2g(g - g') + 2f'(f - f') = c - c'$

4. Show that the circles $x^2 + y^2 + 2ax + c = 0$ and $x^2 + y^2 + 2by + c = 0$ touch each other if $1/a^2 + 1/b^2 = 1/c$.

Sol. $S = x^2 + y^2 + 2ax + c = 0$

$S^1 = x^2 + y^2 + 2by + c = 0$

The centre of the circles C_1 (–a, 0) and C_2(0, –b) respectively.

Radii $r_1 = \sqrt{a^2 - c}$, $r_2 = \sqrt{b^2 - c}$

Given circles are touching circles,

$\Rightarrow C_1C_2 = r_1 + r_2$

$\Rightarrow (C_1C_2)^2 = (r_1 + r_2)^2$

$\Rightarrow (a^2 + b^2) = a^2 - c + b^2 - c + 2\sqrt{a^2 - c}\sqrt{b^2 - c}$

$\Rightarrow c = \sqrt{a^2 - c}\sqrt{b^2 - c}$

$\Rightarrow c^2 = (a^2 - c)(b^2 - c)$

$\Rightarrow c^2 = -c(a^2 + b^2) + a^2b^2 + c^2$

$\Rightarrow c(a^2 + b^2) = a^2b^2 \Rightarrow \frac{1}{c} = \frac{1}{a^2} + \frac{1}{b^2}$.

5. Show that the circles $x^2 + y^2 - 2x = 0$ and $x^2 + y^2 + 6x - 6y + 2 = 0$ touch each other. Find the coordinates of the point of contact. Is the contact external or internal?

Sol. $S = x^2 + y^2 - 2x = 0$

Centre $C_1 = (1, 0)$, Radius $= r_1 = \sqrt{1+0} = 1$

$S' = x^2 + y^2 + 6x - 6y + 2 = 0$

Centre $C_2 = (-3,3)$, $r_2 = \sqrt{9 + 9 - 2} = 4$

$C_1C_2 = \sqrt{(1+3)^2 + (0-3)^2} = \sqrt{16+9} = \sqrt{25} = 5$ $r_1 + r_2 = 1 + 4 = 5$

As $C_1C_2 = r_1 + r_2$ the two circles touch each other externally, the point of contact P divides line of centres internally in the ratio

$r_1 : r_2 = 1 : 4$

Hence point of contact

$$P = \left(\frac{1(-3) + 4(1)}{1+4}, \frac{1(3) + 4(0)}{1+4}\right) = \left(\frac{1}{5}, \frac{3}{5}\right).$$

6. Find the equation of circle which cuts the following circles orthogonally.

i) $x^2 + y^2 + 2x + 4y + 1 = 0$, $2x^2 + 2y^2 + 6x + 8y - 3 = 0$, $x^2 + y^2 - 2x + 6y - 3 = 0$.

Sol.

$S \equiv x^2 + y^2 + 2x + 4y + 1 = 0$

$S^1 \equiv 2x^2 + 2y^2 + 6x + 8y - 3 = 0$

$S^{11} \equiv x^2 + y^2 - 2x + 6y - 3 = 0$

Radical axis of $S = 0$, $S^1 = 0$ is $S - S^1 = 0$

$$-x + \frac{5}{2} = 0 \Rightarrow x = \frac{5}{2} \text{ ----(1)}$$

Radical axis of $S = 0$, $S^{11} = 0$ is $S - S^{11} = 0$

$4x - 2y + 4 = 0 \Rightarrow 2x - y + 2 = 0$

$x = \frac{5}{2} \Rightarrow 5 - y + 2 = 0 \Rightarrow y = 7$ ----(2)

Solving (1) and (2),

Radical centre is P(5/2, 7)

PT = Length of the tangent from P to S = 0

$= \sqrt{\frac{25}{4} + 49 + 5 + 28 + 1} = \sqrt{\frac{25}{4} + 83} \quad = \sqrt{\frac{25 + 332}{4}} = \frac{\sqrt{357}}{2}$

Equation of the circles cutting the given circles orthogonally is

$\left(x - \frac{5}{2}\right)^2 + (y - 7)^2 = \frac{357}{4}$

$\Rightarrow x^2 - 5x + \frac{25}{4} + y^2 - 14y + 49 = \frac{357}{4}$

$\Rightarrow x^2 + y^2 - 5x - 14y + \frac{25}{4} + 49 - \frac{357}{4} = 0$

$\Rightarrow x^2 + y^2 - 5x - 14y + \frac{25 + 196 - 357}{4} = 0$

$\Rightarrow x^2 + y^2 - 5x - 14y + \frac{136}{4} = 0$

$\Rightarrow x^2 + y^2 - 5x - 14y - 34 = 0$

ii) $x^2 + y^2 + 2x + 17y + 4 = 0, x^2 + y^2 + 7x + 6y + 11 = 0, x^2 + y^2 - x + 22y + 3 = 0$

Ans. $x^2 + y^2 - 6x - 4y - 44 = 0$

iii) $x^2 + y^2 + 4x + 2y + 1 = 0, 2(x^2 + y^2) + 8x + 6y - 3 = 0, x^2 + y^2 + 6x - 2y - 3 = 0.$

Ans. $x^2 + y^2 - 14x - 5y - 34 = 0.$

COAXAL SYSTEM OF CIRCLES

DEFINITION

A system of circles is said to be a system of coaxial circles or coaxial system of circles if every pair of circles has the same radical axis.

THEOREM

If $S = 0$, $S' = 0$ are two circles then $\lambda_1 S + \lambda_2 S' = 0$ where λ_1, λ_2 are parameters such that $\lambda_1 + \lambda_2 \neq 0$, represents the coaxial system of all circles containing $S = 0$, $S' = 0$.

Note 1. If S = 0, S′ = 0 are two circles then the coaxial system $\lambda_1 S + \lambda_2 S' = 0$ is called the coaxial system determined by the circles S = 0, S′ = 0.

2 . If two circles intersect, then the radical axis is the common chord and hence $\lambda_1 S + \lambda_2 S' = 0$ represents a coaxial system of circles passing through the points of intersection of the circles S = 0, S′ = 0.

3 . If $\lambda_1 + \lambda_2 = 0$ then $\lambda_1 S + \lambda_2 S' = 0$ reduces to S – S′ = 0, the radical axis of S = 0, S′ = 0.

THEOREM

If S = 0 is a circle and L = 0 is a line then S+λL = 0 where λ is a parameter, represents the coaxial system of all circles of which S = 0 is a member and L = 0 is the radical axis of the system.

THEOREM

If S = 0 is a circle in the coaxial system having radical axis L = 0 then every circle in the system is of the form S + λL = 0 for some constant λ.

Note 1 : If a circle S = 0 and a line L = 0 intersect in two points then S + λL = 0 represents a coaxial system of circles passing through the points of intersection of S = 0, L = 0.

Note 2 : If S = 0, S′ = 0 be two circles then every circle in the coaxial system $\lambda_1 S + \lambda_2 S' = 0$ except S′ = 0 can be taken as S + λL = 0 for some constant λ where L = S – S′.

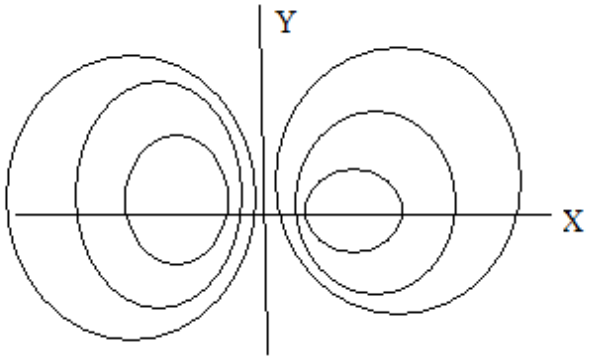
SIMPLEST FORM OF A COAXIAL SYSTEM

A coaxial system of circles is said to be in the simplest form if its line of centers and radical axis are the coordinate axes.

THEOREM

The equation to the system of coaxial circles in the simplest form is $x^2 + y^2 + 2\lambda x + c = 0$ where λ is a parameter and c is a fixed constant.

Proof :



Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, $S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0$ be two circles in the coaxial system.

Centres of the circles are $(-g, -f)$, $(-g', -f')$

Since the line of centers is x-axis, $f = 0$, $f' = 0$.

The radical axis is $S\text{-}S^1=0$

$2(g-g')x + 2(f - f')y + c - c' = 0$

$\Rightarrow 2(g - g')x + (c - c') = 0$

Since the radical axis is y-axis, $c - c' = 0 \Rightarrow c' = c$

The equations of the circles reduce to

$x^2 + y^2 + 2gx + c = 0$, $x^2 + y^2 + 2g'x + c = 0$

$\therefore$ The equation of any circles in the system is of the form $x^2 + y^2 + 2\lambda x + c = 0$ for some λ.

Thus the equation to the system of coaxial circles in the simplest form is $x^2 + y^2 + 2\lambda x + c = 0$ where λ is a parameter and c is a fixed constant.

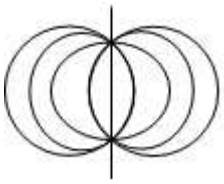
NATURE OF THE COAXIAL SYSTEM

$x^2 + y^2 + 2\lambda x + c = 0$

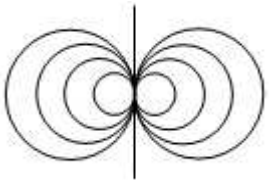
The radical axis of the coaxial system $x^2 + y^2 + 2\lambda x + c = 0$ where λ is a parameter and c is a fixed constant is $x = 0$. Let P be a point of intersection of $x^2 + y^2 + 2\lambda x + c = 0$, $x = 0$. Then $y^2 + c = 0 \Rightarrow y = \pm\sqrt{-c}$.

$\therefore P = (0, \pm\sqrt{-c})$

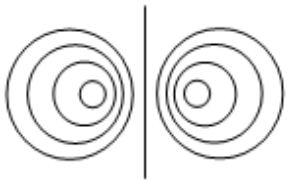
1. If $c < 0$ then every circle in the coaxial system intersects the radical axis at two points. Hence the system of circles is an intersecting system as shown in the figure.



2. If $c = 0$ then $P = (0, 0)$. Thus every circle in the coaxial system touches the radical axis $x = 0$ at the origin. Hence the system of circles is a touching system as shown in the figure



3. If $c > 0$ then P is imaginary. Thus every circle in the coaxial system does not intersect the radical axis. Hence the system of circles is a non-intersecting system as shown in the figure



LIMITING POINTS OF A COAXIAL SYSTEM

DEFINITION

The point circles in a coaxial system are called the limiting points of the coaxial system.

THEOREM

The limiting points of the coaxial system $x^2 + y^2 + 2\lambda x + c = 0$ are $(\mp\sqrt{c}, 0)$.

Proof :

Let $x^2 + y^2 + 2\lambda x + c = 0$ …(1)

Center of (1) is $(-\lambda, 0)$, Radius is $\sqrt{\lambda^2 - c}$

If (1) is a point circle, then its radius = 0

$\Rightarrow \sqrt{\lambda^2 - c} = 0 \Rightarrow \lambda^2 - c = 0 \Rightarrow \lambda^2 = c \Rightarrow \lambda = \pm\sqrt{c}$.

$\therefore$ Limiting points are $(\mp\sqrt{c}, 0)$.

Note 1 : If $c > 0$ then the coaxial system has two limiting points. Then a non-intersecting coaxial system has two limiting points.

Note 2 : If $c = 0$ then the coaxial system has only one limiting point (0, 0). Thus a touching coaxial system has only one limiting point which is the point of contact of the circles of the system.

Note 3 : If $c < 0$ then the coaxial system has no limiting points. Thus an intersecting coaxial system has no limiting point.

THEOREM

If P and Q are the two limiting points of a coaxial system them Q is the image of P with respect to the radical axis.

THEOREM

If P, Q are the two limiting points of a coaxial system then P, Q are inverse points with respect to the coaxial system.

Proof :

Let the coaxial system be $x^2 + y^2 + 2\lambda x + c = 0$

The limiting points are $P(\sqrt{c}, 0), Q(-\sqrt{c}, 0)$

A circle in the coaxial system is $x^2+y^2+2\lambda_1 x+c = 0$

Centre of the circle is $C(-\lambda_1, 0)$

Radius of the circle is $r = \sqrt{\lambda_1^2 - c}$

$$CP = \sqrt{(\sqrt{c} + \lambda_1)^2} = \sqrt{c} + \lambda_1$$

$$CQ = \sqrt{(-\sqrt{c} + \lambda_1)^2} = -\sqrt{c} + \lambda_1$$

$$PQ = \sqrt{(\sqrt{c} + \sqrt{c})^2} = 2\sqrt{c}$$

PQ + QC = CP.

$\therefore$ P, Q, C are collinear and P, Q lie on the same side of C.

$$CP \cdot CQ = (\sqrt{c} + \lambda_1)(-\sqrt{c} + \lambda_1) = \lambda_1^2 - c = r^2$$

$\therefore$ P, Q are inverse points of a coaxial system are conjugate points with respect to every circle in the coaxial system.