CIRCLES
EXERCISE – 1(f)

I.**1. Find the equation of the sphere whose centre is (2, –3, 4) and radius is 5.**

Sol. Centre C= (2, –3, 4), radius r =5.

Equation of the sphere is $(x-a)^2+(y-b)^2+(z-c)^2=r^2$

$(x-2)^2+(y+3)^2+(z-4)^2=25$

$x^2-4x+4+y^2+6y+9+z^2-8z+16-25=0$

$x^2+y^2+z^2-4x+6y-8z+4=0.$

2. Find the centre and radius of the sphere $x^2+y^2+z^2-2x+4y-6z-2=0$.

Sol. Equation of the sphere is $x^2+y^2+z^2-2x+4y-6z-2=0$

Comparing with $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ then

$2u=-2,\ 2v=4,\ 2w=-6,\ d=-2 \Rightarrow u=-1,\ v=2,\ w=-3$

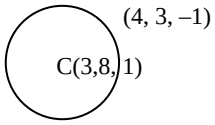
centre is $(-u,-v,-w)=(1,-2,3)$

$r=\sqrt{u^2+v^2+w^2-d}=\sqrt{1+4+9+2}=4$

Radius = 4 units.

3, Find the equation of the sphere that passes through the point (4, 3, –1) and having its centre at (3, 8, 1).

Sol. Centre is (3, 8, 1). Point on the sphere is (4, 3, –1)



$r=CP=\sqrt{(3-4)^2+(8-3)^2+(1+1)^2}=\sqrt{1+25+4}=\sqrt{30}$

Equation of the sphere is $(x-3)^2+(y-8)^2+(z-1)^2=30$

$\Rightarrow x^2-6x+9+y^2-16y+64+z^2-2z+1-30=0$

$\Rightarrow x^2+y^2+z^2-6x-16y-2z+44=0.$

4. Find the equation of the sphere having (2, 3, 4) and (–5, 6, 7) as the extremities of one of its diameter.

Sol. Ends of the diameter are (2, 3, 4) and (–5, 6, 7)

Equation of the sphere having $A(x_1, y_1, y_2)$, $B(x_2, y_2, z_2)$ as the extremities of diameters is

$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$

$\Rightarrow (x-2)(x+5)+(y-3)(y-6)+(z-4)(z+7)=0$

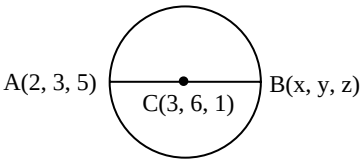
$\Rightarrow x^2+3x-10+y^2-9y+18+z^2+3z-28=0$

$\Rightarrow x^2 + y^2 + z^2 + 3x - 9y + 3z - 20 = 0.$

5. If (2, 3, 5) is one end of a diameter of the sphere $x^2 + y^2 + z^2 - 6x - 12y - 2z + 20 = 0$, then find the coordinates of the other end of the diameter.

Sol. Equation of the sphere is

$x^2 + y^2 + z^2 - 6x - 12y - 2z + 20 = 0$



Centre is C(3, 6, 1)

One end of the diameter is A(2, 3, 5)

Let (x, y, z) be the other end of the diameter

Centre C = mid point of AB

$$\Rightarrow \left[\frac{2+x}{2}, \frac{3+y}{2}, \frac{5+z}{2}\right] = (3,6,1)$$

$$\frac{2+x}{2} = 3 \Rightarrow 2 + x = 6 \Rightarrow x = 4$$

$$\Rightarrow \frac{3+y}{2} = 6 \Rightarrow 3 + y = 12 \Rightarrow y = 9$$

$$\Rightarrow \frac{5+z}{2} = 1 \Rightarrow 5 + z = 2 \Rightarrow z = -3$$

Other end of the diameter is B(4, 9, –3).

II.

1. If the plane $2ax - 3ay + 4az + 6 = 0$ passes through the mid-point of the line segment joining the centres of the spheres $x^2 + y^2 + z^2 + 6x - 8y - 2z = 13$ and $x^2 + y^2 + z^2 - 10x + 4y - 2z = 8$ then find the value of 'a'.

Sol. Equations of the spheres are $x^2 + y^2 + z^2 + 6x - 8y - 2z = 13$ and $x^2 + y^2 + z^2 - 10x + 4y - 2z = 8$

Centre of the spheres are respectively A(–3,4,1), B(5,–2,1)

C is med point of AB

$$\Rightarrow \text{C} = \left(\frac{-3+5}{2}, \frac{4-2}{2}, \frac{1+1}{2}\right) = (1,1,1)$$

Equation of the plane is $2ax - 3ay + 4az + 6 = 0$

The plane is passing through C(1, 1, 1), $2a - 3a + 4a + 6 = 0$

$3a + 6 = 0 \Rightarrow a + 2 = 0 \Rightarrow a = -2.$

2. Find the centre and radius of the sphere passing through the point (0, 0, 0), (0, 4, 0), (–2, 0, 0) and (0, 0, –4).

Sol. Let the equation of the sphere be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

The sphere is passing through A(0, 0, 0)

$\therefore\ d = 0$

B(0, 4, 0) $\Rightarrow 16 + 8v + d = 0$

$\Rightarrow 8v + 16 = 0 \Rightarrow v = -2$

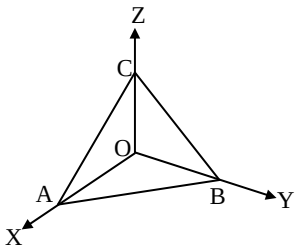
C(–2, 0, 0) $\Rightarrow 4 - 4u = 0 \Rightarrow 4u = 4 \Rightarrow u = 1$

D(0, 0, –4) $\Rightarrow 16 - 8w = 0 \Rightarrow 8w = 16 \Rightarrow w = 2$

Centre = (–u, –v, –w) = (–1, 2, –2).

3. Find the equation of the sphere circumscribing the tetrahedron whose faces are x = 0, y = 0, z = 0 and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

Sol. Equations of the faces are x = 0, y = 0, z = 0 and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$



Solving this equation, the vertices are

O(0, 0, 0), A(a, 0, 0), B(0, b, 0), C(0, 0, c)

Let the equation of the sphere OABC be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

This sphere is passing through O(0, 0, 0), A(a, 0, 0), B(0, b, 0), C(0, 0, c).

O(0, 0, 0) $\Rightarrow d = 0$

A(a, 0, 0) $\Rightarrow a^2 + 2ua = 0 \Rightarrow 2ua = -a^2 \Rightarrow 2u = -a$

B(0, b, 0)$\Rightarrow b^2 + 2vb = 0 \Rightarrow 2vb = -b^2 \Rightarrow 2v = -b$

C(0, 0, c)$\Rightarrow c^2 + 2wc = 0 \Rightarrow 2wc = -c^2 \Rightarrow 2w = -c$

Equation of the sphere circumscribing the tetrahedron :

$x^2 + y^2 + z^2 - ax - by - cz = 0.$

4. Find the equation of the sphere containing the points (1, 0, 0), (0, 1, 0), (0, 0, 1) and having the smallest radius.

Sol. Let the equation of the sphere be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$.

This sphere is passing through (1, 0, 0), (0, 1, 0), (0, 0, 1)

A(1, 0, 0) $\Rightarrow 1 + 2u + d = 0$

B(0, 1, 0) $\Rightarrow 1 + 2v + d = 0$

C(0, 0, 1) $\Rightarrow 1 + 2w + d = 0$

$\therefore$ u = v = w and d = –(2u + 1)

Radius of the sphere = $\sqrt{u^2 + v^2 + w^2 - d}$

$= \sqrt{u^2 + u^2 + u^2 + 2u + 1} = \sqrt{3u^2 + 2u + 1}$

Least value of $3u^2 + 2u + 1 = \dfrac{4ac - b^2}{4a}$

$= \dfrac{12 - 4}{12} = \dfrac{8}{12} = \dfrac{2}{3}$

$\therefore r = \sqrt{\dfrac{2}{3}}, u = -\dfrac{2}{6} = -\dfrac{1}{3}$

$d = -(2u + 1) = -\left(-\dfrac{2}{3} + 1\right) = -\dfrac{1}{3}$

Equation of the required sphere is

$$x^2 + y^2 + z^2 - \frac{2}{3}x - \frac{2}{3}y - \frac{2}{3}y - \frac{1}{3} = 0$$

$$\Rightarrow x^2 + y^2 + z^2 - \frac{2}{3}(x + y + z) - \frac{1}{3} = 0$$

$$3(x^2 + y^2 + z^2) - 2(x + y + z) - 1 = 0$$

5. Find the equation of the sphere passing through the point (0, 0, 0), (0, 1, –1), (–1, 2, 0) and (1, 2, 3).

Sol. Let the equation of the sphere be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

This sphere is passing through (0, 0, 0), (0, 1, –1), (–1, 2, 0) and (1, 2, 3).

O(0, 0, 0) $\Rightarrow$ d = 0 …(i)

A(0, 1, –1) $\Rightarrow$ $1 + 1 + 2v - 2w = 0$

$\Rightarrow$ v – w = –1 …(ii)

B(–1, 2, 0) $\Rightarrow 1 + 4 + 0 - 2u + 4v = 0$

$\Rightarrow$ 2u – 4v = 5 …(iii)

C(1, 2, 3) $\Rightarrow$ 1 + 4 + 9 + 2u + 4v + 6w = 0

$\Rightarrow$ 2u + 4v + 6w = –14…(iv)

Subtracting (iv) – (iii) :

8v + 6w = –19 …(v)

6v – 6w = –6 [(ii) × 6]

14v = –25 $\Rightarrow$ $v = \frac{-25}{14}$

From (i), $w = v + 1 = \frac{-25}{14} + 1 = \frac{-11}{14}$

From (iii), $2u = 4v + 5 = 4\left(\frac{-25}{14}\right) + 5 = \frac{-30}{14}$

Equation of the required sphere OABC is

$$x^2 + y^2 + z^2 - \frac{15}{7}x - \frac{25}{7}y - \frac{11}{7}z = 0$$

$$\Rightarrow 7(x^2 + y^2 + z^2) - 15x - 25y - 11z = 0$$

III.

1. **Find the equation of the sphere containing the points (4, –1, 2), (0, –2, 3), (1, –5, –1) and (2, 0, 1).**

Ans; $x^2 + y^2 + z^2 + -4x + 6y - 2z + 5 = 0$

2. **Find the equation of the sphere containing the points (1, –3, 4), (1, –5, 2), (1, –3, 0) and having its centre on the plane x + y + z = 0**.

Ans: $x^2 + y^2 + z^2 + -2x + 6y - 4z + 10 = 0$

3. **A sphere of radius 2r (r > 0) passes through the origin and cuts the coordinate axes at A, B and C. Prove that the centroid of the tetrahedron OABC lies on the sphere $x^2 + y^2 + z^2 = r^2$.**

4. **A sphere of radius r (r > 0) passes through the origin and meets the coordinate axes at A, B, C. Prove that the centroid of ΔABC lies on the sphere of $(x^2 + y^2 + z^2) = 4r^2$.**

5. **A sphere of radius r (r > 0) passes through the origin and meets the coordinate axes at A, B, C. Prove that the foot of the perpendicular from the origin O to the plane $\overleftrightarrow{ABC}$ satisfies the equations $(x^2 + y^2 + z^2)(x^{-2} + y^{-2} + z^{-2}) = 4r^2$.**

PROBLEMS FOR PRACTICE

1. Find the centre and radius of the circle $x^2 + y^2 + 2x – 4y – 4 = 0$.

Ans. Centre = (–1, 2), Radius = 3

2. Find the centre and radius of the circle $3x^2 + 3y^2 – 6x + 4y – 4 = 0$.

Ans. Centre = (1, –2/3), Radius = 5/3

3. Find the equation of the circle whose centre is (–1, 2) and which passes through (5, 6).

Ans. $x^2 + y^2 + 2x – 4y – 47 = 0$

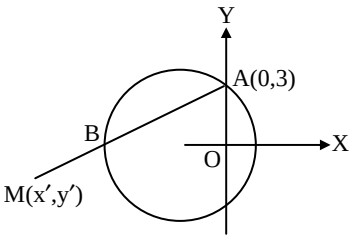
4. Find the equation of the circle passing through (2, 3) and concentric with the circle $x^2 + y^2 + 8x + 12y + 15 = 0$.

Ans. $x^2 + y^2 + 8x + 12y – 65 = 0$

5. From the point A(0, 3) on the circle $x^2 + 4x + (y – 3)^2 = 0$, a chord AB is drawn and extended to a point M such that AM = 2AB. Find the equation of the locus of M.

Sol. Let M = (x', y')

Given that AM = 2AB



$AB + BM = 2AB + AB \Rightarrow BM = AB$

B is the mid point of AM

Coordinates of B are $\left(\frac{x'}{2} - \frac{y'+3}{2}\right)$

B is a point on the circle :

$$\left(\frac{x'}{2}\right)^2 + 4\left(\frac{x'}{2}\right) + \left(\frac{y'+3}{2} - 3\right)^2 = 0$$

$$\frac{x'^2}{4} + 2x' + \frac{y'^2 - 6y' + 9}{4} = 0$$

$$x'^2 + 8x' + y'^2 - 6y' + 9 = 0$$

Locus of M(x', y') is $x^2 + y^2 + 8x – 6y + 9 = 0$ which is a circle.

6. If the circle $x^2 + y^2 + ax + by - 12 = 0$ has the centre at (2, 3) then find a, b and the radius of the circle.

Ans. a = –4, b = –6, radius = 5

7. If the circle $x^2 + y^2 - 4x + 6y + a = 0$ has radius 4, then find a.

Ans. a = –3

8. Find the equation of the circle passing through (4, 1), (6, 5) and having the centre on the line $4x + y - 16 = 0$.

Ans. $x^2 + y^2 - 6x - 8y + 15 = 0$

9. Suppose a point (x_1, y_1) satisfies $x^2 + y^2 + 2gx + 2fy + c = 0$ then show that it represents a circle whenever g, f and c are real.

Sol. Comparing with the general equation of second degree co-efficient of x^2 = coefficient of y^2 and coefficient of xy = 0.

The given equation represents a circle if $g^2 + f^2 - c \geq 0$

(x_1, y_1) is a point on the given equation $x^2 + y^2 + 2gx + 2fy + c = 0$, we have

$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \Rightarrow g^2 + f^2 - c = \quad g^2 + f^2 + x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$

$= (x_1 + g)^2 + (y_1 + f)^2 \geq c$

g, f and c are real.

$\therefore$ The given equation represents a circle.

10. Find the equation of the circle whose extremities of a diameter are (1, 2) and (4, 5).

Ans. $x^2 + y^2 - 5x - 7y + 14 = 0$

11. Find the other end of the diameter of the circle $x^2 + y^2 - 8x - 8y + 27 = 0$, if one end of its is (2, 3).

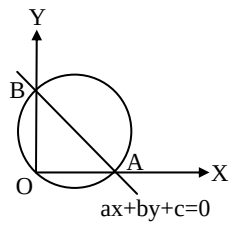
Ans. (6, 5)

12. Find the equation of the circum circle of the triangle formed by $ax + by + c = 0$ $(a, b, c \neq 0)$ and the coordinate axes.

Sol. Let the line ax + by + c = 0 cut X, Y axes at A and B respectively.

Coordinates of O are (0, 0),

A are $\left(-\frac{c}{a}, 0\right)$, B are $\left(0, -\frac{c}{b}\right)$.



Suppose the equation of the required circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

This circle passes through O(0, 0), $\therefore \quad c = 0$

This circle passes through $A\left(-\frac{c}{a}, 0\right) \Rightarrow \frac{c^2}{a^2} + 0 - 2\frac{gc}{a} = 0$

$$2g \cdot \frac{c}{a} = \frac{c^2}{a^2} \Rightarrow 2g = \frac{c}{a} \Rightarrow g = \frac{c}{2a}$$

The circle passes through $B\left(0, -\frac{c}{b}\right) \Rightarrow$

$$0 + \frac{c^2}{b^2} + 0 - 2f\frac{c}{b} = 0$$

$$2f\frac{c}{b} = \frac{c^2}{b^2} \Rightarrow 2f = \frac{c}{b} \Rightarrow f = \frac{c}{2b}$$

Equation of the circle through O, A, B is

$$x^2 + y^2 + \frac{c}{a}x + \frac{c}{b}y = 0$$

$$ab(x^2 + y^2) + (bx + ay) = 0$$

This is the equation of the circum circle of ΔOAB.

13. Find the equation of the circle which passes through the vertices of the triangle formed by $L_1 = x + y + 1 = 0$, $L_2 = 3x + y - 5 = 0$, $L_3 = 2x + y - 5 = 0$.

Ans. $5x^2 + y^2 - 30x - 10y + 25 = 0$

14. Find the centre of the circle passing through the points (0, 0), (2, 0) and (0, 2).

Ans. Centre of the circle : (1, 1)

15. Obtain the parametric equations of the circle $x^2 + y^2 = 1$.

Ans. $x = \cos\theta$, $y = \sin\theta$, $0 \le \theta \le 2\pi$

16. Obtain the parametric equations of the circle represented by $x^2 + y^2 + 6x + 8y - 96 = 0$.

Ans. $x = -3 + 11\cos\theta$, $y = -4 + 11\sin\theta$

where $0 \le \theta \le 2\pi$

17. Locate the position of the point (2, 4) with respect to the circle $x^2 + y^2 - 4x - 6y + 11 = 0$.

Ans. The point (2, 4) lies inside the circle.

18. Find the length of the tangent from (1, 3) to the circle $x^2 + y^2 - 2x + 4y - 11 = 0$.

Ans. 3

19. If a point P is moving such that the length of tangents drawn from P to $x^2 + y^2 - 2x + 4y - 20 = 0$ and $x^2 + y^2 - 2x - 8y + 1 = 0$ are in the ratio 2 : 1. Then show that the equation of the locus of P is $x^2 + y^2 - 2x - 12y + 8 = 0$.

20. If $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle then show that the straight line $lx + my + n = 0$.

i) Touches the circle S = 0 is $(g^2 + f^2 - c)\dfrac{(gl + mf - n)^2}{l^2 + m^2}$

ii) meets the circle S = 0 in two points if $g^2 + f^2 - c > \dfrac{(gl + mf - n)^2}{l^2 + m^2}$

iii) will not meet the circle if $g^2 + f^2 - c < \dfrac{(gl + mf - n)^2}{l^2 + m^2}$

Sol. Let O be the centre and r be the radius of the circle s = 0.

Then c = (–g, –f) and $r = \sqrt{g^2 + f^2 - c}$

i) The given straight line touches the circle if

$$r = \frac{|l(-g) + m(-f) - n|}{\sqrt{l^2 + m^2}}$$

$$\sqrt{g^2 + f^2 - c} = \frac{|-(lg + mf - n)|}{\sqrt{l^2 + m^2}}$$

Squaring both sides, we get

$$g^2 + f^2 - c = \frac{(lg + mf - n)^2}{l^2 + m^2}$$

ii) The given line lx + my + n = 0 meets the circle s = 0 in two points if

$$g^2 + f^2 - c > \frac{(lg + mf - n)^2}{l^2 + m^2}$$

iii) The given line lx + my + n = 0 will not meet the circle s = 0 if

$$g^2 + f^2 - c < \frac{(lg + mf - n)^2}{l^2 + m^2}$$

21. Find the length of the chord intercepted by the circle $x^2 + y^2 + 8x - 4y - 16 = 0$ on the line $3x - y + 4 = 0$.

Ans. $2\sqrt{26}$

22. Find the equation of tangents to $x^2 + y^2 - 4x + 6y - 12 = 0$ which are parallel to $x + 2y - 8 = 0$.

Ans. $x + 2y + (4 \pm 5\sqrt{5}) = 0$

23. Show that the circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$ touches the (i) x-axis if $g^2 = c$, (ii) y-axis if $f^2 = c$.

24. Find the equation of the tangent to $x^2 + y^2 – 6x + 4y – 12 = 0$ at (–1, 1).

Ans. $4x – 3y + 7 = 0$

25. Find the equation of the tangent to $x^2 + y^2 – 2x + 4y = 0$ at (3, 1). Also find the equation of tangent parallel to it.

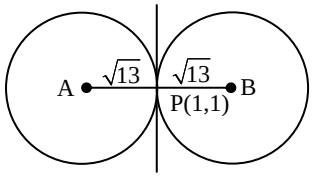
Ans. $2x + y – 5 = 0$.

26. If $4x – 3y + 7 = 0$ is a tangent to the circle represented by $x^2 + y^2 – 6x + 4y – 12 = 0$, then find the point of contact.

Ans. (–1, 1)

27. Find the equations of circles which touch $2x – 3y + 1 = 0$ at (1, 1) and having radius $\sqrt{13}$.

Sol. The centre of required circle lies on a line perpendicular to $2x – 3y + 1 = 0$ and passing through (1, 1).



The equation of the line of centres can be taken as $3x + 2y + k = 0$.(this is perpendicular to given line)

This line passes through (1, 1), $3 + 2 + k = 0 \Rightarrow k = –5$

Equation of AB is $3x + 2y – 5 = 0$

The centres of A and B are situated on $3x + 2y – 5 = 0$ at a distance $\sqrt{13}$ from (1, 1).

The centres A and B are given by $(x_1 \pm r\cos\theta, y_1 \pm r\sin\theta)$

$$\left(1+\sqrt{13}\left(-\frac{2}{\sqrt{13}}\right),1+\sqrt{13}\cdot\frac{3}{\sqrt{13}}\right) \text{and } \left(1-\sqrt{13}\left(-\frac{2}{\sqrt{13}}\right),1-\sqrt{13}\cdot\frac{3}{\sqrt{13}}\right)$$

i.e., (1 – 2, 1 + 3) and (1 + 2, 1 – 3)

(–1, 4) and (3, –2)

Case I.

Centre (–1, 4), $r = \sqrt{13}$

Equation of the circle is

$$(x+1)^2 + (y-4)^2 = 13$$

$$x^2 + 2x + 1 + y^2 - 8y + 16 - 13 = 0$$

$$x^2 + y^2 + 2x - 8y + 4 = 0$$

Case II :

Centre (3, –2), $r = \sqrt{13}$

Equation of the circle is :

$(x-3)^2+(y+2)^2=13$

$x^2-6x+9+y^2+4y+4-13=0$

$x^2+y^2-6x+4y=0$

28. Show that the line $5x + 12y – 4 = 0$ touches the circle $x^2 + y^2 – 6x + 4y + 12 = 0$.

29. If the parametric values of two points A and B lying on the circle $x^2 + y^2 – 6x + 4y – 12 = 0$ are 30° and 60° respectively, then find the equation of the chord joining A and B.

Sol. Equation of the circle is

$x^2 + y^2 – 6x + 4y – 12 = 0$

Equation of the chord joining θ_2 and θ_1 is $(x+g)\cos\frac{\theta_1+\theta_2}{2}+(y+f)\sin\frac{\theta_1+\theta_2}{2}=r\cos\frac{\theta_1-\theta_2}{2}$

$r=\sqrt{9+4+12}=5$

$(x – 3)\cos45° + (y + 2)\sin 45° = 5\cos15°$

$\Rightarrow \frac{(x-3)+(y+2)}{\sqrt{2}}=\frac{5(\sqrt{3}+1)}{2\sqrt{2}}$

$\Rightarrow 2(x + y – 6) = 5\sqrt{3}+5$

$\Rightarrow 2x+2y-7-5\sqrt{3}=0 \Rightarrow 2x+2y-(7+5\sqrt{3})=0$

30. Find the equation of the tangent at the point 30° (parametric value of θ) of the circle is $x^2 + y^2 + 4x + 6y – 39 = 0$.

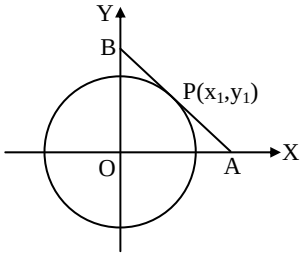
Ans. $\sqrt{3}x+y+(3+2\sqrt{3}-4\sqrt{13})=0$.

31. Find the area of the triangle formed by the tangent at $p(x_1, y_1)$ to the circle $x^2 + y^2 = a^2$ with coordinates axes where $x_1y_1 \neq 0$.

Sol. Equation of the circle is : $x^2 + y^2 = a^2$

Equation of the tangent at $P(x_1, y_1)$ is :

$xx_1 + yy_1 = a^2$ …(i)



This tangent cuts X-axis at A and Y-axis at B.

Changing into intercept form $\frac{xx_1}{a^2}+\frac{yy_1}{a^2}=1 \Rightarrow \frac{x}{\left(\frac{a^2}{x_2}\right)}+\frac{y}{\left(\frac{a^2}{y_1}\right)}=1$

$$OA = \frac{a^2}{x_1},\ OB = \frac{a^2}{y_1}$$

$$OAB = \frac{1}{2}|OA \cdot DB| = \frac{1}{2}\left|\frac{a^2}{x_1} \cdot \frac{a^2}{y_1}\right| = \frac{a^4}{2|x_1y_1|}$$

32. Find the equation of normal to the circle $x^2 + y^2 – 4x – 6y + 11 = 0$ at (3, 2). Also find the other point where the normal meets the circle.

Sol. Equation of the circle is

$x^2 + y^2 – 4x – 6y + 11 = 0$

$g = –2,\ f = –3,\ c = 11$

Radius = $\sqrt{g^2 + f^2 - c} = \sqrt{4+9-11} = \sqrt{2}$

Equation of the normal is $(x – x_1)(y_1 + f) – (y – y_1)(x_1 + g) = 0$

The normal at A(3, 2) is $(x – 3)(2 – 3) – (y – 2)(3 – 2) = 0$

$– x + 3 – y + 2 = 0$

$x + y – 5 = 0$

If the normal at A meets the circle is B, then C is the midpoint of AB.

Centre of the circle = C(–g, –f) = (2, 3)

If coordinates of B are (x, y)

Coordinates of C are $\left(\frac{3+x}{2}, \frac{2+y}{2}\right) = (2,3)$

$$\frac{3+x}{2} = 2 \Rightarrow 3 + x = 4 \Rightarrow x = 1$$

$$\frac{2+y}{2} = 3 \Rightarrow 2 + y = 6 \Rightarrow y = 4$$

$\therefore$ The normal at A(3, 2) meets the circle at B(1, 4).

33. Find the area of the triangle formed by the normal at (3, –4) to the circle $x^2 + y^2 – 22x – 4y + 25 = 0$ with the coordinate.

Ans. $\frac{625}{24}$

34. Show that the line $lx + my + n = 0$ is a normal to the circle $S = 0$ if and only if $gl + mf = n$.

Sol. The straight line $lx + my + n = 0$ is normal to the circle.

$S = x^2 + y^2 + 2gx + 2fy + c = 0$

If the centre (–g, –f) lies on $lx + my + n = 0$

$l(–g) + m(–f) + n = 0$

$gl + fm = n$

35. Find the condition that the tangents drawn from the exterior point (g, f) to $S = x^2 + y^2 + 2gx + 2fy + c = 0$ are perpendicular to each other.

Sol.

Equation of the circle is

$x^2 + y^2 + 2gx + 2fy + c = 0$

$r = \sqrt{g^2 + f^2 - c}$

$S_{11} = g^2 + f^2 + 2g^2 + 2f^2 + c$

$= 3g^2 + 3f^2 + c$

If the angle between the tangents drawn from $P(x_1, y_1)$ to $S = 0$ is θ, then $\left(\frac{\theta}{2}\right) = \frac{r}{\sqrt{S_1}}$

$\theta = 90° \Rightarrow \tan\frac{\theta}{2} = \tan 45 = \frac{\sqrt{g^2 + f^2 - c}}{\sqrt{3g^2 + 3f^2 + c}}$

$1 = \frac{g^2 + f^2 - c}{3g^2 + 3f^2 + c}$

$\Rightarrow 3g^2 + 3f^2 + c = g^2 + f^2 - c$

$\Rightarrow 2g^2 + 2f^2 + 2c = 0$

$\Rightarrow g^2 + f^2 + c = 0$

This is the condition for the tangent drawn from (g, f) to the circle $S = 0$ are perpendicular.

36. If θ_1, θ_2 are the angles of inclination of tangents through a point P to the circle $x^2 + y^2 = a^2$, then find the locus of P are $\cot\theta_1 + \cot\theta_2 = k$.

Sol. Equation of the circle is : $x^2 + y^2 = a^2$

If m is the slope of the tangent, then the equation of tangent passing through $P(x_1, y_1)$ can be taken as :

$y_1 = mx_1 \pm a\sqrt{1 + m^2}$

$(y_1 - mx_1)^2 = a^2(1 + m^2)$

$m^2x_1^2 + y_1^2 - 2mx_1y_1 - a^2 - a^2m^2 = 0$

$m^2(x_1^2 - a^2) - 2mx_1y_1 + (y_1^2 - a^2) = 0$

Let m_1 and m_2 be the roots of this equation ,where m_1 and m_2 are the slopes of the tangents and $m_1 = \tan\theta_1, m_2 = \tan\theta_2$

$$m_1 + m_2 = \tan\theta_1 + \tan\theta_2 = \frac{2x_1y_1}{x_1^2 - a^2}$$

$$m_1m_2 = \tan\theta_1 \tan\theta_2 = \frac{y_1^2 - a^2}{x_1^2 - a^2}$$

Given that $\cot\theta_1 + \cot\theta_2 = k$

$$\frac{1}{\tan\theta_1} + \frac{1}{\tan\theta_2} = k \Rightarrow \frac{\tan\theta_1 + \tan\theta_2}{\tan\theta_1 \tan\theta_2} = k$$

$$\frac{2x_1y_1}{y_1^2 - a^2} = k \Rightarrow 2x_1y_1 = k(y_1^2 - a^2)$$

Locus of $P(x_1, y_1)$ is $2xy = k(y^2 - a^2)$

Also, conversely if $P(x_1, y_1)$ satisfies the condition $2xy = k(y^2 - a^2)$ then it can be show that

$\cot\theta_1 + \cot\theta_2 = k$.

Thus the locus of P is $2xy = k(y^2 - a^2)$.

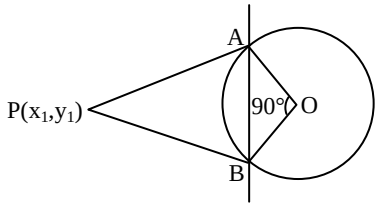
37. Find the chord of contact of (2, 5) with respect to the circle $x^2 + y^2 - 5x + 4y - 2 = 0$.

Ans. $x - 14y - 6 = 0$

38. If the chord of contact of a point P with respect to the circle $x^2 + y^2 = a^2$ meets the circle at A and B such that $\angle AOB = 90°$ then show that P lies on the circle $x^2 + y^2 = 2a^2$.

Sol. Equation of the circle : $x^2 + y^2 = a^2$ …(i)

$P(x_1, y_1)$ is any point on the locus



Chord of contact of P is $xx_1 + yy_1 = a^2$

$$\frac{xx_1 + yy_1}{a^2} = 1 \qquad \ldots(ii)$$

Homogeniousing (i) with the help of (ii), combined equation of OA and OB is

$$x^2 + y^2 = \frac{a^2(xx_1 + yy_1)^2}{a^4}$$

$$a^2x^2 + a^2y^2 = x^2x_1^2 + y^2y_1^2 + 2x_1y_1xy$$

$$(x^2 - a^2) + 2x_1y_1xy + y^2(y_1^2 - a^2) = 0$$

$\angle AOB = 90°$

$\Rightarrow$ Coefficient of x^2 + coefficient of $y^2 = 0$

$$x_1^2 - a^2 + y_1^2 - a^2 = 0$$

$$x_1^2 + y_1^2 = 2a^2$$

$\therefore$ $P(x_1, y_1)$ lies on the circle $x^2 + y^2 = 2a^2$.

39. Find the equation of the polar of (2, 3) with respect to the circle $x^2 + y^2 + 6x + 8y - 96 = 0$.

Ans. $5x + 7y - 78 = 0$

40. Find the pole of $x + y + 2 = 0$ with respect to the circle $x^2 + y^2 - 4x + 6y - 12 = 0$.

Ans. $(-23, -28)$

41. Show that the poles of the tangents to the circle $x^2 + y^2 = a^2$ with respect to the circle $(x + a)^2 + y^2 = 2a^2$ lie on $y^2 + 4ax = 0$.

Ans. $y^2 + 4ax = 0$

42. Show that (4, –2) and (3, –6) are conjugate with respect to the circle $x^2 + y^2 - 24 = 0$.

43. If (4, k) and (2, 3) are conjugate points with respect to the circle $x^2 + y^2 = 17$, then find k.

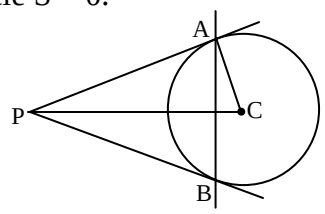
Ans. $k = 3$

44. Show that the lines $2x + 3y + 11 = 0$ and $2x - 2y - 1 = 0$ are conjugate with respect to the circle $x^2 + y^2 + 4x + 6y - 12 = 0$.

45. Show that the area of the triangle formed by two tangents through $P(x_1, y_1)$ to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ and the chord of contact of P with respect to the circle $S = 0$ is $\frac{r(S_{11})^{3/2}}{S_{11} + r^2}$ where r is the radius of the circle.

Sol. PA and PB are two tangents from P to the circle $S = 0$.

AB is the chord of contact.



Let θ be the angle between the tangent B.

$$\tan\frac{\theta}{2} = \frac{r}{\sqrt{S_{11}}}$$

$$\Delta PAB = \frac{1}{2} PA \cdot PB \sin\theta$$

$$= \frac{1}{2}\sqrt{S_{11}}\sqrt{S_{11}}\,\frac{2\tan\theta/2}{1+\tan^2\theta/2} = \frac{1}{2}S_{11}\,\frac{2\frac{r}{\sqrt{S_{11}}}}{1+\frac{r^2}{S_{11}}}$$

$$= S_{11}\frac{r}{\sqrt{S_{11}}}\frac{S_{11}}{S_{11}+r^2} = \frac{r(S_{11})^{3/2}}{S_{11}+r^2}$$

46. Find the mid point of the chord intercepted by $x^2 + y^2 - 2x - 10y + 1 = 0$ on the line $x - 2y + 7 = 0$.

Ans. (7/5, 21/5)

47. Find the locus of mid points of the chords of contact of $x^2 + y^2 = a^2$ from the points lying on the line $lx + my + n = 0$.

Ans. $n(x^2 + y^2) + a^2(lx + my) = 0$.

48. Show that the four common tangents can be drawn for the circles given by $x^2 + y^2 - 14x + 6y + 33 = 0$ and $x^2 + y^2 + 30x - 2y + 1 = 0$ and find the internal and external centres of similitude.

Ans. Internal centre of similitude : (3/2, –2)

External centre of similitude : (18, –5)

49. Prove that the circles $x^2+y^2 - 8x - 6y + 21 = 0$ and $x^2 + y^2 - 2y - 15 = 0$ have exactly two common tangents. Also find the intersection of those tangents.

Ans. (8, 5)

50. Show that the circles $x^2+ y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 18y + 26 = 0$ touch each other. Also find the point of contact and common tangent at this point of circle.

Ans. $\left(\frac{1}{13}, \frac{-21}{13}\right)$

51. Show that the circles $x^2+ y^2 - 4x - 6y - 12 = 0$ and $5(x^2 + y^2) - 8x - 14y - 32 = 0$ touch each other and find their point of contact.

Ans. (–1, –1)

52. Find the equation of the pair of tangents from (10, 4) to the circle $x^2 + y^2 = 25$.

Ans. $9x^2 + 80xy - 75y^2 - 500x - 200y + 2900 = 0$

53, Find the equation to all possible common tangents of the circles $x^2 + y^2 - 2x - 6y + 6 = 0$ and $x^2 + y^2 = 1$.

Ans. $x + 1 = 0$ and $4x - 3y - 5 = 0$

54. Find the equation of the sphere whose centre is (2, –3, 4) and radius is 5.

Ans. $x^2 + y^2 + z^2 - 4x + 6y - 8z + 4 = 0$

55. Find the centre and radius of the sphere $x^2 + y^2 + z^2 - 2x - 4y - 6z = 11$.

Ans. Centre is (1, 2, 3), r = 5

56. Find the equation of the sphere containing the points O(0, 0, 0), A(a, 0, 0), B(0, b, 0) and C(0, 0, c). Also find the centre and radius of the sphere.

Sol. Suppose equation of the sphere is :

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

This sphere passes through O(0, 0, 0)

$\Rightarrow d = 0$

This sphere passes through A(a, 0, 0)

$a^2 + 2ua = 0 \Rightarrow 2u = -a$

This sphere passes through B(0, b, 0)

$b^2 + 2vb = 0 \Rightarrow 2v = -b$

This sphere passes through C(0, 0, c)

$c^2 + 2wc = 0 \Rightarrow 2w = -c$

Equation of the sphere OABC is

$x^2 + y^2 + z^2 - ax - by - cz = 0$

Centre $(-u, -v, -w) = \left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$

$r = \sqrt{u^2 + v^2 + w^2 - d} = \sqrt{\frac{a^2}{4} + \frac{b^2}{4} + \frac{c^2}{4}} = \frac{\sqrt{a^2 + b^2 + c^2}}{2}$

57. **Find the equation of the circum scribing the tetrahedron whose faces are $\frac{y}{b} + \frac{z}{c} = 0$, $\frac{z}{c} + \frac{x}{a} = 0$, $\frac{x}{a} + \frac{y}{b} = 0$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.**

Ans. $x^2 + y^2 + z^2 - (a^2 + b^2 + c^2)\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right) = 0$

58. A plane passes through a fixed point (a, b, c). Show that the foot of the perpendicular from the origin to the plane lies on the sphere $x^2 + y^2 + z^2 - ax - by - cz = 0$.

Sol. Let A = (a, b, c) and let $P(x_0, y_0, z_0)$ be the foot of perpendicular from the origin to the plane.

d.r.s of AP are : $(x_0 - a, y_0 - b, z_0 - c)$

d.r.s of OP are : (x_0, y_0, z_0)

OP and AP are perpendicular

$x_0(x_0 - a) + y_0(y_0 - b) + z_0(z_0 - c) = 0$

$x_0^2 + y_0^2 + z_0^2 - ax_0 - by_0 - cz_0 = 0$

Locus of P (x_0, y_0, z_0) is

$x^2 + y^2 + z^2 - ax - by - cz = 0$.

59. A variable plane passes through a fixed point (a, b, c) and intersect X, Y, Z coordinate axes at A, B, C respectively. Show that the centre of the sphere OABC lies on the surface $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2$.

Sol. Let the variable plane passing through (a, b, c) meets X-axis at A, Y-axis at B and Z-axis at C.

Let OA = p, OB = q, OC = r

Coordinates of A are (p, 0, 0), B are (0, q, 0) and C are (0, 0, r).

Centre of the sphere OABC is $\left(\frac{p}{2},\frac{q}{2},\frac{r}{2}\right)$

Equation of the plane ABC is $\frac{x}{p}+\frac{y}{q}+\frac{z}{r}=1$

Thus plane passes through P(a, b, c)

$\therefore \frac{a}{p}+\frac{b}{q}+\frac{c}{r}=1$

$$\Rightarrow \frac{a}{2\left(\frac{p}{2}\right)}+\frac{b}{2\left(\frac{q}{2}\right)}+\frac{c}{2\left(\frac{r}{2}\right)}=1$$

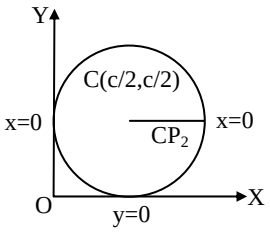
$$\Rightarrow \frac{a}{\left(\frac{p}{2}\right)}+\frac{b}{\left(\frac{q}{2}\right)}+\frac{c}{\left(\frac{r}{2}\right)}=2$$

The centre of the sphere $\left(\frac{p}{2},\frac{q}{2},\frac{r}{2}\right)$ lies on the surface $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2$.

60. Find the equations of circles which touch the lines x = 0, y = 0 and x = c.

Sol. Centre of the circle is $\left(\pm\frac{c}{2},\pm\frac{c}{2}\right)$

r = radius = CP_2



Equation of the ciecle is

$$\left(x\pm\frac{c}{2}\right)^2+\left(y\pm\frac{c}{2}\right)^2=c^2$$

$$x^2\pm cx+\frac{c^2}{4}+y^2\pm cy+\frac{c^2}{4}-\frac{c^2}{4}=0$$

$x^2 + y^2 \pm cx \pm cy + \frac{c^2}{4} = 0$.

61. Find the condition that the two circles $x^2 + y^2 + 2a_1x + 2b_1y = 0$ and $x^2 + y^2 + 2a_2x + 2b_2y = 0$ touch each other.

Sol. Equations of the circles are

$x^2 + y^2 + 2a_1x + 2b_1y = 0$

$x^2 + y^2 + 2a_2x + 2b_2y = 0$

Centre are $A(-a_1, -b_1)$, $B(-a_2, -b_2)$

$r_1 = \sqrt{a_1^2 + b_1^2}$, $r_2 = \sqrt{a_2^2 + b_2^2}$

If the circles touch each other

$AB = r_1 \pm r_2$

$AB^2 = r_1^2 + r_2^2 \pm 2r_1r_2$

$(a_2 - a_1)^2 + (b_2 - b_1)^2 = a_1^2 + b_1^2 + a_2^2 + b_2^2 \pm 2r_1r_2$

$a_1^2 + a_2^2 - 2a_1a_2 + b_1^2 + b_2^2 - 2b_1b_2 = a_1^2 + b_1^2 + a_2^2 + b_2^2 \pm 2r_1r_2$

$\rightarrow 2(a_1a_2 + b_1b_2) = \pm 2r_1r_2$

$\Rightarrow (a_1a_2 + b_1b_2)^2 = r_1^2r_2^2 = (a_1^2 + b_1^2)(a_2^2 + b_2^2)$

$a_1^2a_2^2 + b_1^2b_2^2 + 2a_1a_2b_1b_2 = a_1^2a_2^2 + a_1^2b_2^2 + a_2^2b_1^2 + b_1^2b_2^2$

$\Rightarrow a_1^2b_2^2 + a_1^2b_2^2 - 2a_1a_2b_1b_2 = 0$

$\Rightarrow [a_1b_2 - a_2b_1]^2 = 0$

$a_1b_2 - a_2b_1 = 0$ is the required condition.

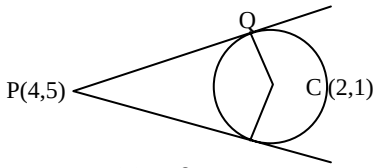
62. Find the inverse of the point (1, 2) with respect to the circle $x^2 + y^2 - 4x - 6y + 9 = 0$.

Ans: Q(0, 1).

63. Find the area of the triangle formed by positive y-axis, the normal and tangent to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$.

Ans; $\frac{2}{\sqrt{3}}$

64. Let $x^2 + y^2 – 4x – 2y – 11 = 0$ be a circle. A pair of tangents are drawn from the point (4,5).Find the area of the quadrilateral formed by these two tangents and a pair of radii of the circle which are lines joining the centre and the point of contact of these two tangents.



Sol. Equation of the circle is $x^2 + y^2 – 4x – 2y – 11 = 0$

Centre is C(2, 1), r = $\sqrt{4+1+11} = 4$

P(4, 5) is the given point, PQ is the tangent from P to the circle.

$PQ = \sqrt{S_{11}} = \sqrt{16+25-16-10-11} = 2$

Area of the Quadrilateral = $2 \cdot \Delta CPQ$

$= 2 \cdot \frac{1}{2} CP \cdot PQ = 4 \cdot 2 = 8$ sq.units