CIRCLES
EXERCISE – 1(D)

1) Find the condition that the tangents drawn from (0,0) to $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ be perpendicular to each other.

Sol. Let θ be the angle between the pair of tangents then $\tan\frac{\theta}{2} = \frac{r}{\sqrt{S_{11}}}$

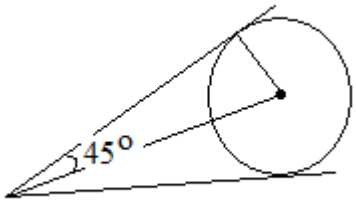
given $\theta = \frac{\pi}{2}$, radius r= $\sqrt{g^2 + f^2 - c}$

Point (0, 0) then S_{11} =0+0+0+0+0+c =c

$\tan 45^0 = \frac{\sqrt{g^2 + f^2 - c}}{\sqrt{c}}$

$\Rightarrow 1 = \frac{\sqrt{g^2 + f^2 - c}}{\sqrt{c}}$

$\Rightarrow g^2 + f^2 - c = c$



$\Rightarrow g^2 + f^2 = 2c$

This is the required condition

2) Find the chord of contact of (0, 5) with respect to the circle

Sol .

Equation of the circle is $S = x^2 + y^2 - 5x + 4y - 2 = 0$

Equation of the chord of contact is $S_1 = 0$

$\Rightarrow x.0 + y.5 - \frac{5}{2}(x + 0) + 2(y+5) - 2 = 0$

$\Rightarrow 10y - 5x + 4y + 20 - 4 = 0$

$\Rightarrow -5x + 14y + 16 = 0$

$\Rightarrow 5x - 14y - 16 = 0$

3) Find the chord of contact of (1, 1) to the circle $x^2 + y^2 = 9$.

Ans:. x+ y = 9

4) Find the polar of (1,2) with respect to $x^2 + y^2 = 7$.

Sol.

point (1,2) and circle is $S = x^2 + y^2 = 7$

The Polar of $P(x_1, y_1)$ w.r. to s = 0 is S_1=0

$\Rightarrow$ x + 2y – 7 = 0 is the polar equation.

5) Find the polar of (3, -1) with respect to $2x^2 + 2y^2=11$

Sol.

Equation of circle is $2x^2 + 2y^2=11$

$\Rightarrow x^2 + y^2= \frac{11}{2}$. Point is (3,-1)

Equation of polar is $S_1=0 \Rightarrow xx_1 + yy_1 = a^2$

$\Rightarrow x(3) + (-1)\, y = \frac{11}{2}$

$\Rightarrow \quad 6x - 2y - 11 = 0.$

6) Find the polar of (1, -2) with respect to $x^2 + y^2 - 10x - 10y + 25 = 0$

Sol .

Equation of the circle is

$x^2 + y^2 - 10x - 10y +25 = 0$

Polar of P (1, -2) is $S_1 = 0$

$\Rightarrow x.1 + y(-2) -5(x+1) -5(y-2) + 25 = 0$

$\Rightarrow x - 2y - 5x - 5 - 5y + 10 + 25 = 0$

$\Rightarrow -4x - 7y + 30 = 0$

$\Rightarrow 4x + 7y - 30 = 0$

7) Find the pole of $ax + by + c = 0$ ($c \neq 0$) With respect to $x^2 + y^2 = r^2$.

Sol.

Let (x_1, y_1) be pole. Then the polar equation is $S_1=0$.

$\Rightarrow xx_1 + yy_1 - r^2 = 0$ _____ (i)

But polar is $ax + by + c = 0$_____(ii)

Equations (i) and (ii) are representing same lines

$$\Rightarrow \frac{x_1}{a} = \frac{y_1}{b} = \frac{-r^2}{c} \Rightarrow x_1 = -\frac{-a}{c} r^2, y_1 = \frac{-br^2}{c}$$

$$\therefore pole\left(\frac{-ar^2}{c}, \frac{-br^2}{c} \right)$$

8) Find the pole of $3x + 4y - 45 = 0$ with respect to $x^2 + y^2 -6x -8y+ 5 =0$.

Ans: (6, 8).

9) Find the pole of $x-2y + 22 = 0$ with respect to $x^2 + y^2 -5x + 8y + 6 = 0$

Ans: (2, -3)

10) Show that the points (-6, 1), (2, 3) are Conjugate points with respect to the circle $x^2 + y^2 - 2x + 2y + 1 = 0$

Sol.

Polar of (2, 3) w.r.t $S= x^2 + y^2 - 2x + 2y + 1 = 0$ is $S_1 =0$

$\Rightarrow 2x+3y-1(x+2)+1(y+3)+1=0$
$\Rightarrow x+4y+2=0$……(1)
Substituting (-6,1) in (i), then
$(-6)+4(1)+2=0$
The point (-6,1) is a point on the polar of 2,3).
$\therefore$ (-6, 1) and (2,3) are conjugate w.r.t.circle.

II method.

$S= x^2+y^2-2x+2y+1=0$
Points are (-6, 1), (2, 3)
Now $S_{12} = -6.2+1.3-(-6+2)+(1+3)+1$
$= -12+3+4+4+1 = 0.$
Therefore given points are conjugate points.

11. Find the value of k if kx + 3y – 1 = 0, 2x +y + 5 = 0 are conjugate lines with respect to the circle $x^2 + y^2-2x – 4y – 4 = 0$

Sol.

Given $S = x^2 + y^2-2x – 4y – 4 = 0$
Lines are kx + 3y – 1 = 0 and 2x +y + 5 = 0
Let (x_1, y_1) be the pole
Then polar is $S_1=0. \Rightarrow xx_1 +yy_1 -1(x+x_1) -2 (y+y_1) – 4 = 0$
$\Rightarrow x(x_1-1) + y (y_1 – 2) - x_1 – 2y_1 – 4 = 0$ –(i)

Comparing (i) with 2x + y + 5 = 0

$$\frac{x_1-1}{2}=\frac{y_1-2}{1}=-\frac{x_1-2y_1-4}{5}$$

$$\frac{x_1-1}{2}=\frac{2(y-2)}{2}=\frac{-x-2y_1-4}{5}=\frac{x_1-1+2y_1-4-x_1-2y_1-4}{2+2+5}$$

$= \frac{-9}{9} = -1$
$x_1 = -1, y_1 = 1 \Rightarrow$ Pole (-1, 1)

Given lines are conjugate lines , (-1, 1) satisfies kx + 3y – 1 = 0
$\Rightarrow K(-1) + 3(1)-1 = 0$
$\Rightarrow$ - k + 2 = 0
$\Rightarrow$ K = 2

12) Find the value of k if x + y – 5 = 0, 2x + ky – 8 = 0 are conjugate with respect To the circle $x^2 + y^2 – 2x – 2y – 1 =0$

Ans; k = 2

14) Find the value of k if the points (1,3)And (2,k) are conjugate with respect to the circle $x^2 + y^2=35$.

Ans; k = 11

15) Find the value of k if the points (4, 2) and (k, - 3) are conjugate points with respect to the circle $x^2 + y^2 – 5x + 8y + 6 = 0$

Sol.

Equation of the circle is $x^2 + y^2 – 5x + 8y + 6 = 0$

Let P(4, 2) and Q(k, - 3)

Given points are conjugate points w.r.t the circle S =0, therefore $S_{12} = 0$

$$\Rightarrow 4k + 2(-3) - \frac{5}{2}(4 + k) + 4(2 - 3) + 6 = 0$$

$$\Rightarrow \frac{3k}{2} - 14 = 0$$

$$\Rightarrow k = \frac{28}{3}$$

II

1) **Find the angle between the tangents drawn from (3, 2) to the circle $x^2 + y^2 – 6x + 4y – 2 = 0$.**

Sol.

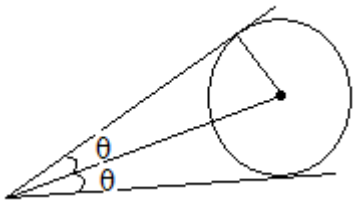
Equation of the circle is $S\equiv x^2 + y^2 – 6x + 4y – 2 = 0$

$r = \sqrt{9 + 4 + 2} = \sqrt{15}$

point P(3,2) $\Rightarrow S_{11} = 9 + 4 – 18 + 8 - 2 = 1$

Let 2θ be the angle between the tangents. Then

$$\tan\theta = \frac{r}{\sqrt{S_{11}}} = \frac{\sqrt{15}}{1} = \sqrt{15}$$



$$\Rightarrow \text{Cos } 2\theta = \frac{1 - \tan^2\theta}{1 + \tan^2\theta} = \frac{1 - 15}{1 + 16} = -\frac{7}{8}$$

Angle between the tangent at P = $\cos^{-1}\left(\frac{7}{8}\right)$.

2) **Find the angle between the pair of tangents drawn from (1,3) to the circle $x^2 + y^2 – 2x + 4y – 11 = 0$.**

Ans. Angle between the tangents = $\cos^{-1}\left(\frac{7}{25}\right)$

3) Find the angle between the pair of tangents drawn from (0, 0) to the circle $x^2 + y^2 - 14x + 2y + 25 = 0$.

Angle between the tangents = $2\theta = \frac{\pi}{4}$

4) Find the locus of P where the tangents drawn from to $x^2 + y^2 = a^2$ include an angle α

Sol.

Equation of the circle is $S = x^2 + y^2 = a^2$, radius = a

let (x_1, y_1) be any point . $\Rightarrow S_{11} = x_1^2 + y_1^2 - a^2$

let $2\theta(=\alpha)$ be the angle between the tangents. Then

$$\Rightarrow \tan\theta = \frac{r}{\sqrt{s_{11}}} = \frac{a}{\sqrt{x_1^2 + y_1^2 - a^2}} \Rightarrow \cos 2\theta = \frac{1 - \frac{a}{\sqrt{x_1^2 + y_1^2 - a^2}}}{1 + \frac{a}{\sqrt{x_1^2 + y_1^2 - a^2}}}$$

$$\Rightarrow \cos\alpha = \frac{x_1^2 + y_1^2 - 2a^2}{x_1^2 + y_1^2}$$

$$\Rightarrow \left(x_1^2 + y_1^2\right)\cos\alpha = x_1^2 + y_1^2 - 2a^2$$

Locus of (x_1, y_1) is $(x^2 + y^2)\cos\alpha = x^2 + y^2 - 2a^2$

$\Rightarrow 2a^2 = (x^2 + y^2)(1 - \cos\alpha)$

$\Rightarrow 2a^2 = (x^2 + y^2)(2\sin^2\alpha/2)$

$$\Rightarrow x^2 + y^2 = \frac{a^2}{\sin^2\frac{\alpha}{2}} = a^2 \operatorname{cosec}\frac{\alpha}{2}$$

5) Find the locus of P where the tangents drawn from P to $x^2 + y^2 = a^2$ are at right angle.

Sol.

$S = x^2 + y^2 - a^2 = 0$. Radius = a

Let $(x_1\ y_1)$ be any point on the locus $\Rightarrow S_{11} = x_1^2 + y_1^2 - a^2$

let 2θ be the angle between the tangents. Then

$$\Rightarrow \tan\theta = \frac{r}{\sqrt{s_{11}}} = \frac{a}{\sqrt{x_1^2 + y_1^2 - a^2}}$$

Given $2\theta = \frac{\pi}{2} \Rightarrow \tan\theta = \tan\frac{\pi}{4} = 1$

$\therefore \Rightarrow \tan\theta = \frac{r}{\sqrt{s_{11}}} = \frac{a}{\sqrt{x_1^2 + y_1^2 - a^2}} = 1 \Rightarrow a^2 = x_1^2 + y_1^2 - a^2$

$\Rightarrow x_1^2 + y_1^2 - 2a^2 = 0$

Locus of P (x_1, y_1) is $x^2 + y^2 = 2a^2$

6) Find the slope of the polar of (1,3) with respect to the circle $x^2 + y^2 - 4x - 4y - 4 = 0$. Also find the distance from the centre to it. ?

Sol.

Equation of the circle is

$S = x^2 + y^2 - 4x - 4y - 4 = 0$, center $C = (2,2)$.

Polar of P (1,3) is $S_1 = 0 \Rightarrow x.1 + y.3 - 2(x+1) - 2(y+3) - 4 = 0$

$\Rightarrow x + 3y - 2x - 2 - 2y - 6 - 4 = 0$

$\Rightarrow -x + y - 12 = 0$

Slope of the polar = 1.

Distance from the centre C(2,2) to the line $= \left|\frac{-2 + 2 - 12}{\sqrt{1+1}}\right|$

$= \frac{12}{\sqrt{2}} = 6 = \sqrt{2}$

6) If $ax + by + c = 0$ is the polar of (1,1) with respect to the circle $x^2 + y^2 - 2x + 2y + 1 = 0$ and H.C.F. of a,b,c is equal to one then find $a^2 + b^2 + c^2$.

Sol.

Equation of the circle is $S = x^2 + y^2 - 2x + 2y + 1 = 0$.

Polar of (1, 1) w.r.to the circle is $S_1 = 0$.

$\Rightarrow x.1 + y.1 - |(x+1)+|\,(y+1) + 1 = 0$

$\Rightarrow x + y - x - 1 + y + 1 + 1 = 0$

$\Rightarrow 2y + 1 = 0$

Given equation of the line $ax + by + c = 0$

Comparing (1) and (2)

$\frac{a}{0} = \frac{b}{2} = \frac{c}{1} = k$, say

$a = 0, b = 2k, c = k$

$a^2 + b^2 + c^2 = 0 + 4k^2 + k^2 = 5k^2$

H.C.F of (a,b,c) = 1 $\Rightarrow k = 1$

$a^2 + b^2 + c^2 = 5(1)^2 = 5$

III).

1) Find the coordinates of the point of intersection of tangents at the points where $x + 4y - 14 = 0$ meets the circle $x^2 + y^2 - 2x + 3y - 5 = 0$.

Sol. Equation of the given circle is $S = x^2 + y^2 - 2x + 3y - 5 = 0$.

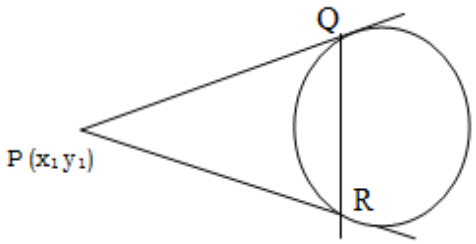
Equation of the line is $x + 4y - 14 = 0$----(i)

Let $P(x_1, y_1)$ be the point of intersection the tangents , then the equation of chord of contact P is $S_1 = 0$.

$\Rightarrow x\,x_1 + y\,y_1 - 1(x + x_1) + \frac{3}{2}(y + y_1) - 5 = 0$

$\Rightarrow 2x\,x_1 + 2y\,y_1 - 2x - x_1 + \frac{3}{2}(y + y_1) - 5 = 0$

$\Rightarrow 2(x_1 - 1)x + (2x_1 + 3)y + 2x_1 - 3y_1 + 10 = 0$ -----(2)



Comparing (1) and (2)

$$\frac{2(x_1-1)}{1}=\frac{2y_1+3}{4}=\frac{2x_1-3y_1+10}{14}$$

$$2(x_1-1)=\frac{2y_1+3}{4}$$

$$\frac{2(x_1-1)}{1}=\frac{2y_1+3}{4}=\frac{2x_1-3y_1+10}{14}$$

$$2(x_1-1)=\frac{2y_1+3}{4}$$

$\Rightarrow 8x_1 - 8 = 2y_1 + 3 \Rightarrow 8x_1 - 2y_1 = 11$ - (1)

$\Rightarrow 2(x_1-1) = \frac{2x_1 - 3y_1 + 10}{15}$

$\Rightarrow 28x_1 - 28 = 2x_1 - 3y_1 + 10$

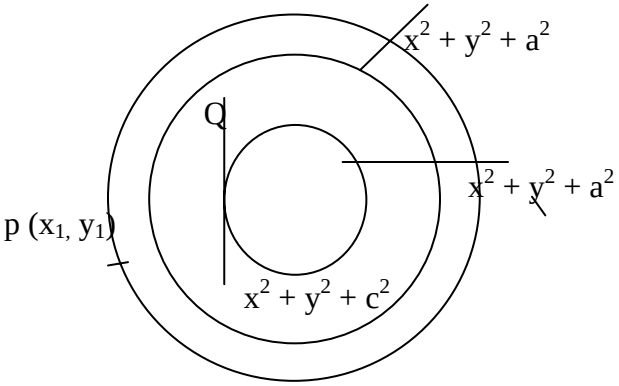
$\Rightarrow 26x_1 + 3y_1 = 38$-------(2)

Solving above equations,

$x_1 = \frac{109}{76}$ $y_1 = \frac{9}{38}$

∴ Co – ordinates of p are $\left(\frac{109}{76}, \frac{9}{38}\right)$

2) If the polar of the points on the circle $x^2 + y^2 = a^2$ with respect to the circle $x^2 + y^2 = b^2$ touches the circle $x^2 + y^2 = c^2$ .Then prove that a,b,c are in Geometrical Progression.



Sol. let P (x_1, y_1) bea point on the circle $x^2 + y^2 = a^2$

$\Rightarrow x_1^2 + y_1^2 = a^2$ - (1)

Polar of P w.r.to the circle $x^2 + y^2 = b^2$ is $xx_1 + yy_1 = b^2$

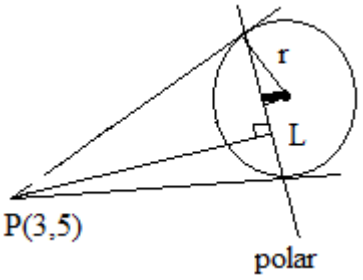
This is a tangent to the circle $x^2 + y^2 = c^2$

$\frac{|0+0-b^2|}{\sqrt{x\frac{2}{1}+ y\frac{2}{1}}} = c \Rightarrow \frac{b^2}{a} = c \quad \Rightarrow b^2 = ac$

$\therefore$ a, b, c are in Geometric Progression

3) Tangents are drawn to the circle $x^2 + y^2 = 16$ from the point P (3,5). Find the area of the triangle formed by these tangents and the chord of contact of P?

Sol : Equation of the circle is $S=x^2 + y^2 -16=0$.



Polar of (3 ,5) is $3x + 5y = 16$

PL = length of the perpendicular from P to its polar $= \frac{|9+25-16|}{\sqrt{9+25}} = \frac{16}{\sqrt{34}}$

Centre of the circle = C (0, 0)

d = Length of the perpendicular from c to polar

$= \frac{|0+0-16|}{\sqrt{34}} = \frac{16}{\sqrt{34}}$

Length of the chord $= 2\sqrt{r^2-d^2} = 2\sqrt{16-\frac{256}{34}} = 2\sqrt{\frac{544-256}{34}}$

$= 2\sqrt{\frac{288}{34}} = 24\sqrt{\frac{2}{34}}$

Area of Δ PQR $= \frac{1}{2}$ base.height$= \frac{1}{2} . 24\sqrt{\frac{2}{34}} . \frac{18}{\sqrt{34}} = \frac{216\sqrt{2}}{34}$

$= \frac{108\sqrt{2}}{17}$ Sq. units .

4) Find the locus of the point whose polars with respect to the circles $x^2 + y^2 – 4x - 4y - 8 = 0$ and $x^2 + y^2 – 2x + 6y – 2 = 0$ are mutually perpendicular.

Sol.

Equation of the circles is

$S= x^2 + y^2 – 4x - 4y - 8 = 0$ - (1)

$S'=x^2 + y^2 – 2x + 6y – 2 = 0$ - (2)

let P (x, y) be any point in the locus.

Equation of the polar of p w.r.to circle (1) is $S_1 = 0$

$\Rightarrow xx_1 \; yy_1 – 2 (x + x_1) – 2 (y + y_1) – 8 = 0$

$\Rightarrow x(x_1 -2) +y (y_1 -2) – (2 x_1 + 2 y_1 + 8) = 0$ -------(3)

Polar of P w.r. to circle (2) is $xx_1 + \; yy_1 – 1 (x+ x_1) – 3 (y + Y_1) – 2 = 0$

$\Rightarrow xx_1 + \; yy_1 – x - x_1 + 3y + 3y_1 – 2 = 0$

$\Rightarrow x(x_1 – 1) + y (y_1 + 3) –(x_1 + 3 y_1 + 2) =0$

(3) and (4) are perpendicular

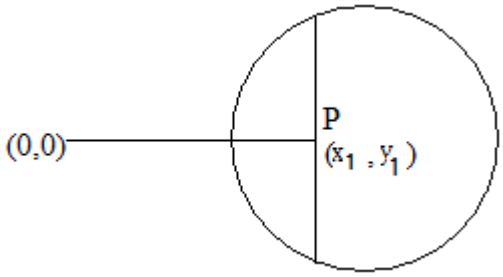
$\Rightarrow a_1 a_2 + b_1 b_2 = 0$

$(x_1 – 2) (x_1 – 1) + (y_1 – 2) (y_1 +3) = 0$

$\Rightarrow x_1^2 + y_1^2 – 3x_1 + y_1 – 6 = 0$

Locus of $p(x_1, y_1)$ is $x^2 + y^2 - 3x + y – 4 = 0$.

5) Find the locus of the foot of the perpendicular drawn from the origin to any chord of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ which subtends a right angle at the origin. ?



Sol.

Equation of the circle is $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$

Let $P (x_1, y_1)$ be the foot of the perpendicular from the origin on the chord.

Slop OP = $\frac{y_1}{x_1}$

$\Rightarrow$ Slop of chord = $-\frac{x_1}{y_1}$ (chord is perpendicular to OP)

$\Rightarrow$ Equation of the chord is $y-y_1 = -\frac{x_1}{y_1} (x - x_1)$

$\Rightarrow yy_1 \; xx_1 = - xx_1 + x_1^2$

$\Rightarrow xx_1 + yy_1 = x_1^2 + y_1^2$

$\Rightarrow \frac{xx_1 + yy_1}{x_1^2 + y_1^2} = 1$--------(1)

Equation of the circle is $x^2 + y^2 + 2fy + c = 0$ - (2)

Hamogenising (2) with the help of (1). Then

$$x^2 + y^2 + (2gx + 2fy)\frac{xx_1 + yy_1}{x_1^2 + y_1^2} + \frac{(xx_1 + yy_1)^2}{(x_1^2 + y_1^2)^2} = 0$$

$$x^2\left[1 + \frac{2gx_1}{x_1^2 + y_1^2} + \frac{cx_1^2}{(x_1^2 + y_1^2)}\right] + y^2\left[1 + \frac{2fy_1}{x_1^2 + y_1^2} + \frac{cy_1^2}{(x_1^2 + y_1^2)}\right] + (\ldots\ldots\ldots\ldots\ldots..)\ xy = 0$$

but above equation is representing a pair of perpendicular lines ,

Co – eff. of x^2 + co-eff of $y^2 = 0$

$$1 + \frac{2gx_1}{x_1^2 + y_1^2} + \frac{cx_1^2}{(x_1^2 + y_1^2)^2} + 1 + \frac{2fy_1}{x_1^2 + Y_1^2} + \frac{cy_1^2}{(x_1^2 + y_1^2)^2} = 0$$

$$2 + \frac{2gx_1 + 2fy_1}{x_1^2 + y_1^2} + \frac{(x_1^2 + y_1^2)}{(x_1^2 + y_1^2)} = 0$$

$$2 + \frac{2gx_1 + 2fy_1}{x_1^2 + y_1^2} + \frac{c}{x_1^2 + y_1^2} = 0$$

$$2\ (x_1 1\uparrow 2 + y_1 1\uparrow 2) + 2gx_{\downarrow}1 + 2fy_{\downarrow}1 + c = 0$$

Locus of L (x_1, y_1) is

$$2(x^2 + y^2) + 2g\,x + 2fy + c = 0$$