CIRCLES

EXERCISE -1 (B)

I.

1 Locate the position of the point P with respect to the circle S=0 when

i) P(3,4) and $S \equiv x^2 + y^2 - 4x - 6y - 12 = 0$

Sol. $S \equiv x^2 + y^2 - 4x - 6y - 12$

Given point P(3,4) $\Rightarrow$ $S_{11} = 3^2 + 4^2 - 4.3 - 6.4 - 12$

$= 9 + 16 - 12 - 24 - 12$

$= -23 < 0$

P (3,4) lies inside the circle

ii) **P(1,5) and $S \equiv x^2 + y^2 - 2x - 4y + 3 = 0$**

Sol.

$S_{11} = (1)^2 + (5)^2 - 2(-1) - 4(5) + 3 = 7$

$S_{11} > 0$ $\therefore$ P is outside the circle

II) P (4,2) and $S \equiv 2x^2 + 2y^2 - 5x - 4y - 3 = 0$

Sol.

$S_{11} = 2(4)^2 + 2(2)^2 - 5(4) - 4(2) - 3 = 9$

$S_{11} > 0$ (P lies outside the circle)

2) Find the power of the point P with Respect to the circle S = 0 When

i) P = (5,-6), and $S \equiv x^2 + y^2 + 8x + 12y + 15$

Sol. power of the point $= S_{11} = 25 + 36 + 40 - 72 + 15 = 116 - 72 = 44$

ii) P =(-1,1) and $S \equiv x^2 + y^2 - 6x + 4y - 12$

Power of the point $= S_{11} = 1 + 1 + 6 + 4 - 12 = 0$

iii) P = (2,4) and $S \equiv x^2 + y^2 - 4x - 6y - 12$

Power of the point = 4 + 16 -8 -24 -12 = - 24.

3) Find the length of tangent from P to the circle S = 0 when

i) P =(-2,5) and S ≡ x^2+y^2-25.

Sol. Length of tangent = $\sqrt{S_{11}}=\sqrt{(-[2])^2+(5)^2-25}$

= 2 units

ii) P = (0, 0), S ≡ $x^2+y^2-14x+2y+25$

$S_{11}=\sqrt{0+0-0+0+25}$

= 5 units

iii) P =(2,5) and S ≡ $x^2+y^2-5x+4y-5$

Sol. Length of the tangent $\sqrt{S_{11}}=\sqrt{4+25-10+20-5}$

$=\sqrt{34}$ units

II.

1) If the length of the tangent from (5,4)to the circle $X^2+y^2+2ky=0$ is 1 then find k.

Sol.

Given point (5,4).

Length of tangent $=\sqrt{S_{11}}=\sqrt{([5])^2+(4)^2+8k}$

But length of tangent =1

$\therefore\ 1=\sqrt{25+16+8k}$

$\Rightarrow$ 1 =41 + 8K

$\Rightarrow$ K = -5 units.

2. If the length of the tangent from (2,5) to the circle $x^2+y^2-5x+4y+k=0$ is $\sqrt{37}$ then fine k.

Sol. given point (2,5)

Length of tangent = $\sqrt{S_{11}}$ = $\sqrt{([2])^2+(5)^2-5\,x\,2+4\,x5+k}$

$\Rightarrow$ 37 = 39 +k

$\Rightarrow$ K = - 2 units.

III.

1. **If a point P is moving such that the Lengths of tangents drawn from P to $X^2 + y^2 + 6x + 18y + 26 = 0$ are in the Ratio 2:3, then find the equation of the Locus of P.**

Sol. Let p(x, y) be any point on the locus.

Let $S \equiv X^2 + y^2 + 4x + 6y - 12 = 0$

Lengths of tangents from P to S=0 is $PT_1 = \sqrt{x^2 + y^2 - 4x - 6y - 12}$

Let $S1 = x^2 + y^2 + 6x + 18y + 26 = 0$

Length Tangent from P to $S^1=0$ is $PT_2 = \sqrt{x^2 + y^2 + 6x + 18y + 26}$

Given $\frac{PT_1}{PT_2} = \frac{2}{3} \Rightarrow \frac{PT_1^2}{PT_2^2} = \frac{4}{9}$

$9\ PT_1^2 = 4.PT_2^2$

$9(x^2 + y^2 - 4x - 6y - 12) = 4(x^2 + y^2 + 6x + 18y + 26)$

$9x^2 + 9y^2 - 36x - 54y - 108 = 4x^2 + 4y^2 + 24x + 72y + 104$

Locus of P is $5x^2 + 5y^2 - 60x - 126y - 212 = 0$

2. **If a point P is Moving such that the Lengths of the tangents drawn from P to the circles $x^2 + y^2 + 8x + 12y + 15 = 0$ and $x^2 + y^2 - 4x - 6y - 12 = 0$ are equal then find the equation of the locus of p**

Sol.

Equations of the circles are $S \equiv x^2 + y^2 + 8x + 12y + 15 = 0$

$S^1 \equiv x^2 + y^2 - 4x - 6y - 12 = 0$

P (x_1, y_1) is any point on the locus and PT_1, PT_2 are the tangents from P to the two circles.

Given condition is

$PT_1 = PT_2 \Rightarrow PT_1^2 = PT_2^2$

$x_1^2 + y_1^2 + 8x_1 + 12y_1 + 15 = x_1^2 + y_1^2 - 4x_1 - 6y_1 - 12$

$12x_1 + 18y_1 + 27 = 0 \Rightarrow 4x_1 + 6y_1 + 9 = 0$

Locus of $P(x_1, y_1)$ is $4x + 6y + 9 = 0$

THEOREM

The equation of the tangent to the circle S = 0 at $P(x_1, y_1)$ is $S_1 = 0$.

NORMAL:

Let S=0 be a circle and P be a point in the circle S=0. The line passing through P and perpendicular to the tangent of S = 0 at P is called the normal to S = 0 at P.

THEOREM

The equation of the normal to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ at $P(x_1, y_1)$ is $(y_1 + f)(x - x_1) - (x_1 + g)(y - y_1) = 0$.

COROLLARY

The equation of the normal to the circle $x^2+y^2 = a^2$ at $P(x_1, y_1)$ is $y_1x - x_1y = 0$.

THEOREM

The condition that the straight line $lx + my + n = 0$ may touch the circle $x^2 + y^2 = r^2$ is $n^2 = r^2(l^2 + m^2)$ and the point of contact is $\left(\frac{-a^2l}{n}, \frac{-a^2m}{n}\right)$.

Proof:

The given line is $lx + my + n = 0$… (1)

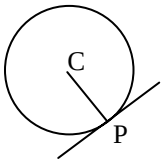
The given circle is $x^2 + y^2 = r^2$…(2)

Centre C = (0,0), radius =r

Line (1) is a tangent to the circle (2)

$\Leftrightarrow$ The perpendicular distance from the centre C to the line (1) is equal to the radius r.

$\Leftrightarrow \left|\frac{0 - n}{\sqrt{l^2 + m^2}}\right| = r$



$\Leftrightarrow (n)^2 = r^2(l^2 + m^2)$

Let $P(x_1,y_1)$ be the point of contact.

Equation of the tangent is $S_1=0$, => $x_1x+y_1y -r^2=0$.----(3)

Equations (1)and (3) are representing the same line, therefore,

$$\frac{x_1}{l} = \frac{y_1}{m} = \frac{-a^2}{n} \Rightarrow x_1 = \frac{-a^2l}{n}, y_1 = \frac{-a^2m}{n}$$

Therefore, point of contact is $\left(\frac{-a^2l}{n}, \frac{-a^2m}{n}\right)$

THEOREM

The condition for the straight line $lx + my + n = 0$ may be a tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $(g^2 + f^2 – c)(l^2 + m^2) = (lg + mf – n)^2$.

Proof:

The given line is $lx + my + n = 0$ … (1)

The given circle is $x^2 + y^2 + 2gx + 2fy + c = 0$… (2)

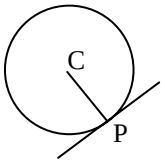
Centre C = (–g, –f), radius $r = \sqrt{g^2 + f^2 - c}$

Line (1) is a tangent to the circle (2)

$\Leftrightarrow$ The perpendicular distance from the centre C to the line (1) is equal to the radius r.

$$\Leftrightarrow \left|\frac{-lg-mf+c}{\sqrt{l^2+m^2}}\right| = \sqrt{g^2+f^2-c}$$

$$\Leftrightarrow (lg+mf-n)^2 = (g^2+f^2-c)(l^2+m^2)$$



COROLLARY

The condition for the straight line y = mx + c to touch the circle $x^2 + y^2 = r^2$ is $c^2 = r^2(1 + m^2)$.

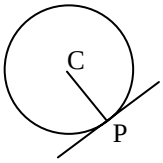
The given line is y=mx+c i.e.,mx-y+c=0 …(1)

The given circle is $S=x^2 + y^2 = r^2$

Centre C = (0,0), radius =r.

If (1) is a tangent to the circle, then

radius of the circle = perpendicular distance from centre of the circle to the line.



$$\Rightarrow r = \frac{|c|}{\sqrt{m^2+1}} \Rightarrow r^2 = \frac{c^2}{m^2+1} \Rightarrow r^2\left(m^2+1\right) = c^2$$

COROLLARY

If the straight line y = mx + c touches the circle $x^2 + y^2 = r^2$, then their point of contact is $\left(-\frac{r^2m}{c}, \frac{r^2}{c}\right)$.

Proof:

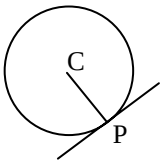
The given line is y=mx+c i.e.,mx-y+c=0 …(1)

The given circle is $S=x^2 + y^2 = r^2$…(2)

Centre C = (0,0), radius =r

Let $P(x_1,y_1)$ be the point of contact.

Equation of the tangent is $S_1=0$, => $x_1x+y_1y-r^2=0$. ----(3)



Equations (1)and (3) are representing the same line, therefore,

$$\frac{x_1}{m} = \frac{y_1}{-1} = \frac{-r^2}{c} \Rightarrow x_1 = \frac{-r^2m}{c}, y_1 = \frac{r^2}{c}$$

Point of contact is $\left(x_1,y_1\right) = \left(-\frac{r^2m}{c}, \frac{r^2}{c}\right)$

THEOREM

If P(x, y) is a point on the circle with centre C(α,β) and radius r, then $x = \alpha + r\cos\theta$, $y = \beta + r\sin\theta$ where $0 \le \theta < 2\pi$.

Note 1 : The equations $x = \alpha + r\cos\theta$, $y = \beta + r\sin\theta$, $0 \le \theta < 2\pi$ are called parametric equations of the circle with centre (α, β) and radius r.

Note 2 : A point on the circle $x^2 + y^2 = r^2$ is taken in the form (r cos θ, r sin θ). The point (r cos θ, r sin θ) is simply denoted as point θ.

THEOREM

The equation of the chord joining two points θ_1 and θ_2 on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $(x+g)\cos\frac{\theta_1+\theta_2}{2}+(y+f)\sin\frac{\theta_1+\theta_2}{2}=r\cos\frac{\theta_1-\theta_2}{2}$ **where r is the radius of the circle.**

Note 1 : The equation of the chord joining the points θ_1 and θ_2 on the circle $x^2 + y^2 = r^2$ is $x\cos\frac{\theta_1+\theta_2}{2}+y\sin\frac{\theta_1+\theta_2}{2}=r\cos\frac{\theta_1-\theta_2}{2}$.

Note 2 : The equation of the tangent at P(θ) on the circle $(x+g)\cos\theta+(y+f)\sin\theta=\sqrt{g^2+f^2-c}$.

Note 3 : The equation of the tangent at P(θ) on the circle $x^2 + y^2 = r^2$ is

$x\cos\theta + y\sin\theta = r$.

Note 4 : The equation of the normal at P(θ) on the circle $x^2 + y^2 = r^2$ is

$x\sin\theta - y\cos\theta = r$.

EXERCISE – 1(C)

1. **Find the equation of the tangent at P of the circle S = 0 where P and S are given by**

i) P = (7,-5), S≡ $x^2 + y^2 - 6x + 4y - 12$

Sol.

Equation of the circle is S≡ $x^2 + y^2 - 6x + 4y - 12 = 0$

Equation of the tangent at P (7,5) is $S_1=0$

$\Rightarrow x(7) + y(-5) - 3(x+7) + 2(y-5) - 12 = 0$

$\Rightarrow 7x - 5y - 3x - 21 + 2y - 10 - 12 = 0$

$\Rightarrow \quad 4x - 3y - 43 = 0$

ii) P = (-1,1), S≡ $x^2 + y^2 - 6x + 4y - 12$

Ans : $4x - 3y + 7 = 0$

iii) P = (-6,-9), S≡ $x^2 + y^2 + 4x + 6y – 39$
Ans. $2x + 3y + 39 = 0$

iv) P = (3,4), S ≡ $x^2 + y^2 – 4x – 6y + 11$
Ans. $x + y – 7 = 0$

2) Find the equation of the normal at P of the circle S = 0 where P and S are given by
i) P = (3, - 4), S≡ $x^2 + y^2 + x + y – 24$

Sol. Equation of the normal passing through(x_1,y_1) is
$(x – x_1)(y_1 + f) –(y - y_1)(x_1 + g) = 0$
$\Rightarrow (x – 3) (-4 + \frac{1}{2}) – (y + 4) (3 + \frac{1}{2}) = 0$
$\Rightarrow \frac{7}{2} (x – 3) - \frac{7}{2} (y +4) = 0$
$\Rightarrow (x – 3) + (y + 4) = 0$
$\Rightarrow$ $x – 3 + y + 4 = 0$ i.e., $x + y + 1 = 0$

ii) P = (3,5), S≡ $x^2 + y^2 – 10x – 2y + 6$
Sol . Equation of the normal passing through (x_1,y_1) is
$(x - x_1) (y_1 + f) – (y - y_1) (x_1 + g)$
$(x – 3) (5 – 1) – (y – 5) (3 – 5) = 0$
$\Rightarrow 4x – 12 + 2y – 10 = 0 \Rightarrow 4x + 2y – 22 = 0$
$\Rightarrow 2x + y – 11 = 0$

iii) P = (1,3), S≡ $3(x^2 + y^2) – 19x - 29 y + 76.$
iv) Ans. $11x – 13y + 28 = 0$

v) P = (1, 2) S≡ $x^2 + y^2 – 22x – 4y + 25.$
Ans. $y = 2$

II.

1) Find the length of the chord intercepted by the circle $x^2 + y^2 – 8x – 2y – 8 = 0$ the line $x + y + 1 = 0$

Sol. Equation of the circle is $x^2 + y^2 - 8x – 2y – 8 = 0$
Centre is C(4,1) and radius r = $\sqrt{16 + 1 + 8} = 5$
Equation of the line is $x + y + 1 = 0$
d = distance from the centre to the line = $\left|\frac{4 + 1 + 1}{\sqrt{1 + 1}}\right| = \frac{6}{\sqrt{2}} = 3\sqrt{2}$
Length of the chord = $2\sqrt{r^2 – d^2} = 2\sqrt{25 - 18} = 2\sqrt{7}$ units.

2) Find the length of the chord intercepted by the circle $x^2 + y^2 – x + 3y – 22 = 0$ on the line $y = x - 3$
Sol. Ans: $4\sqrt{6}$ units.

3) Find the length of the chord formed by $x^2 + y^2 = a^2$ on the line $x \cos\alpha + y \sin\alpha = p$.

Sol. Equation of the circle is $x^2 + y^2 = a^2$

Centre C (0, 0), radius = a

Equation of the line is $X \cos\alpha + y \sin\alpha - p = 0$

distance from the centre to the line = $\frac{|0 + 0 - p|}{\sqrt{\cos^2\alpha + \sin^2\alpha}} = p$

Length of the chord = $2\sqrt{r^2 - d^2} = 2\sqrt{a^2 - p^2}$

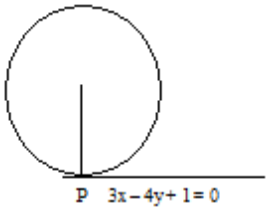
4) Find the equation of circle with centre (2, 3) and touching the line 3x - 4y + 1= 0.

Sol.

Centre C=(2,3).

Radius r =Perpendicular distance from C to 3x-4y +1=0

$= \left|\frac{3(2) - 4(3) + 1}{\sqrt{3^2 + 4^2}}\right|$

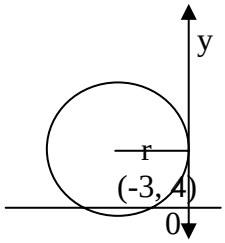


Equation of circle $(x-h)^2 + (y-k)^2 = r^2$

$(x-2)^2 + (y-3)^2 = 1$

$x^2 + y^2 - 4x - 6y + 12 = 0$.

5) Find the equation of the circle with Centre (-3,4) and touching y-axis.



Sol. Centre of the circle is C (-3,4).

Since the circle is touching y – axis, radius r = distance of c from y-axis = $|3| = 3$

Equation of the circle is $(x+3)^2 + (y-4)^2 = 9$

$x^2 + 6x + 9 + y^2 - 8y + 16 - 9 = 0$

$x^2 + y^2 + 6x - 8y + 16 = 0$

6) Find the equation of tangents of the Circle $x^2 + y^2 + 8x - 2y + 12 = 0$ at the points Whose ordinates are 1.

Sol. Equation of the circle is $S \equiv x^2 + y^2 - 8x - 2y + 12 = 0$

Let the point P be $(\alpha, 1)$

Since P is a point on the circle , $\alpha^2 + 1 - 8\alpha - 2 + 12 = 0$

$a^2 - 8a + 11 = 0$

$a = \frac{8 \pm \sqrt{64 - 44}}{2} = \frac{8 \pm 2\sqrt{5}}{2} = 4 \pm \sqrt{5}$

$a_1 = 4 + \sqrt{5}$, $a_2 = 4 - \sqrt{5}$

Co – ordinates of P are $(4 + \sqrt{5}, 1)$ and Q $(4 - \sqrt{5}, 1)$

Equation of the tangent at P$(4 + \sqrt{5},1)$ is $S_1 = 0$

$x(4 + \sqrt{5}) + y.1 - 4(x + 4 + \sqrt{5}) - (y + 1) + 12 = 0$

$\Rightarrow 4x + \sqrt{5}x + y - 4x - 16 - 4\sqrt{5} - y - 1 + 12 = 0$

$\Rightarrow \sqrt{5}x - 5 - 4\sqrt{5} = 0 \Rightarrow \sqrt{5}(x - \sqrt{5} - 4) = 0$

$\Rightarrow x - \sqrt{5} - 4 = 0 \quad \Rightarrow \quad x = 4 + \sqrt{5}$

Equation of the tangent at Q$(4 - \sqrt{5}, 1)$ is $S_2 = 0$

$\Rightarrow x(4 - \sqrt{5}) + y.1 - 4(x + 4 - \sqrt{5}) - (y+1) + 12 = 0$

$\Rightarrow 4x - \sqrt{5}x + y - 4x - 16 + 4\sqrt{5} - y - 1 + 12 = 0$

$\Rightarrow -\sqrt{5}x + 4\sqrt{5} - 5 = 0 \Rightarrow -\sqrt{5}(x - 4 + \sqrt{5}) = 0$

$\Rightarrow x - 4 + \sqrt{5} = 0 \Rightarrow \quad x = 4 - \sqrt{5}$

7) Find the equation of tangents the circle $x^2 + y^2 - 10 = 0$ at the points whose abscissa are 1?

Sol. Equation of the circle is $S = x^2 + y^2 = 10$

Let the point be (1, y) , then $1 + y^2 = 10 \Rightarrow y^2 = 9$

$Y = \pm 3$.

Co – ordinates of P are (1,3) and (1, -3)

Equation of the tangent at P (1, 3) is $S_1 = 0$. $\Rightarrow x.1 + y.3 = 10$

$\Rightarrow x + 3y - 10 = 0$

Equation of the tangent of P(1, -3) is $S_2 = 0$

$\Rightarrow x.1 + y(-3) = 10 \Rightarrow x - 3y - 10 = 0$

III.

1) If $x^2 + y^2 = c^2$ and $\frac{x}{a} + \frac{y}{b} = 1$ intersect at A and B, then find AB. Hence deduce the condition, that the line touches the circle.

Sol. Equation of the line is $\frac{x}{a} + \frac{y}{b} = 1$

Equation of the circle $x^2 + y^2 = c^2$.

centre C = (0,0) and radius r = c

Perpendicular from c to the line is $d = \left|\frac{\frac{0.1}{a} + 0.\frac{1}{b} - 1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}}\right| = \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}}$

Length of chord AB = $2\sqrt{r^2 - d^2}$ = $2\sqrt{c^2 - \frac{1}{\frac{1}{a^2} + \frac{1}{b^2}}}$ = $2\sqrt{c^2 - \left(\frac{a^2b^2}{a^2 + b^2}\right)}$

If the line touches the circle, then Length of the chord = 0

$\rightarrow$ c = $\sqrt{\frac{a^2b^2}{a^2 + b^2}}$

2. If y = mx + c and $x^2 + y^2 = a^2$ i) intersect at A and B. ii) AB = 2λ, then show that $C^2 = (1 + m^2)(a^2 - \lambda^2)$

Sol.

Equation of circle $x^2 + y^2 = a^2$

C = (0, 0) r = a

Length of chord = $2\sqrt{r^2 - d^2}$

$\Rightarrow 2\sqrt{r^2 + d^2} = 2\lambda$ $\Rightarrow$ $a^2 - d^2 = \lambda^2$ ----------(*1)

Equation of the line y = mx + c $\Rightarrow$ mx – y + c = 0

Perpendicular distance from c to line is d = $\left|\frac{0 - 0 + c}{\sqrt{m^2 + 1}}\right|$

$\Rightarrow d^2 = \frac{c^2}{m^2 + 1}$

Therefore, from (1)

$\Rightarrow a^2 - \frac{c^2}{m^2 + 1} = \lambda^2$

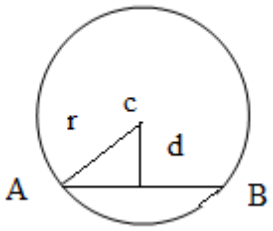
$\therefore$ $c^2 = (a^2 - \lambda^2)(1 + m^2)$ which is the required condition

3) Find the equation of the circle with centre (-2, 3) cutting a chord length 2 units on 3x + 4y + 4 = 0

Sol. Equation of the line is 3x + 4y + 4 = 0

d = Length of the perpendicular form (-2, 3) to 3x+4y +4 = 0

= $\frac{|3(-2) + 4.3 + 4|}{\sqrt{9 + 16}}$ = $\frac{10}{5}$ = 2



If r is the radius of the circle then Length of the chord = $2\sqrt{r^2 - d^2}$ = 2 (given)

$r^2 - d^2 = 1$ then $r^2 - 2^2 = 1 \Rightarrow r^2 = 5$.

$\therefore$ Equation of the circle is $(x+2)^2 + (y-3)^2 = r^2$

$x^2 + 4x + 4 + y^2 - 6y + 9 = 5$

$x^2 + y^2 + 4x - 6y + 8 = 0$

4) Find the equation of tangent and normal at (3,2) of the circle $x^2 + y^2 - x - 3y - 4 = 0$

Sol.

Equation of the circle is $S \equiv x^2 + y^2 - x - 3y - 4 = 0$

Equation of the tangent at P(3,2) is $S1 = 0$

$\Rightarrow x.3 + y.2 - \frac{1}{2}(x + 3) - \frac{3}{2}(y + 2) - 4 = 0$

$6x + 4y - x - 3 - 3y - 6 - 8 = 0 \Rightarrow 5x + y - 17 = 0$

Let the normal be $x - 5y + c = 0$

Since normal passes through P (3, 2)

$\Rightarrow 3 - 10 + c = 0 \Rightarrow c = 7$

Hence Equation of the normal is $x - 5y + 7 = 0$

5) Find the equation of the tangent and normal at (1,1) to the circle $2x^2 + 2y^2 - 2x - 5y + 3 = 0$

Ans: $2x - y - 1 = 0, x + 2y - 3 = 0$

6) Prove that the tangent at (3, -2) of the circle $x^2 + y^2 = 13$ touches the circle $x^2 + y^2 + 2x - 10y - 26 = 0$ and find its point of contact. ?

Sol.

Equation of the circle is $S \equiv x^2 + y^2 = 13$

Equation of the tangent at P(3, -2) is $S_1 = 0 \Rightarrow x.3 + y(-2) = 13$

$\Rightarrow \quad 3x - 2y - 13 = 0$ ----------(1)

Equation of the second circle is $x^2 + y^2 + 2x - 10y - 26 = 0$

Centre is c (-1, 5) and

Radius r = $\sqrt{1 + 25 + 26} = \sqrt{52} = 2\sqrt{13}$

d = length of the perpendicular from c (-1, 5) to (1)

$d = \frac{|-3 - 10 - 13|}{\sqrt{9 + 4}} = \frac{26}{\sqrt{13}} = 2\sqrt{13}$ = radius

$\therefore$ The tangent to the first circle also touches the second circle.

Equation of the circle $x^2 + y^2 + 2x - 10y - 26 = 0$

Center c = (-1,5)

Equation of normal is $2x + 3y + k = 0$. This normal is passing through (-1,5)

$\Rightarrow 2(-1) + 3(5) + k = 0 \Rightarrow K = -13$

$\therefore$ Normal is $2x + 3y - 13 = 0$ ---------- (2)

Solving (1) & (2)

Point of intersection is (5,1)

$\therefore$ Point of contact is (5,1)

7) Show that the tangent at (-1,2) the Circle $x^2 + y^2$-4x-8y+7=0 touches the Circle $x^2 + y^2$+4x+6y=0 and also Find its point of contact.

Sol.

S≡ $x^2 + y^2 - 4x - 8y + 7 = 0$ equation of the tangent at (-1, 2) to S =0 is S_1=0

$\Rightarrow x(-1) + y(2) - 2(x-1) - 4(y+2) + 7 = 0$

$\Rightarrow -3x - 2y + 1 = 0 \Rightarrow 3x + 2y - 1 = 0$.

Equation of the second circle is $x^2 + y^2 + 4x + 6y = 0$

Centre C =(-2,-3) radius r $= \sqrt{g^2 + f^2 - c} = \sqrt{4 + 9} = \sqrt{13}$

Perpendicular distance from C to the line is d = $\left|\frac{3(-2) + 2(-3) - 1}{\sqrt{3^2 + 2^2}}\right| = \left|\frac{-13}{\sqrt{13}}\right| = \sqrt{13}$

= radius of the circle.

Hence 3x + 2y – 1 =0 is also tangent to $x^2 + y^2$+4x +6y = 0

Point of contact is the foot of perpendicular from the centre of the circle to the tangent.

Let (h, k) be foot of perpendicular from (-2,-3) to the line 3x+2y-1 =0

$\frac{h+2}{3} = \frac{k+3}{2} = \frac{|3(-2) + 2(-3) - 1|}{9+4} \Rightarrow \frac{h+2}{3} = 1$ and $\frac{k+3}{2} = 1$

h = 1, k = -1 therefore (1, -1) is point of contact.

8) Find the equation of the tangents to the circle $x^2 + y^2$-4x + 6y – 12 = 0 which are parallel to x + y – 8 = 0

Sol.

Equation of the circle is S = $x^2 + y^2 - 4x + 6y - 12 = 0$

Centre is C(2, -3); r = radius = $\sqrt{4 + 9 + 12} = 5$

Equation of the line parallel to the line x +y-8=0 is x+y+k=0

If x+y+k=0 is a tangent to the circle, then radius = perpendicular distance from the centre.

$5 = \frac{|2 - 3 + k|}{\sqrt{1+1}} \Rightarrow |k - 1| = 5\sqrt{2} \Rightarrow k-1 = \pm 5\sqrt{2} \Rightarrow k = 1 \pm 5\sqrt{2}$

Equation of the tangent is $x+y+1 \pm 5\sqrt{2} = 0$

9) Find the equations of the tangents to the circle x^2+y^2 + 2x – 2y – 3 = 0 which are perpendicular to 3x –y+4=0

Sol.

S = x^2+y^2 +2x-2y-3=0, centre C (-1, 1) and radius $r = \sqrt{1+1+3} = \sqrt{5}$

Equation of the line perpendicular to 3x-y+4 =0 is x+ 3y +k = 0

If this line is a tangent, then

$\sqrt{5} = \left|\frac{-1+3+k}{\sqrt{1+9}}\right| \Rightarrow 5 = \frac{(k+2)^2}{10}$

$\Rightarrow 50 = k^2+4k+4 \Rightarrow k^2+4k-46 = 0$

$\Rightarrow k = \dfrac{-4 \pm \sqrt{16+184}}{2}$

$k = \dfrac{-4 \pm 10\sqrt{2}}{2} = -2 \pm 5\sqrt{2}$

Equation of the required tangent is $x + 3y - 2 \pm 5\sqrt{2} = 0$

10) Find the equation of the tangents to the circle $x^2+y^2 - 4x - 6y + 3 = 0$ which makes an angle 45^0 with x – axis.

Sol. Equation of the circle is $S = x^2+y^2-4x-6y+3=0$

Centre C(3, 3), radius $r = \sqrt{4 + 9 - 3} = \sqrt{10}$

Slope of the tangent $m = \tan 45^0 = 1$

Equation of the tangent can be taken as $y=x+c$ i.e., $x-y+c = 0$

Length of the perpendicular from centre c to tangent $= \dfrac{|2 - 3 + c|}{\sqrt{2}}$

$\therefore \sqrt{10} = \dfrac{|c - 1|}{\sqrt{2}} \Rightarrow (c-1)^2 = 20$

$c-1 = \pm \sqrt{20} = \pm 2\sqrt{5}$

$c = 1 \pm 2\sqrt{5}$

Equation of the tangents are $x - y + 1 \pm 2\sqrt{5} = 0$

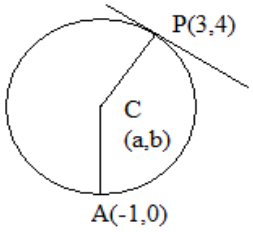
11) Find the equation of the circle passing through (-1, 0) and touching x + y – 7=0 at (3,4)

Sol.

Let C (a,b) be the centre of the circle

Let A(-1,0) and P(3,4)

Equation of the tangent is x+y-7 =0 ----------(i)



Now CA = CP $\Rightarrow CA^2 = CP^2$

$\Rightarrow (a+1)^2 + b^2 = (a-3)^2 + (b-4)^2$

$\Rightarrow$ 8a +8b -24 =0 $\Rightarrow$a+b -3 = 0 ---(2)

Line CP is perpendicular to tangent (1) ,$\therefore$ product of their slopes = -1

$\left(\dfrac{b-4}{a-3}\right)(-1) = -1 \Rightarrow$ a – b +1 =0 ----(3)

Solving (2) and (3) , a=1 and b=2. Centre C =(1, 2)

Radius r = CA = $\sqrt{(1+1)^2 + 2^2} = \sqrt{8}$

Equation of the circle is $(x-1)^2 + (y-2)^2 = 8$

$\Rightarrow x^2+y^2- 2x – 4y – 3 =0$

12) Find the equations of the circles passing through (1, -1)and touching the lines $4x + 3y + 5 = 0$ and $3x – 4y – 10 = 0$

13) Show that $x +y +1 = 0$ touches the circle $x^2+y^2-3x + 7y + 14 = 0$ and find its point of contact.

Ans: (2,-3)

THEOREM

If a line passing through a point $P(x_1, y_1)$ intersects the circle $S = 0$ at the points A and B then $PA.PB = |S_{11}|$.

COROLLARY

If the two lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y+c_2 = 0$ meet the coordinate axes in four distinct points then those points are concyclic $\Leftrightarrow a_1a_2 = b_1b_2$.

COROLLARY

If the lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y+c_2 = 0$ meet the coordinate axes in four distinct concyclic points then the equation of the circle passing through these concyclic points is $(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) – (a_1b_2 + a_2b_1)xy = 0$.

THEOREM

Two tangents can be drawn to a circle from an external point.

Proof: The given circle is $S=x^2 + y^2 = r^2$…(2)

Centre C = (0,0), radius = r

Let $P(x_1, y_1)$ be an external point . then $S_{11} >0 \Rightarrow x_1^2 + y_1^2 - a^2 > 0$

Let m be the slope of the tangent.

Then equation of the tangent is $y = mx + r\sqrt{1+m^2}$

This tangent is passing through $P(x_1, y_1)$, then

$y_1 = mx_1 + r\sqrt{1+m^2} \Rightarrow y_1 - mx_1 = r\sqrt{1+m^2}$

$\Rightarrow (y_1 - mx_1)^2 = r^2(1+m^2)$

$\Rightarrow (x_1^2 - a^2)m^2 - 2x_1y_1m + (y_1^2 - a^2) = 0$ Which is a quadratic equation in m and hence m has two values.

The discriminant of above equation is

$4x_1^2y_1^2 - 4(x_1^2 - a^2)(y_1^2 - a^2) = 4a^2(x_1^2 + y_1^2 - a^2) > 0$

The values of m are real and different.

Hence two tangents can be drawn from an external point to the circle.

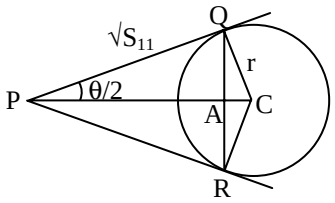
COROLLARY

If m_1, m_2 are the slopes of tangents drawn to the circle $x^2 + y^2 = a^2$ from an external point (x_1, y_1) then $m_1 + m_2 = \frac{2x_1y_1}{x_1^2 - a^2}$, $m_1m_2 = \frac{y_1^2 - a^2}{x_1^2 - a^2}$.

THEOREM

If θ is the angle between the tangents through a point P to the circle S = 0 then $\tan\frac{\theta}{2} = \frac{r}{\sqrt{S_{11}}}$ where r is the radius of the circle.

Proof:

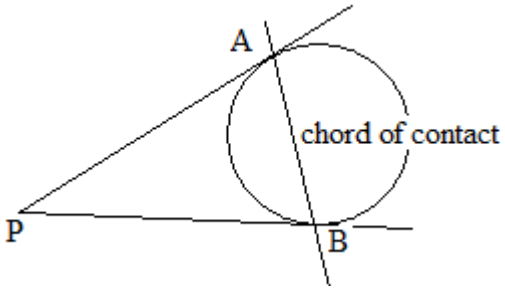


Let the two tangents from P to the circle S = 0 touch the circle at Q, R and θ be the angle between these two tangents. Let C be the centre of the circle. Now QC = r, PQ = $\sqrt{S_{11}}$ and ΔPQC is a right angled triangle at Q.

$\therefore \tan\frac{\theta}{2} = \frac{QC}{PQ} = \frac{r}{\sqrt{S_{11}}}$

CHORD OF CONTACT:

The line joining the points of contact of the tangents to a circle S = 0 drawn from an external point P is called chord of contact of P with respect to S = 0.



THEOREM

The equation to the chord of contact of $P(x_1, y_1)$ with respect to the circle S = 0 is $S_1 = 0$.

POLE AND POLAR

Let S = 0 be a circle and P be a point. If any line drawn through the point P meets the circle in two points A and B then the point of intersection of the tangents drawn at A

and B lie on a line called the polar of P w.r.t. S = 0 and P is called the pole of the polar line.

THEOREM

The equation of the polar of the point $P(x_1, y_1)$ with respect to the circle S = 0 is $S_1 = 0$.

Note 1: If P lies outside the circle S = 0 then the polar of P meets the circle in two points and the polar becomes the chord of contact of P.

Note 2: If P lies on the circle S = 0 then the polar of P becomes the tangent at P to the circle S = 0.

Note 3: If P lies inside the circle S = 0, then the polar of P does not meet the circle.

Note 4: If P is the centre of the circle S = 0, then the polar of P with respect to S = 0 does not exist.

THEOREM

The pole of the line lx + my + n = 0 ($n \neq 0$) with respect to $x^2 + y^2 = a^2$ is $\left(-\frac{la^2}{n}, -\frac{ma^2}{n}\right)$.

Proof :

Let $P(x_1, y_1)$ be the pole of $lx + my + n = 0$ … (1)

The polar of P with respect to the circle is :

$xx_1 + yy_1 - a^2 = 0$ …(2)

Now (1) and (2) represent the same line

$$\therefore \frac{x_1}{\ell} = \frac{y_1}{m} = \frac{-a^2}{n} \Rightarrow x_1 = \frac{-la^2}{n}, y = \frac{-ma^2}{n}$$

$$\therefore \text{Pole } P = \left(-\frac{la^2}{n}, -\frac{ma^2}{n}\right)$$

THEOREM

If the pole of the line lx + my + n = 0 with respect to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is (x_1, y_1) then $\frac{x_1 + g}{\ell} = \frac{y_1 + f}{m} = \frac{r^2}{lg + mf - n}$ where r is the radius of the circle.

Proof :

Let $P(x_1, y_1)$ be the pole of the line $lx + my + n = 0$ …(1)

The polar of P with respect to S = 0 is $S_1 = 0$

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

$\Rightarrow (x_1 + g)x + (y_1 + f)y + gx_1 + fy_1 + c = 0$ …(2)

Now (1) and (2) represent the same line.

$$\therefore \frac{x_1+g}{\ell}=\frac{y_1+f}{m}=\frac{gx_1+gy_1+c}{n}=k(\text{say})$$

$$\frac{x_1+g}{\ell}=k\Rightarrow x_1+g=\ell k\Rightarrow x_1=\ell k-g$$

$$\frac{y_1+f}{m}=k\Rightarrow y_1+f=mk\Rightarrow y_1=mk-f$$

$$\frac{gx_1+gy_1+c}{n}=k\Rightarrow gx_1+gy_1+c=nk$$

$$\Rightarrow g(lk-g)+f(mk-f)+c=nk$$

$\Rightarrow k(lg+mf-n)=g^2+f^2-c=r^2$ where r is the radius of the circle

$$\Rightarrow k=\frac{r^2}{lg+mf-n}.$$

$$\therefore \frac{x_1+g}{\ell}=\frac{y_1+f}{m}=\frac{r^2}{lg+mf-n}.$$

CONJUGATE POINTS

THEOREM

If the polar of a point P with respect to the circle S = 0 passes through a point Q, then the polar of Q with respect to S = 0 passes through P.

DEFINITION

Two points P and Q are said to be conjugate points with respect to the circle S = 0 if the polar of P with respect to S = 0 passes through Q.

CONJUGATE LINES

DEFINITION

Two lines $L_1 = 0$, $L_2 = 0$ are said to be conjugate lines with respect to the circle S = 0 if the pole of $L_1 = 0$ lies on $L_2 = 0$.

THEOREM

The lines $l_1x + m_1y + n_1 = 0$ and $l_2x + m_2y + n_2 = 0$ are conjugate with respect to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ iff $r^2(l_1l_2 + m_1m_2) = (l_1g + m_1f - n_1)(l_2g + m_2f - n_2)$.

Proof :

Suppose $L_1 \equiv l_1x + m_1y + n_1 = 0$ and $L_2 \equiv l_2x + m_2y + n_2 = 0$ are conjugate with respect to $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ and r be the radius of the circle. If (x_1, y_1) is the pole of $l_1x + m_1y + n_1 = 0$ then

$$\frac{x_1+g}{l_1}=\frac{y_1+f}{m_1}=\frac{r^2}{l_1g+m_1f-n_1}$$

$$\Rightarrow x_1 + g = \frac{l_1 r^2}{l_1 g + m_1 f - n_1}, y_1 + f = \frac{m_1 r^2}{l_1 g + m_1 f - n_1}$$

$$\Rightarrow x_1 = -g + \frac{l_1 r^2}{l_1 g + m_1 f - n_1}, y_1 = -f + \frac{m_1 r^2}{l_1 g + m_1 f - n_1}$$

$$\therefore \text{Pole} = \left(-g + \frac{l_1 r^2}{l_1 g + m_1 f - n_1}, -f + \frac{m_1 r^2}{l_1 g + m_1 f - n_1} \right)$$

$L_1 = 0$, $L_2 = 0$ are conjugate lines with respect to$S = 0$

$\Leftrightarrow$ P lies on $L_2 = 0$.

$$\Leftrightarrow l_2 \left(-g + \frac{l_1 r^2}{l_1 g + m_1 f - n_1} \right) + \quad m_2 \left(-f + \frac{m_1 r^2}{l_1 g + m_1 f - n_1} \right) + n_2 = 0$$

$$\Leftrightarrow -l_2 g + \frac{l_1 l_2 r^2}{l_1 g + m_1 f - n_1} - m_2 f + \frac{m_1 m_2 r^2}{l_1 g + m_1 f - n_1} + n_2 = 0$$

$$\Leftrightarrow \frac{(l_1 l_2 + m_1 m_2) r^2}{l_1 g + m_1 f - n_1} = l_2 g + m_2 f - n_2$$

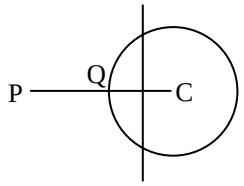
$r^2(l_1 l_2 + m_1 m_2) = (l_1 g + m_1 f - n_1)(l_2 g + m_2 f - n_2)$.

THEOREM

The condition for the lines $l_1x + m_1y + n_1 = 0$ and $l_2x + m_2y + n_2 = 0$ to be conjugate with respect to the circle $x^2 + y^2 = a^2$ is $a^2(l_1l_2 + m_1m_2) = n_1n_2$.

INVERSE POINTS:

Let S = 0 be a circle with centre C and radius r. Two points P, Q are said to be inverse points with respect to S = 0 if (i) C, P, Q are collinear (ii) P, Q lies on the same side of C (iii) CP.CQ = r^2.



Note 1 : Let P, Q be a pair of inverse points with respect to the circle S = 0. If P lies inside the circle S = 0 then Q lies outside the circle S = 0. If P lies on the circle S = 0, then Q = P.

Note 2 : If P, Q are a pair of inverse points with respect to a circle S = 0 then Q is called inverse point of P.

Note 3 :

The inverse point of a point P with respect to a circle S = 0 is unique.

THEOREM

Let S = 0 be a circle with centre C and radius r. The polar of a point P with respect to the circle S = 0 meets $\overleftrightarrow{CP}$ in Q iff P, Q are inverse points with respect to S = 0.

Proof:

Suppose the polar of P with respect to S = 0 meets $\overleftrightarrow{CP}$ in Q.

Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ be the given circle.

$\therefore$ C = (–g, –f), $r = \sqrt{g^2 + f^2 - c}$. Let $P(x_1, y_1)$

The polar of P with respect to S = 0 is $S_1 = 0$

$\Rightarrow xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

$\Rightarrow (x_1 + g)x + (y_1 + f)y + (gx_1 + fy_1 + c) = 0$ …(1)

Slope of (1) is $-\frac{x_1 + g}{y_1 + f}$, Slope of $\overleftrightarrow{CP}$ is $\frac{y_1 + f}{x_1 + g}$.

Product of the slopes = –1.

$\therefore$ $\overleftrightarrow{CP}$ is perpendicular to the polar of P.

CQ = The perpendicular distance from C to (1)

$$= \left| \frac{(x_1 + g)(-g) + (y_1 + f)(-f) + gx_1 + fx_1 + c}{\sqrt{(x_1 + g)^2 + (y_1 + f)^2}} \right|$$

$$= \frac{|-gx_1 - g^2 - fy_1 - f^2 + gx_1 + fy_1 + c|}{CP}$$

$$= \frac{|g^2 + f^2 - c|}{CP} = \frac{r^2}{CP} \Rightarrow CP \cdot CQ = r^2$$

Conversely suppose that P, Q are inverse points with respect to S = 0. Then Q is the inverse point of P with respect to S = 0.

Let the polar of P with respect to S = 0 meets $\overleftrightarrow{CP}$ in R.

Then P, R are inverse points with respect S = 0.

$\therefore$ R is the inverse of P with respect to S = 0.

Since the inverse point of P with respect to S = 0 is unique, R = Q.

$\therefore$ The polar of P with respect to S = 0 meets $\overleftrightarrow{CP}$ in Q.

Note 1 :

If P, Q are inverse points with respect to
S = 0 then P, Q are conjugate points with respect to S = 0.

Note 2 :

If P, Q are inverse points with respect to
S = 0 then Q is the foot of the perpendicular from P on the polar of P with respect
S = 0.

Note 3:

The polar of a point P with respect to a circle with centre C is a perpendicular to $\overleftrightarrow{CP}$.

MIDPOINT OF A CHORD

THEOREM

The equation of the chord of the circle S = 0 having $P(x_1, y_1)$ as its midpoint is $S_1 = S_{11}$.

THEOREM

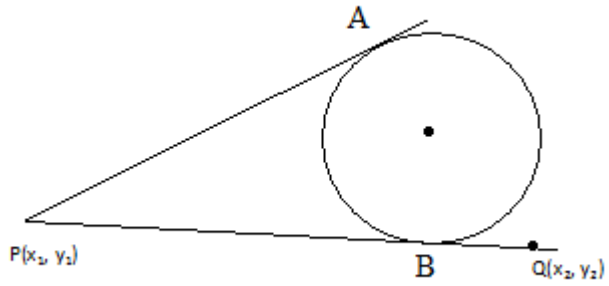
The length of the chord of the circle S = 0 having $P(x_1, y_1)$ as its midpoint is $2\sqrt{|S_{11}|}$.

PAIR OF TANGENTS

THEOREM

The equation to the pair of tangents to the circle S = 0 from $P(x_1, y_1)$ is $S_1^2 = S_{11}S$.

Proof:



Let the tangents from P to the circle S=0 touch the circle at A and B.

Equation of AB is $S_1 = 0$.

i.e., $x_1x + y_1y + g(x + x_1) + f(y + y_1) + c = 0$ =====i)

Let $Q(x_2, y_2)$ be any point on these tangents. Now locus of Q will be the equation of the pair of tangents drawn from P.

The line segment PQ is divided by the line AB in the ratio $-S_{11}:S_{22}$

$$\Rightarrow \frac{PB}{QB} = \left|\frac{S_{11}}{S_{22}}\right| \text{----ii)}$$

BUT $PB = \sqrt{S_{11}}, QB = \sqrt{S_{22}} \quad \Rightarrow \frac{PB}{QB} = \frac{\sqrt{S_{11}}}{\sqrt{S_{22}}}$ ----iii)

From ii) and iii) $\Rightarrow \frac{s^2_{11}}{s^2_{22}} = \frac{S_{11}}{S_{22}}$

$\Rightarrow S_{11}S_{22} = S^2_{12}$

Hence locus of $Q(x_{2,}y_2)$ is $S_{11}S = S^2_1$.