Exponential and Logarithmic Series

1. $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\infty$

2. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

3. $(n+1)^{th}$ **term of** $e^x = T_{n+1} + \frac{x^n}{n!}$

4. $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + ...$

5. $e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$

6. $e^1 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ...\infty$

7. $e^1 = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} - .....\infty$

8. $e^x + e^{-x} = 2\left\{1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ...\infty\right\}$

9. $e^x - e^{-x} = 2\left\{1 + \frac{x^3}{3!} + \frac{x^5}{5!} + ...\infty\right\}$

10. $a^x = 1 + \frac{x \log a}{1!} + \frac{(x \log a)^2}{2!} + ...\infty$

11. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^5}{5} + ...|x| < 1$

12. $\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{5} + ...$

13. $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - .....$

14. $\log\left(\frac{1+x}{1-x}\right) = 2\left\{x + \frac{x^3}{3} + \frac{x^5}{5} .....\right\}$

EXERCISE – 7(a)

1. Find the coefficient of x^2 in the series expansion of e^{3x+4}.

Sol. $e^{3x+4} = e^4 \cdot e^{3x}$

$$= e^4\left[1+\frac{3x}{1!}+\frac{(3x)^2}{2!}+...+\frac{(3x)^n}{n!}+...\right]$$

$\therefore$ Coefficient of x^2 in $e^{3x+4} = e^4\left(\frac{9}{2}\right) = \frac{9}{2}e^4$.

2. Find the coefficient of x^3 in the series expansion of e^{2x+3}.

Sol. $e^{2x+3} = e^3 \cdot e^{2x}$

$$= e^3\left[1+\frac{(2x)}{1!}+\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}+...+\frac{(2x)^n}{n!}+...\right]$$ $\therefore$ Coefficient of x^3 in e^{2x+3}

$$= e^3\frac{(2^3)}{3!} = \frac{8}{6}e^3 = \frac{4}{3}e^3$$

II.

1. Find the sum of the series $1+\frac{3}{1!}+\frac{5}{2!}+\frac{7}{3!}+....$

Hint : $\sum_{n=0}^{\infty}\frac{1}{n!} = \sum_{n-1=0}^{\infty}\frac{1}{(n-1)!} = e$

Sol. $1+\frac{3}{1!}+\frac{5}{2!}+\frac{7}{3!}+...$

$$= \sum_{n=0}^{\infty}\frac{2n+1}{n!} = \sum_{n=0}^{\infty}\left[\frac{2}{(n-1)!}+\frac{1}{n!}\right]$$

$$= 2\sum_{n-1=0}^{\infty}\frac{1}{(n-1)!}+\sum_{n=0}^{\infty}\frac{1}{n!}$$

$$2e+e = 3e$$

$$\therefore 1+\frac{3}{1!}+\frac{5}{2!}+\frac{7}{3!}+... = 3e$$

2. Find the sum of the series $1+\frac{4}{2!}+\frac{7}{3!}+\frac{10}{4!}+...$

Sol. $1+\frac{4}{2!}+\frac{7}{3!}+\frac{10}{4!}+...$

$$= \sum_{n=1}^{\infty}\frac{3n-2}{n!} = \sum_{n=1}^{\infty}\left[\frac{3}{(n-1)!}-\frac{2}{n!}\right]$$

$$= \sum_{n=1}^{\infty}\left[\frac{3}{(n-1)!}\right]-\sum_{n=1}^{\infty}\frac{2}{n!} = 3\sum_{n=1}^{\infty}\frac{1}{(n-1)!}-2\sum_{n=1}^{\infty}\frac{1}{n!}$$

$$= 3(e)-2(e-1)$$

$= 3e - 2e + 2 = e + 2$

$\therefore 1+\frac{4}{2!}+\frac{7}{3!}+\frac{10}{4!}+...=e+2$

3. Show that $\frac{2}{3!}+\frac{4}{5!}+\frac{6}{7!}+\frac{7}{8!}+...=\frac{1}{e}$.

Sol. $\frac{2}{3!}+\frac{4}{5!}+\frac{6}{7!}+\frac{7}{8!}+...$

$=\sum_{n=1}^{\infty}\frac{2n}{(2n+1)!}=\sum_{n=1}^{\infty}\frac{(2n+1)-1}{(2n+1)!}$

$=\sum_{n=1}^{\infty}\left[\frac{1}{(2n)!}-\frac{1}{(2n+1)!}\right]$

$=\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}-\frac{1}{7!}+...$

$=1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+...$

$=e^{-1}=\frac{1}{e}$

$\therefore \frac{2}{3!}+\frac{4}{5!}+\frac{6}{7!}+\frac{8}{9!}+...=\frac{1}{e}$

4. Show that $1+\frac{2^2}{2!}+\frac{3^2}{3!}+\frac{4^2}{4!}+...=2e$.

Sol. $1+\frac{2^2}{2!}+\frac{3^2}{3!}+\frac{4^2}{4!}+...$

$=\sum_{n=1}^{\infty}\frac{n^2}{n!}==\sum_{n=1}^{\infty}\frac{n}{(n-1)!}$

$=\sum_{n=1}^{\infty}\frac{n-1+1}{(n-1)!}=\sum_{n=1}^{\infty}\left[\frac{1}{(n-2)!}+\frac{1}{(n-1)!}\right]$

$=\sum_{n=1}^{\infty}\frac{1}{(n-2)!}+\sum_{n=1}^{\infty}\frac{1}{(n-1)!}$

$=e+e=2e$

$\therefore 1+\frac{2^2}{2!}+\frac{3^2}{3!}+\frac{4^2}{4!}+...=2e$

II.

1. Find the sum of the series

$1+\frac{2x}{1!}+\frac{3x^2}{2!}+\frac{4x^3}{3!}+...$

Sol. $=\sum_{n=0}^{\infty}\frac{n+1}{n!}x^n=\sum_{n=0}^{\infty}\left[\frac{1}{(n-1)!}+\frac{1}{n!}\right]x^n$

$$= \sum_{n=0}^{\infty} \frac{1}{(n-1)!} + \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$= x \sum_{n-1=0}^{\infty} \frac{1}{(n-1)!} x^{n-1} + \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$$= x \cdot e^x + e^x = (x+1)e^x$$

$$\therefore 1 + \frac{2x}{1!} + \frac{3x^2}{2!} + \frac{4x^3}{3!} + \ldots = (x+1)e^x$$

2. Find the sum of the series.

$$1 + \frac{2^3}{1!} + \frac{3^3}{3!} + \frac{4^3}{4!} + \ldots$$

Sol. $1 + \frac{2^3}{1!} + \frac{3^3}{3!} + \frac{4^3}{4!} + \ldots$

$$= \sum_{n=1}^{\infty} \frac{n^3}{n!} = \sum_{n=1}^{\infty} \frac{n^2}{(n-1)!}$$

$$= \sum_{n=1}^{\infty} \frac{n^2 - 1 + 1}{(n-1)!} = \sum_{n=1}^{\infty} \left[\frac{(n-1)(n+1)}{(n-1)!} + \frac{1}{(n-1)!} \right]$$

$$= \sum_{n=2}^{\infty} \frac{n+1}{(n-2)!} + \sum_{n=1}^{\infty} \frac{1}{(n-1)!}$$

$$= \sum_{n=2}^{\infty} \frac{n-2+3}{(n-2)!} + \sum_{n=1}^{\infty} \frac{1}{(n-1)!}$$

$$= \sum_{n=3}^{\infty} \frac{1}{(n-3)!} + 3\sum_{n=2}^{\infty} \frac{3}{(n-2)!} + \sum_{n=1}^{\infty} \frac{1}{(n-1)!}$$

$$= \sum_{n-3=0}^{\infty} \frac{1}{(n-3)!} + 3 \sum_{n-2=0}^{\infty} \frac{1}{(n-2)!} + \sum_{n-1=0}^{\infty} \frac{1}{(n-1)!}$$

Hint : $\sum_{n-2=0}^{\infty} \frac{1}{(n-2)!} = e$, $\sum_{n-3=0}^{\infty} \frac{1}{(n-3)!} = e$

$= e + 3e + e = 5e$

$$\therefore 1 + \frac{2^3}{1!} + \frac{3^3}{3!} + \frac{4^3}{4!} + \ldots = 5e$$

3. Show that

$$\frac{1}{2!} + \frac{1+2}{3!} + \frac{1+2+2^2}{4!} + \ldots = \frac{1}{2}(e-1)^2$$

Hint : $1 + 2 + 2^2 + \ldots 2^{n-1} = 2^n - 1$

Sol. $\frac{1}{2!} + \frac{1+2}{3!} + \frac{1+2+2^2}{4!} + \ldots$

$$= \sum_{n=1}^{\infty} \frac{1+2+2^2+\ldots+2^{n-1}}{(n+1)!}$$

$$= \sum_{n=1}^{\infty} \frac{(2^n-1)}{(2-1)(n+1)!} = \sum_{n=1}^{\infty} \left[\frac{2^n}{(n+1)!} - \frac{1}{(n+1)!} \right]$$

$$= \left(\frac{2}{2!} + \frac{2^2}{3!} + \frac{2^3}{4!} + \ldots \right) - \left(\frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots \right)$$

$$= \frac{1}{2} \left[\frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^3}{4!} + \ldots \right] - \left[\left(1 + \frac{1}{1!} + \frac{1}{2!} + \ldots \right) - 2 \right]$$

$$= \frac{1}{2} \left[\left(1 + \frac{2}{1!} + \frac{2^3}{3!} + \ldots \right) \right] - 3 - (e-2)$$

$$= \frac{1}{2} \left[e^2 - 3 \right] - (e-2)$$

$$= \frac{1}{2} [e^2 - 3 - 2e + 4]$$

$$= \frac{1}{2} (e^2 - 2e + 1) = \frac{1}{2} (e-1)^2$$

$$\therefore \frac{1}{2!} + \frac{1+2}{3!} + \frac{1+2+2^2}{4!} + \ldots = \frac{1}{2} (e-1)^2$$

4. Show that

$$\frac{1}{1!} + \frac{1+5}{2!} + \frac{1+5+5^2}{3!} + \frac{1+5+5^2+5^3}{4!} + \ldots = \frac{e^5 - e}{4}$$

Sol. $\frac{1}{1!} + \frac{1+5}{2!} + \frac{1+5+5^2}{3!} + \frac{1+5+5^2+5^3}{4!} + \ldots$

$$= \sum_{n=1}^{\infty} \frac{1+5+5^2+\ldots+5^{n-1}}{n!}$$

$$= \sum_{n=1}^{\infty} \frac{1(5^n-1)}{(5-1)n!}$$

$$= \frac{1}{4} \sum_{n=1}^{\infty} \left(\frac{5^n}{n!} - \frac{1}{n!} \right)$$

$$= \frac{1}{4} \left[\sum_{n=1}^{\infty} \frac{5^n}{n!} - \sum_{n=1}^{\infty} \frac{1}{n!} \right]$$

$$= \frac{1}{4}\left[\left(\frac{5}{1!}+\frac{5^2}{2!}+\frac{5^3}{3!}+\ldots\right)-(e-1)\right]$$

$$= \frac{1}{4}\left[\left(1+\frac{5}{1!}+\frac{5^2}{2!}+\ldots\right)-e\right]$$

$$= \frac{1}{4}(e^5-e)$$

$$\therefore \frac{1}{1!}+\frac{1+5}{2!}+\frac{1+5+5^2}{3!}+\ldots=\frac{e^5-e}{4}$$

5. Find the coefficient of x^n in the series expansion of $\frac{1+2x+3x^2}{e^x}$ in powers of x.

Sol. $\frac{1+2x+3x^2}{e^x} = (1+2x+3x^2)e^{-x}$

$$= (1+2x+3x^2)$$

$$\left(1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\ldots+\frac{(-1)^n x^n}{n!}+\ldots\right)$$

Coefficient of x^n in $\frac{1+2x+3x^2}{e^x}$

$$= 1\times\frac{(-1)^n}{n!}+2\times\frac{(-1)^{n-1}}{(n-1)!}+3\times\frac{(-1)^{n-2}}{(n-2)!}$$

$$= \frac{(-1)^n}{n!}\left[1+2(-1)n+3(1)n(n-1)\right]$$

$$= \frac{(-1)^n}{n!}[3n^2-5n+1]$$

$\therefore$ Coefficient of x^n in $\frac{1+2x+3x^2}{e^x}$

$$= \frac{(-1)^n}{n!}[3n^2-5n+1]$$

6. Find the coefficient of x^n in the expansion of $(3 + 2x)e^{3x}$ in powers of x.

Sol. $(3+2x)e^{3x} = (3+2x)$

$$\left[1+\frac{3x}{1!}+\frac{(3x)^2}{2!}+\ldots+\frac{(3x)^n}{n!}+\ldots\right]$$

$$= (3+2x)\left[1+\frac{3}{1!}x+\frac{3}{2!}x^2+\ldots+\frac{3^n}{n!}x^n+\ldots\right]$$

$\therefore$ Coefficient of x^n in $(3 + 2x)e^{3x}$

$$= 3\cdot\frac{(3^n)}{n!} + 2\cdot\frac{3^{n-1}}{(n-1)!}$$

$$= \frac{3\cdot 3\cdot 3^{n-1}}{n!} + \frac{2\cdot 3^{n-1} n}{n!} = \frac{3^{n-1}}{n!}(9+2n)$$

$\therefore$ Coefficient of x^n in $(3 + 2x)e^{3x}$ in powers of $x = \frac{3^{n-1}}{n!}(9+2n)$.

7. Show that the coefficient of x^n in the series expansion of $\frac{2+3x}{3^{2x}}$ is $\frac{(-1)^n 2^{n-1}(4-3n)}{n!}$.

Sol. $\frac{2+3x}{3^{2x}} = (2+3x)(e^{-2x})$

$$= (2+3x)\left[1 - \frac{2x}{1!} + \frac{(2x)^2}{2!} - \frac{(2x)^3}{3!} + ...+(-1)^n\frac{(2x)^n}{n!} + ...\right]$$

$$= (2+3x)\left[1 - \frac{2x}{1!} + \frac{2^2x^2}{2!} - \frac{2^3x^3}{3!} + ...+(-1)^n\frac{2^n x^n}{n!} + ...\right]$$

$\therefore$ Coefficient of x^n of $\frac{2+3x}{3^{2x}}$ is $2\cdot\frac{(-1)^n 2^n}{n!} + 3\cdot\frac{(-1)^{n-1}2^{n-1}}{(n-1)!}$

$$= \frac{(-1)^n}{n!}\left[2\cdot 2\cdot 2^{n-1} + (-1)3\cdot 2^{n-1}(n)\right]$$

$$= \frac{(-1)^n 2^{n-1}(4-3n)}{n!}$$

8. Sum of the following series using second method $1 + \frac{2x}{1!} + \frac{3x^2}{2!} + \frac{4x^3}{3!} + ...$.

Sol. The general term of the given series is $\frac{n.x^{n-1}}{(n-1)!}$

n = 1, 2, 3, ……

$$1 + \frac{2x}{1!} + \frac{3x^2}{2!} + ... = \sum_{n=1}^{\infty}\frac{n\cdot x^{n-1}}{(n-1)!} = \frac{1}{x}\sum_{n=1}^{\infty}\frac{n^2\cdot x^n}{n!}$$

Here $U_n = n^2$

Let $n^2 = a_0 + a_1 n + a_2(n)(n-1)$

$\Rightarrow n^2 = a_0 + (a_1 - a_2)n + a_n n^2$

Equating coefficients of like terms, we get

$a_0 = 0$, $a_1 - a_2 = 0$, $a_2 = 1$

$\Rightarrow a_0 = 0$, $a_1 = 1$, $a_2 = 1$

The required sum

$$= \frac{1}{x}\sum_{n=0}^{\infty}\frac{n^2 \cdot x^n}{n!} = \frac{1}{x}(a_0 + a_1 x + a_2 x^2)e^x$$

$$= \frac{1}{x}(0 + 1 \cdot x + 1 \cdot x^2)e^x$$

$$= (1 + x)e^x$$

9. $1 + \frac{2^3}{2!} + \frac{3^3}{3!} + \frac{4^3}{4!} + \ldots$

Sol. The general term of the given series is

$$\frac{n^3}{n!},\ n = 1, 2, 3\ldots$$

$$\therefore 1 + \frac{2^3}{2!} + \frac{3^3}{3!} + \frac{4^3}{4!} + \ldots = \sum_{n=1}^{\infty}(n^3)\frac{1}{n!}$$

Here $U_n = n^2$, and x = 1

Let $n^2 = a_0 + a_1 n + a_2(n)(n-1)$

$\Rightarrow n^2 = a_0 + (a_1 - a_2)n + a_2 n^2$

Comparing coefficients, we get

$a_0 = 0,\ a_1 - a_2 = 0,\ a_2 = 1$

$\Rightarrow a_0 = 0,\ a_1 = 1,\ a_2 = 1$

$$\therefore \sum_{n=0}^{\infty} n^2 \frac{1}{n!} = (a_0 + a_1 x + a_2 x^2)e^x$$

$$\Rightarrow 0 + \sum_{n=0}^{\infty} n^2 \frac{1}{n!} = (0 + 1 \cdot 1 + 1 \cdot 1^2)e^1$$

$\because x = 1$

$$\Rightarrow \sum_{n=1}^{\infty}(n^2)\frac{1}{n!} = 2e$$

12. $1 + \frac{3}{1!} + \frac{5}{2!} + \frac{7}{3!} + \ldots$

Sol. 1, 3, 5, 7, …… are in A.P.

a = 1, d = 2

$\therefore$ The general term of the given series

$$= \frac{1 + (n-1)2}{(n-1)!} = \frac{2n-1}{(n-1)!}$$

$$\therefore 1 + \frac{3}{1!} + \frac{5}{2!} + \frac{7}{3!} + \ldots\ldots = \sum_{n=1}^{\infty}(2n-1)\frac{1}{(n-1)!}$$

$$= \sum_{n=1}^{\infty}(n)(2n-1)\frac{1}{n!}$$

Here $U_n = n(2n - 1)$ and $x = 1$

Let $n(2n - 1) = a_0 + a_1 n + a_2(n)(n - 1)$

$\Rightarrow 2n^2 - n = a_0 + (a_1 - a_2)n + a_2 n^2$

Comparing coefficients of like terms

$a_0 = 0, a_1 - a_2 = -1, a_2 = 2$

$\Rightarrow a_0 = 0, a_1 = 1, a_2 = 2$

$\therefore \sum_{n=0}^{\infty} n(2n-1)\frac{1}{n!} = (a_0 + a_1 x + a_2 x^2)e^x$

$\Rightarrow 0 + \sum_{n=1}^{\infty} n(2n-1)\frac{1}{n!} = [0 + (1)\times 1 + 2(1)^2]e^1$

$\therefore$ The required sum $\sum_{n=1}^{\infty} n(2n-1)\frac{1}{n!} = 3e$.

13. $\frac{3\cdot 5}{1!}x + \frac{4\cdot 6}{2!}x^2 + \frac{5\cdot 7}{3!}x^3 + \ldots$

Sol. 3, 4, 5, … are in A.P. with a = 3, d = 1

Its nth term = 3 + (n – 1)1 = n + 2

Again 5, 6, 7, … are in A.P. with a = 5, d = 1

Its n^{th} term = 5 + (n – 1)1 = n + 4

$\therefore$ The general term of the given series is $\frac{(n+2)(n+4)}{n!}x^n = (n^2 + 6x + 8)\frac{x^n}{n!}$

n = 1, 2, 3, …

Here $U_n = n^2 + 6n + 8$

Let $n^2 + 6n + 8 = a_0 + a_1 n + a_2(n)(n - 1)$

$\Rightarrow n^2 + 6n + 8 = a_0 + (a_1 - a_2)n + a_2 n^2$

Comparing the coefficients, we get

$a_0 = 8, a_1 - a_2 = 6, a_2 = 1$

$\Rightarrow a_0 = 8, a_1 = 7, a_2 = 1$

$\sum_{n=0}^{\infty} (n^2 + 6n + 8)\frac{x^n}{n!} = (a_0 + a_1 x + a_2 x^2)e^x$

$8 + \sum_{n=1}^{\infty} (n^2 + 6n + 8)\frac{x^n}{n!} = (8 + 7x + 1\cdot x^2)e^x$

$\Rightarrow \sum_{n=1}^{\infty} (n^2 + 6n + 8)\frac{x^n}{n!} = (x^2 + 7x + 8)e^x - 8$

14. $\frac{1\cdot 2}{1!}+\frac{2\cdot 3}{2!}+\frac{3\cdot 4}{3!}+...$

Sol. The general term of the given series is

$\frac{n(n+1)}{n!}=(n^2+n)\frac{1}{n!},\ n=1,2,3,...$

$\therefore \frac{1\cdot 2}{1!}+\frac{2\cdot 3}{2!}+\frac{3\cdot 4}{3!}+.....=\sum_{n=1}^{\infty}(n^2+n)\frac{1}{n!}$

Here $U_n=n^2+n$ and x = 1

Let $n^2 + n = a_0 + a_1n + a_2(n)(n - 1)$

$\Rightarrow n^2 + n = a_0 + (a_1 - a_2)n + a_2n^2$

Comparing coefficients of like terms

$a_0=0, a_1-a_2=1, a_2=1$

$\Rightarrow a_0=0, a_1=2, a_2=1$

$\therefore \sum_{n=0}^{\infty}(n^2+n)\frac{1}{n!}=(a_0+a_1x+a_2x^2)e^x$

$\Rightarrow \sum_{n=1}^{\infty}(n^2+n)\frac{1}{n!}=(0+2\cdot 1+1\cdot 1^2)e^1=3e$

15. $\frac{1}{1!}+\frac{1+2}{2!}+\frac{1+2+3}{3!}+...$

Sol. The general term of the given series

$\frac{1+2+3+...+n}{n!}=\left[\frac{(n)(n+1)}{2}\right]\frac{1}{n!}$

Here $U_n=\frac{1}{2}(n^2+n), x=1$

Let $\frac{1}{2}(n^2+n)=a_0+a_1n+a_2(n)(n-1)$

$\Rightarrow \frac{1}{2}(n^2+n)=a_0+(a_1-a_2)n+a_2n^2$

Comparing coefficients of like terms

$a_0=0, a_1-a_2=\frac{1}{2}, a_2=\frac{1}{2}$

$\Rightarrow a_0=0, a_1=1, a_2=\frac{1}{2}$

$\therefore \sum_{n=0}^{\infty}\frac{1}{2}(n)(n+1)\frac{1}{n!}=(a_0+a_1x+a_2x^2)e^x$

$\Rightarrow 0+\sum_{n=1}^{\infty}\frac{1}{2}(n)(n+1)\frac{1}{n!}=(0+1\cdot 1+\frac{1}{2}\cdot 1^2)e^1$

$\therefore$ The general sum $\sum_{n=1}^{\infty}\frac{1}{2}(n)(n+1)\frac{1}{n!}=\frac{3}{2}e$

16. $1+\frac{1^2+2^2}{2!}+\frac{1^2+2^2+3^2}{3!}+...$

Sol. The general term of the given series is

$$\frac{1^2+2^2+3^2+...+n^2}{n!}$$

$$=\frac{n(n+1)(2n+1)}{6}\left(\frac{1}{n!}\right)$$

$$\therefore 1+\frac{1^2+2^2}{2!}+\frac{1^2+2^2+3^2}{3!}+...$$

$$=\sum_{n=1}^{\infty}\frac{1}{6}(n)(n+1)(2n+1)\frac{1}{n!}$$

Here $U_n=\frac{1}{6}(2n^3+3n^2+n)$ and $x=1$

Let $\frac{1}{6}(2n^3+3n^2+n)$

$$=a_0+a_1n+a_2(n)(n-1)+a_3(n)(n-1)(n-2)$$

$$\Rightarrow\frac{1}{3}\cdot n^3+\frac{1}{2}n^2+\frac{1}{6}n$$

$$=a_0+(a_1-a_2+2a_3)n+(a_2-3a_3)n^2+a_3n^3$$

Comparing the coefficients of like terms we get

$$a_0=0, a_1-a_2+2a_3=\frac{1}{6}, a_2-3a_3=\frac{1}{2}, a_3=\frac{1}{3}$$

$$a_0=0, a_1=1, a_2=\frac{3}{2}, a_3=\frac{1}{3}$$

$$\therefore\sum_{n=0}^{\infty}\frac{1}{6}(n)(n+1)(2n+1)\frac{1}{n!}$$

$$=\left(a_0+a_1x+a_2x^2+a_3x^3\right)e^x$$

$$=0+\sum_{n=1}^{\infty}\frac{1}{6}\frac{(n)(n+1)(2n+1)}{n!}$$

$$=\left(0+1\cdot1+\frac{3}{2}\cdot1^2+\frac{1}{3}\cdot1^3\right)e^1$$

$$=\left(1+\frac{3}{2}+\frac{1}{3}\right)e=\left(\frac{6+9+2}{6}\right)e=\frac{17}{6}e$$

$\therefore$ The required sum

$$\sum_{n=1}^{\infty}\frac{1}{6}(n)(n+1)(2n+1)\frac{1}{n!}=\frac{17}{6}e$$

EXERCISE – 7(B)

I.

1. Find the sum of the infinite series.

$$\frac{1}{5}+\frac{1}{2\cdot5^2}+\frac{1}{3\cdot5^3}+\frac{1}{4\cdot5^4}+\ldots$$

Sol. $\frac{1}{5}+\frac{1}{2\cdot5^2}+\frac{1}{3\cdot5^3}+\frac{1}{4\cdot5^4}+\ldots$

$$=\frac{1}{5}+\frac{1}{2}\left(\frac{1}{5}\right)^2+\frac{1}{3}\left(\frac{1}{5}\right)^3+\frac{1}{4}\left(\frac{1}{5}\right)^4+\ldots$$

Hint : $\log_e(1-x)=$

$$-\left\{x+\frac{1}{2}\left(\frac{1}{x}\right)^2+\frac{1}{3}\left(\frac{1}{x}\right)^3+\ldots\right\},|x|<1$$

Here $x=\frac{1}{5}\in(-1,1)$

$$=-\log_e\left(1-\frac{1}{5}\right)=-\log_e\left(\frac{4}{5}\right)$$

$$=\log_e\left(\frac{4}{5}\right)^{-1}=\log_e\left(\frac{5}{4}\right)$$

$$\therefore \frac{1}{5}+\frac{1}{2\cdot5^2}+\frac{1}{3\cdot5^3}+\frac{1}{4\cdot5^4}+\ldots=\log_e\left(\frac{5}{4}\right)$$

2. Show that $\frac{1}{2\cdot3}+\frac{1}{4\cdot5}+\frac{1}{6\cdot7}+\ldots=1-\log_e 2$**.**

Sol. $\frac{1}{2\cdot3}+\frac{1}{4\cdot5}+\frac{1}{6\cdot7}+\ldots$

$$=\frac{3-2}{2\cdot3}+\frac{5-4}{4\cdot5}+\frac{7-6}{6\cdot7}+\ldots$$

$$=\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\frac{1}{7}+\ldots$$

$$=1-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\frac{1}{7}+\ldots$$

$$=1-\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots\right)$$

Hint : $\log_e 2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots=1-\log_e 2$

$$\therefore \frac{1}{2\cdot3}+\frac{1}{4\cdot5}+\frac{1}{6\cdot7}+\ldots=1-\log_e 2$$

3. Show that $\frac{1}{2\cdot 2^2}+\frac{2}{3\cdot 2^3}+\frac{3}{4\cdot 2^4}+...=1-\log_e 2$

Sol. In the given series

$$T_n=\frac{n}{(n+1)(2)^{n+1}}$$

$$=\left(\frac{n+1-1}{n+1}\right)\left(\frac{1}{2}\right)^{n+1}$$

$$=\left(1-\frac{1}{n+1}\right)\left(\frac{1}{2}\right)^{n+1}$$

$$=\left(\frac{1}{2}\right)^{n+1}-\left(\frac{1}{n+1}\right)\left(\frac{1}{2}\right)^{n+1}$$

Sum of the infinite series

$$\sum_{n=1}^{\infty}T_n=$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n+1}+\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)^{n+1}}{n-1}\left(\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+...\right)$$

$$=\frac{1}{2^2}\left(1+\frac{1}{2}+\frac{1}{2^2}+...\infty\right)$$

$$=\left[\left\{\frac{\left(\frac{1}{2}\right)}{1}+\frac{\left(\frac{1}{2}\right)^2}{2}+\frac{\left(\frac{1}{2}\right)^3}{3}+...\right\}-\frac{1}{2}\right]$$

$$=\frac{1}{4}\left(\frac{1}{1-\frac{1}{2}}\right)-\left[-\log\left(1-\frac{1}{2}\right)\right]+\frac{1}{2}$$

$$=\frac{1}{4}(2)+\log_e\left(\frac{1}{2}\right)+\frac{1}{2}$$

$$=1+\log_e\left(\frac{1}{2}\right)=1-\log_e 2$$

Note : i) Here $a+ar+ar^2+...\infty=\frac{a}{1-r}$ if $|r|<1$

ii) $\log_e(1-x)=-\left(x+\frac{x^2}{2}+\frac{x^3}{3}+...\right)$

II.

1. Show that $\frac{2x}{x^2+1}+\frac{1}{3}\left(\frac{2x}{x^2+1}\right)^3+\frac{1}{5}\left(\frac{2x}{x^2+1}\right)^5+\ldots=\log_e\left(\frac{x+1}{x-1}\right)$

Sol. $\frac{2x}{x^2+1}+\frac{1}{3}\left(\frac{2x}{x^2+1}\right)^3+\frac{1}{5}\left(\frac{2x}{x^2+1}\right)^5+\ldots$

Hint : $\frac{1}{2}\log_e\left(\frac{x+1}{x-1}\right)=x+\frac{1}{3}x^4+\frac{1}{5}x^5+\ldots$

$=\frac{1}{2}\log_e\left(\frac{1+\frac{2x}{x^2-1}}{1-\frac{2x}{x^2+1}}\right)=\frac{1}{2}\log_e\left(\frac{\frac{x^2-1+2x}{x^2+1}}{\frac{x^2+1-2x}{x^2-1}}\right)$

$=\frac{1}{2}\log_e\frac{(x+1)^2}{(x-1)^2}=\frac{1}{2}\log_e\left(\frac{x+1}{x-1}\right)^2$

$=\log_e\left(\frac{x+1}{x-1}\right)$

2. Show that $\frac{1}{2x+1}+\frac{1}{3(2x+1)^3}+\frac{1}{5(2x+1)^5}+\ldots=\log_e\sqrt{\frac{x+1}{x}}$

Sol. $\frac{1}{2x+1}+\frac{1}{3(2x+1)^3}+\frac{1}{5(2x+1)^5}+\ldots$

$=\frac{1}{2x+1}+\frac{1}{3}\left(\frac{1}{2x+1}\right)^3+\frac{1}{5}\left(\frac{1}{2x+1}\right)^5+\ldots$

$=\frac{1}{2}\log_e\left(\frac{1+\frac{1}{2x+1}}{1-\frac{1}{2x+1}}\right)=\frac{1}{2}\log_e\left(\frac{\frac{2x+1+1}{2x+1}}{\frac{2x+1-1}{2x+1}}\right)$

$=\frac{1}{2}\log_e\left(\frac{2x+2}{2x}\right)=\frac{1}{2}\log_e\left(\frac{x+1}{x}\right)$

$=\log_e\sqrt{\frac{x+1}{x}}$

3. Show that $\frac{1}{3}+\frac{1}{3\cdot3^3}+\frac{1}{5\cdot3^5}+\ldots=\frac{1}{2}\log_e 2$**.**

Sol. L.H.S. $=\frac{1}{3}+\frac{1}{3\cdot3^3}+\frac{1}{5\cdot3^5}+\ldots$

Hint : $x+\frac{1}{3}(x)^3+\frac{1}{5}(x)^5+\ldots=\frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right)$

$$= \frac{1}{2}\log\left(\frac{1+\frac{1}{3}}{1-\frac{1}{3}}\right) = \frac{1}{2}\log_e\left(\frac{4}{2}\right) = \frac{1}{2}\log_e(2)$$

III.

1. Show that $\frac{1}{1\cdot2\cdot3}+\frac{1}{3\cdot4\cdot5}+\frac{1}{5\cdot6\cdot7}+\ldots = \frac{1}{2}\log_e 2-\frac{1}{2}$

Sol. L.H.S.= $\frac{1}{1\cdot2\cdot3}+\frac{1}{3\cdot4\cdot5}+\frac{1}{5\cdot6\cdot7}+\ldots$

$$= \sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n)(2n+1)}$$

$$= \sum_{n=1}^{\infty}\left(\frac{1}{2(2n-1)}-\frac{1}{2n}+\frac{1}{2(2n+1)}\right)$$

$$= \frac{1}{2}\left[\sum_{n=1}^{\infty}\left(\frac{1}{2n-1}-\frac{1}{n}+\frac{1}{2n+1}\right)\right]$$

$$= \frac{1}{2}\left[\sum_{n=1}^{\infty}\left(\frac{1}{2n-1}-\frac{1}{2n}-\frac{1}{2n}+\frac{1}{2n+1}\right)\right]$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{2n-1}-\frac{1}{2n}\right)+\frac{1}{2}\sum_{n=1}^{\infty}\left(-\frac{1}{2n}+\frac{1}{2n+1}\right)$$

$$= \frac{1}{2}\left[1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\ldots\right]+\frac{1}{2}\left[-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\right]$$

$$= \frac{1}{2}\log_e 2+\frac{1}{2}\left[\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\right)-1\right]$$

$$= \frac{1}{2}\log_e 2+\frac{1}{2}\log_e 2-\frac{1}{2} = \log_e 2-\frac{1}{2}$$

2. Show that $\frac{1}{2}+\frac{3}{2}\cdot\frac{1}{4}+\frac{5}{3}\cdot\frac{1}{8}+\frac{7}{4}\cdot\frac{1}{16}+\ldots = 2-\log_e 2$

Sol. $\frac{1}{2}+\frac{3}{2}\cdot\frac{1}{4}+\frac{5}{3}\cdot\frac{1}{8}+\frac{7}{4}\cdot\frac{1}{16}+\ldots$

$$= \frac{1}{2}+\frac{3}{2}\left(\frac{1}{2}\right)^2+\frac{5}{3}\left(\frac{1}{2}\right)^3+\frac{7}{4}\left(\frac{1}{2}\right)^4+\ldots$$

$$= \sum_{n=1}^{\infty}\frac{(2n-1)}{n}\left(\frac{1}{2}\right)^n$$

$$= \sum_{n=1}^{\infty}\left(2-\frac{1}{n}\right)\left(\frac{1}{2}\right)^n$$

$$= \sum_{n=1}^{\infty}\left[2\left(\frac{1}{2}\right)^n - \frac{1}{n}\left(\frac{1}{2}\right)^n\right]$$

$$= 2\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n - \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{2}\right)^n$$

$$= 2\left[\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\ldots\right]-\left[\frac{1}{1}\left(\frac{1}{2}\right)+\frac{1}{2}\left(\frac{1}{2}\right)^2+\frac{1}{3}\left(\frac{1}{2}\right)^3+\ldots\right]$$

$$= 2\left(\frac{\frac{1}{2}}{1-\frac{1}{2}}\right)-\left[-\log_e\left(1-\frac{1}{2}\right)\right]$$

$$= 2\left[\frac{\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)}\right]+\log_e\left(\frac{1}{2}\right) = 2-\log_e(2)$$

Note : i) $\log(1-x) = -\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots\right)$

ii) $a+ar+ar^2+\ldots\infty = \frac{a}{1-r}$, if $|r|<1$

3. Show that $1+\frac{1}{3\cdot 2^2}+\frac{1}{5\cdot 2^4}+\frac{1}{7\cdot 2^6}+\ldots = \log_e 3$

Sol. $1+\frac{1}{3\cdot 2^2}+\frac{1}{5\cdot 2^4}+\frac{1}{7\cdot 2^6}+\ldots$

$$= 1+\frac{1}{3}\left(\frac{1}{2}\right)^2+\frac{1}{5}\left(\frac{1}{2}\right)^4+\frac{1}{7}\left(\frac{1}{2}\right)^6+\ldots$$

$$= 2\left[\left(\frac{1}{2}\right)+\frac{1}{3}\left(\frac{1}{2}\right)^3+\frac{1}{5}\left(\frac{1}{2}\right)^5+\frac{1}{7}\left(\frac{1}{2}\right)^7+\ldots\right]$$

$$= \log_e\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right) = \log_e\left(\frac{\frac{3}{2}}{\frac{1}{2}}\right) = \log_e 3$$

4. If $y = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots$**, show that** $x = y - \frac{y^2}{2!} + \frac{y^3}{3!} - \frac{y^4}{4!} + \ldots$

Sol. $y = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots$

$= -\log_e(1-x)$

$\Rightarrow -y = \log_e(1-x)$

$\Rightarrow (1-x) = e^{-y}$

i.e. $x = 1 - e^{-y}$

$= 1 - \left(1 - \frac{y}{1!} + \frac{y^2}{2!} - \frac{y^3}{3!} + \ldots\right)$

$= y - \frac{y^2}{2!} + \frac{y^3}{3!} - \ldots$

$\therefore x = y - \frac{y^2}{2!} + \frac{y^3}{3!} - \frac{y^4}{4!} + \ldots$