SOLVED PROBLEMS

1. Find the sum of the infinite series.

$$1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\ldots$$

Sol. $e^1=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots$

$$e^{-1}=1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}+\ldots$$

$$\Rightarrow e+e^{-1}=2\left[1+\frac{1}{2!}+\frac{1}{4!}+\ldots\right]$$

$$\therefore 1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\ldots=\frac{e+e^{-1}}{2}$$

2. a) Find the sum of the series.

$$\frac{1}{1!}+\frac{1+2}{2!}+\frac{1+2+3}{3!}+\frac{1+2+3+4}{4!}+\ldots$$

Sol. $\frac{1}{1!}+\frac{1+2}{2!}+\frac{1+2+3}{3!}+\frac{1+2+3+4}{4!}+\ldots$

$$=\sum_{n=1}^{\infty}\frac{1+2+3+\ldots+n}{n!}$$

$$=\sum_{n=1}^{\infty}\frac{n(n+1)}{2\cdot n!}$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\frac{n+1}{(n-1)!}$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\frac{n-1+2}{(n-1)!}$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{n-1}{(n-1)!}+\frac{2}{(n-1)!}\right]$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{(n-2)!}+\sum_{n=1}^{\infty}\frac{1}{(n-1)!}$$

$$=\frac{1}{2}\sum_{n-2=0}^{\infty}\frac{1}{(n-2)!}+\sum_{n-1=0}^{\infty}\frac{1}{(n-1)!}$$

$$=\frac{1}{2}e+e=\frac{3}{2}e$$

$$\therefore \frac{1}{1!}+\frac{1+2}{2!}+\frac{1+2+3}{3!}+\ldots=\frac{3}{2}e$$

b) Find the sum of the infinite series.

$$1+\frac{1+2}{2!}+\frac{1+2+2^2}{3!}+\frac{1+2+2^2+2^3}{4!}+\ldots$$

Sol. $1+\frac{1+2}{2!}+\frac{1+2+2^2}{3!}+\frac{1+2+2^2+2^3}{4!}+\ldots$

$$=\sum_{n=1}^{\infty}\frac{1+2+2^2+\ldots+2^{n-1}}{n!}$$

$$=\sum_{n=1}^{\infty}\frac{(2^n-1)}{(2-1)n!}=\sum_{n=1}^{\infty}\frac{2^n-1}{n!}$$

$$=\sum_{n=1}^{\infty}\frac{2^n}{n!}-\sum_{n=1}^{\infty}\frac{1}{n!}$$

$$=\left(\frac{2}{1!}+\frac{2^2}{2!}+\frac{2^3}{3!}+\ldots\right)-(e-1)$$

$$=\left(1+\frac{2}{1!}+\frac{2^2}{2!}+\frac{2^3}{3!}+\ldots\right)-e=e^2-e$$

3. Find the sum of the series $1+\frac{2^3}{1!}x+\frac{3^3}{2!}x^2+\ldots$

Sol. $1+\frac{2^3}{1!}x+\frac{3^3}{2!}x^2+\ldots$

$$=\sum_{n=0}^{\infty}\frac{(n+1)^3}{n!}x^n$$

$$=\sum_{n=0}^{\infty}\frac{n^3+3n^2+3n+1}{n!}x^n$$

$$=\sum_{n=0}^{\infty}\left[\frac{n(n^2+3n+3)}{n!}+\frac{1}{n!}\right]x^n$$

$$=\sum_{n=0}^{\infty}\left[\frac{n^2+3n+3}{(n-1)!}+\frac{1}{n!}\right]x^n$$

$$=\sum_{n=0}^{\infty}\left[\frac{(n-1)(n+4)}{(n-1)!}+\frac{1}{n!}\right]x^n$$

$$=\sum_{n=0}^{\infty}\left[\frac{n+4}{(n-2)!}+\frac{7}{(n-1)!}+\frac{1}{n!}\right]x^n$$

$$=\sum_{n=0}^{\infty}\left[\frac{n-2+6}{(n-2)!}+\frac{7}{(n-1)!}+\frac{1}{n!}\right]x^n$$

$$=\sum_{n=0}^{\infty}\left[\frac{1}{(n-3)!}+\frac{6}{(n-2)!}+\frac{7}{(n-1)!}+\frac{1}{n!}\right]x^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{(n-3)!} x^n + \sum_{n=0}^{\infty} \frac{6}{(n-2)!} x^n + \sum_{n=0}^{\infty} \frac{7}{(n-1)!} x^n + \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$$= x^3 \sum_{n=0}^{\infty} \frac{1}{(n-3)!} x^{n-3} + 6x^2 \sum_{n=0}^{\infty} \frac{1}{(n-2)!} x^{n-2} + 7 \sum_{n=0}^{\infty} \frac{1}{(n-1)!} x^{n-7} + \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$$= x^3 e^x + 6x^2 e^x + 7x e^x + e^x$$

$$= (x^3 + x^2 + x + 1) e^x$$

$$\therefore 1 + \frac{2^3}{1!} x + \frac{3^3}{2!} x^2 + \ldots = (x^3 + x^2 + x + 1) e^x$$

4. Show that $\frac{1 \cdot 2}{1!} + \frac{2 \cdot 3}{2!} + \frac{3 \cdot 4}{3!}$ …= 3e.

Sol. $\frac{1 \cdot 2}{1!} + \frac{2 \cdot 3}{2!} + \frac{3 \cdot 4}{3!} \ldots$

$$= \sum_{n=1}^{\infty} \frac{n(n+1)}{n!} = \sum_{n=1}^{\infty} \frac{n+1}{(n-1)!}$$

$$= \sum_{n=1}^{\infty} \frac{n-1+2}{(n-1)!}$$

$$= \sum_{n=1}^{\infty} \left[\frac{1}{(n-2)!} + \frac{2}{(n-1)!} \right]$$

$$= \sum_{n-2=0}^{\infty} \frac{1}{(n-2)!} + 2 \sum_{n-1=0}^{\infty} \frac{1}{(n-1)!}$$

$$= e + 2e = 3e$$

$$\therefore \frac{1 \cdot 2}{1!} + \frac{2 \cdot 3}{2!} + \frac{3 \cdot 4}{3!} + \ldots = 3e$$

5. Show that $\frac{3 \cdot 5}{1!} + \frac{4 \cdot 6}{2!} x^2 + \frac{5 \cdot 7}{3!} x^3 + \ldots = (x^2 + 7x + 8) e^x - 8$

Sol. $\frac{3 \cdot 5}{1!} + \frac{4 \cdot 6}{2!} x^2 + \frac{5 \cdot 7}{3!} x^3 + \ldots$

$$= \sum_{n=1}^{\infty} \frac{(n+2)(n+4)}{n!} x^n$$

$$= \sum_{n=1}^{\infty} \frac{n^2 + 6x + 8}{n!} x^n$$

$$= \sum_{n=1}^{\infty} \left[\frac{n(n+6)}{n!} + \frac{8}{n!} \right] x^n$$

$$= \sum_{n=1}^{\infty}\left[\frac{n+6}{(n-1)!}+\frac{8}{n!}\right]x^n$$

$$= \sum_{n=1}^{\infty}\left[\frac{n-1+2}{(n-1)!}+\frac{8}{n!}\right]x^n$$

$$= \sum_{n=1}^{\infty}\left[\frac{1}{(n-2)!}+\frac{7}{(n-1)!}+\frac{8}{n!}\right]x^n$$

$$= \sum_{n=1}^{\infty}\frac{1}{(n-2)!}x^n+\sum_{n=1}^{\infty}\frac{7}{(n-1)!}x^n+\sum_{n=1}^{\infty}\frac{8}{n!}x^n$$

$$= x^2\sum_{n-2=0}^{\infty}\frac{1}{(n-2)!}x^{n-2}+7x\sum_{n-1=0}^{\infty}\frac{1}{(n-1)!}x^{n-1}+8\sum_{n=1}^{\infty}\frac{1}{n!}x^n$$

$$= x^2e^x+7xe^x+8(e^x-1)$$

$$= x^2e^x+7xe^x+8e^x-8$$

$$= (x^2+7x+8)e^x-8$$

$$\frac{3\cdot5}{1!}+\frac{4\cdot6}{2!}x^2+\frac{5\cdot7}{3!}x^3+\ldots=(x^2+7x+8)e^x-8$$

6. Find the sum of the series $\frac{4}{1!}+\frac{11}{2!}+\frac{22}{3!}+\frac{37}{4!}+\frac{56}{5!}+\ldots$

Sol. $\frac{4}{1!}+\frac{11}{2!}+\frac{22}{3!}+\frac{37}{4!}+\frac{56}{5!}+\ldots$

$$= \sum_{n=1}^{\infty}\frac{2n^2+n+1}{n!}$$

$$= \sum_{n=1}^{\infty}\frac{n(2n+1)+1}{n!}$$

$$= \sum_{n=1}^{\infty}\left[\frac{2n+1}{(n-1)!}+\frac{1}{n!}\right]$$

$$= \sum_{n=1}^{\infty}\frac{2(n-1)+3}{(n-1)!}+\sum_{n=1}^{\infty}\frac{1}{n!}$$

$$= \sum_{n=1}^{\infty}\left[\frac{2}{(n-2)!}+\frac{3}{(n-1)!}\right]+\sum_{n=1}^{\infty}\frac{1}{n!}$$

$$= 2\sum_{n-2=0}^{\infty}\frac{1}{(n-2)!}+3\sum_{n-1=0}^{\infty}\frac{1}{(n-1)!}+\sum_{n=1}^{\infty}\frac{1}{n!}$$

$$= 2e+3e+3-1=6e-1$$

$$\therefore \frac{4}{1!}+\frac{11}{2!}+\frac{22}{3!}+\frac{37}{4!}+\frac{56}{5!}+\ldots=6e-1$$

7. Find the sum of the series $1+\frac{1^2+2^2}{2!}+\frac{1^2+2^2+3^2}{3!}+...$

Sol. $1+\frac{1^2+2^2}{2!}+\frac{1^2+2^2+3^2}{3!}+...$

$$=\sum_{n=1}^{\infty}\left(\frac{1^2+2^2+3^2+...+n^2}{n!}\right)$$

$$=\sum_{n=1}^{\infty}\frac{n(n+1)(2n+1)}{6n!}$$

$$=\sum_{n=1}^{\infty}\frac{2n^2+3n+1}{6(n-1)!}$$

$$=\sum_{n=1}^{\infty}\left[\frac{n^2}{3(n-1)!}+\frac{n}{2(n-1)!}+\frac{1}{6(n-1)!}\right]$$

$$=\sum_{n=1}^{\infty}\left[\frac{n^2-1+1}{3(n-1)!}+\frac{n-1+1}{2(n-1)!}+\frac{1}{6(n-1)!}\right]$$

$$=\sum_{n=1}^{\infty}\left[\frac{(n-1)(n+1)}{3(n-1)!}+\frac{1}{3(n-1)!}+\frac{n-1}{2(n-2)!}+\frac{1}{2(n-1)!}+\frac{1}{6(n-1)!}\right]$$

$$=\sum_{n=1}^{\infty}\left[\frac{n+1}{3(n-2)!}+\frac{1}{3(n-1)!}+\frac{1}{2(n-2)!}+\frac{1}{2(n-1)!}+\frac{1}{6(n-1)!}\right]$$

$$=\sum_{n=1}^{\infty}\left[\frac{n-2+3}{3(n-2)!}+\frac{1}{2(n-2)!}+\frac{1}{(n-1)!}\left(\frac{1}{3}+\frac{1}{2}+\frac{1}{6}\right)\right]$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{3(n-3)!}+\frac{1}{(n-2)!}+\frac{1}{2(n-2)!}+\frac{1}{(n-1)!}\left(\frac{2+3+1}{6}\right)\right]$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{3(n-3)!}+\frac{3}{2(n-2)!}+\frac{1}{(n-1)!}\right]$$

$$=\frac{1}{3}\sum_{n-3=0}^{\infty}\frac{1}{(n-3)!}+\frac{3}{2}\sum_{n-2=0}^{\infty}\frac{1}{(n-2)!}+\sum_{n-1=0}^{\infty}\frac{1}{(n-1)!}$$

$$=\frac{1}{3}e+\frac{3}{2}e+e$$

$$=\left(\frac{1}{3}+\frac{3}{2}+1\right)e=\frac{2+9+6}{6}e=\frac{17}{6}e$$

$$\therefore 1+\frac{1^2+2^2}{2!}+\frac{1^2+2^2+3^2}{3!}+...=\frac{17}{6}e$$

8. Find the coefficient of x^k in the expansion of $\frac{1-ax-x^2}{e^x}$.

Sol. $\frac{1-ax-x^2}{e^x} = (1-ax-x^2)e^{-x}$

$= (1-ax-x^2)$

$$\left[1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\ldots+\frac{(-1)^n x^n}{n!}+\ldots\right]$$

Coefficient of x^k in $\frac{1-ax-x^2}{e^x}$

$= \frac{1(-1)^k}{k!} - a\frac{(-1)^{k-1}}{(k-1)!} - \frac{(-1)^{k-2}}{(k-2)!}$

$= \frac{(-1)^k}{k!}\left[1-a(-k)-k(k-1)\right]$

$= \frac{(-1)^k}{k!}\left[1+ak-k(k-1)\right]$

$\therefore$ Coefficient of x^k in $\frac{1-ax-x^2}{e^x}$

$= \frac{(-1)^k}{k!}\left[1+ak-k(k-1)\right]$

9. Find the coefficient of x^n in the series expansion $\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots\right)^2$, when n is odd and when n is even.

Sol. $\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots\right)^2 = \left(\frac{e^x+e^{-x}}{2}\right)^2$

$= \frac{1}{4}\left[e^{2x}+2+e^{-2x}\right]$

$$= \frac{1}{4}\left[\left\{1+\frac{(2x)}{1!}+\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}+\ldots\right\}+\left\{1-\frac{(2x)}{1!}+\frac{(2x)^2}{2!}-\frac{(2x)^3}{3!}+\ldots\right\}+2\right]$$

$\therefore$ Coefficient of x^n in the series expansion is

$$= \frac{1}{4}\left[\frac{2^n}{n!} + \frac{(-1)^n 2^n}{n!}\right]$$

$$= \begin{cases} 0, \text{ if n is odd} \\ \frac{1}{4}\left(2 \cdot \frac{2^n}{n!}\right), \text{ if n is even} \end{cases}$$

$$= \begin{cases} 0, \text{ if n is odd} \\ \frac{2^{n-1}}{n!}, \text{ if n is even} \end{cases}$$

10. Find the sum of the infinite series $1 + \frac{1+3}{2!} + \frac{1+3+3^2}{3!} + \frac{1+3+3^2+3^3}{4!} + \ldots$

Sol. The n^{th} term of the given series is

$$T_n = \frac{1+3+3^2+\ldots+3^{n-1}}{n!}$$

$$= \left(\frac{3^n-1}{3-1}\right)\frac{1}{n!} = \frac{1}{2}\left(\frac{3^n}{n!} - \frac{1}{n!}\right)$$

Sum of the infinite series

$$\sum_{n=1}^{\infty} T_n = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{3^n}{n!} - \frac{1}{n!}\right)$$

$$= \frac{1}{2}\left[\sum_{n=1}^{\infty}\frac{3^n}{n!} - \sum_{n=1}^{\infty}\frac{1}{n!}\right]$$

$$= \frac{1}{2}\left[(e^3-1) - (e-1)\right]$$

$$= \frac{1}{2}(e^3 - e) = \frac{e}{2}(e^2-1)$$

11. Show that $1 + \frac{1+a}{2!} + \frac{1+a+a^2}{3!} + \frac{1+a+a^2+a^3}{4!} + \ldots = \frac{e^a - e}{a-1}$

Sol. The n^{th} term of the given series is

$$T_n = \frac{1+a+a^2+\ldots+a^{n-1}}{n!} = \left(\frac{a^n-1}{a-1}\right)\frac{1}{n!}$$

$\therefore$ Sum of the given series

$$\sum_{n=1}^{\infty} T_n = \sum_{n=1}^{\infty}\left(\frac{a^n-1}{a-1}\right)\frac{1}{n!}$$

$$= \frac{1}{a-1}\left[\frac{a-1}{1!} + \frac{a^2-1}{2!} + \frac{a^3-1}{3!} + \ldots\right]$$

$$= \frac{1}{a-1}\left[\left(\frac{a}{1!}+\frac{a^2}{2!}+\frac{a^3}{3!}+\ldots\right)-\left(\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots\right)\right]$$

$$= \frac{1}{a-1}\left[(e^a-1)-(e-1)\right] = \frac{e^a-e}{a-1}$$

12. Find the sum of the series $\frac{4}{1!}+\frac{11}{2!}+\frac{22}{3!}+\frac{32}{4!}+\frac{56}{5!}+\ldots$ using 2nd method.

Sol. The general term of the given series is

$$\frac{2n^2+n+1}{n!},\ n=1,2,3,\ldots$$

$$\therefore \frac{4}{1!}+\frac{11}{2!}+\frac{22}{3!}+\frac{32}{4!}+\frac{56}{5!}+\ldots$$

$$= \sum_{n=1}^{\infty}(2n^2+n+1)\frac{1}{n!}$$

Here $U_n = 2n^2+n+1$ and x = 1

$\therefore$ Let $2n^2 + n + 1 = a_0 + a_1 n + a_2 n(n-1)$

$= a_0 + (a_1 - a_2)n + a_2 n^2$

Comparing the coefficient of like terms, we get $a_0 = 1$, $a_1 - a_2 = 1$, $a_2 = 2$.

$\Rightarrow a_0 = 1,\ a_1 = 3,\ a_2 = 2$

$$\therefore \sum_{n=0}^{\infty}(2n^2+n+1)\frac{1}{n!} = (a_0+a_1x+a_2x^2)e^x$$

$$\therefore 1+\sum_{n=1}^{\infty}(2n^2+n+1)\frac{1}{n!} = (1+3\cdot1+2\cdot1^2)e^1$$

$$\because x = 1$$

$$\Rightarrow \sum_{n=1}^{\infty}(2n^2+n+1)\frac{1}{n!} = 6e-1$$

Hence the required sum

$$\sum_{n=1}^{\infty}\frac{(2n^2+n+1)}{n!}\frac{1}{n!} = 6e-1.$$

13. Sum the infinite series $\frac{1}{1\cdot2}-\frac{1}{2\cdot3}+\frac{1}{3\cdot4}-\frac{1}{4\cdot5}+\ldots$

Sol. $\frac{1}{1\cdot2}-\frac{1}{2\cdot3}+\frac{1}{3\cdot4}-\frac{1}{4\cdot5}+\ldots$

$$= \frac{2-1}{1\cdot2}-\frac{3-2}{2\cdot3}+\frac{4-3}{3\cdot4}-\frac{5-4}{4\cdot5}+\ldots$$

$$= \left(1-\frac{1}{2}\right)-\left(\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)-\left(\frac{1}{4}+\frac{1}{5}\right)+\ldots$$

$$= 1+2\left(-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\right)$$

$$= 2\left[1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\right]-1$$

$$= 2\log_e(1+1)-\log_e e$$

$$= \log_e 4-\log_e e = \log_e\left(\frac{4}{e}\right)$$

$$\therefore \frac{1}{1\cdot 2}-\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}-\frac{1}{4\cdot 5}+\ldots = \log_e\left(\frac{4}{e}\right)$$

14. Sum the infinite series $\frac{5}{1\cdot 2\cdot 3}+\frac{7}{3\cdot 4\cdot 5}+\frac{9}{5\cdot 6\cdot 7}+\ldots$

Sol. $\frac{5}{1\cdot 2\cdot 3}+\frac{7}{3\cdot 4\cdot 5}+\frac{9}{5\cdot 6\cdot 7}+\ldots$

$$= \sum_{n=1}^{\infty}\frac{(2n+3)}{(2n-1)(2n)(2n+1)}$$

$$= \sum_{n=1}^{\infty}\left(\frac{2}{2n-1}-\frac{3}{2n}+\frac{1}{2n+1}\right)$$

$$= \sum_{n=1}^{\infty}\left(\frac{2}{2n-1}-\frac{2}{2n}-\frac{1}{2n}+\frac{1}{2n+1}\right)$$

$$= 2\sum_{n=1}^{\infty}\left(\frac{1}{2n-1}-\frac{1}{2n}\right)+\sum_{n=1}^{\infty}\left(-\frac{1}{2n}+\frac{1}{2n+1}\right)$$

$$= 2\left[1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\right]+\left[-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\ldots\right]$$

$$= 2\log_e(1+1)+\left[1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\right]-1$$

$$= \log_e 4+\log_e 2-1$$

$$= \log_e 8-\log_e 2 = \log_e\left(\frac{8}{e}\right)$$

$$\therefore \frac{5}{1\cdot 2\cdot 3}+\frac{7}{3\cdot 4\cdot 5}+\frac{9}{5\cdot 6\cdot 7}+\ldots = \log_e\left(\frac{8}{e}\right)$$

15. Find the sum of the infinite series $\frac{1}{2}-\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}-\frac{1}{4\cdot 2^4}+\ldots$

Sol. $\frac{1}{2}-\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}-\frac{1}{4\cdot 2^4}+\ldots$

$$= \frac{\left(\frac{1}{2}\right)}{1}-\frac{\left(\frac{1}{2}\right)^2}{2}-\frac{\left(\frac{1}{2}\right)^3}{3}-\frac{\left(\frac{1}{2}\right)^4}{4}+\ldots$$

$$= \log_e\left(1+\frac{1}{2}\right) = \log_e\left(\frac{3}{2}\right)$$

$$\therefore \frac{1}{2}-\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}-\frac{1}{4\cdot 2^4}+\ldots = \log_e\left(\frac{3}{2}\right)$$

16. Find the sum of the infinite series $\frac{1}{1\cdot 3}+\frac{1}{2}\left(\frac{1}{3\cdot 5}\right)+\frac{1}{3}\left(\frac{1}{5\cdot 7}\right)+...$

Sol. $\frac{1}{1\cdot 3}+\frac{1}{2}\left(\frac{1}{3\cdot 5}\right)+\frac{1}{3}\left(\frac{1}{5\cdot 7}\right)+...$

$$=\sum_{n=1}^{\infty}\frac{1}{n}\left[\frac{1}{(2n-1)(2n+1)}\right]$$

$$=\sum_{n=1}^{\infty}\left(\frac{1}{2n-1}-\frac{1}{n}+\frac{1}{2n+1}\right)$$

Hint : $\left(\frac{1}{n}=\frac{1}{2n}+\frac{1}{2n}\right)$

$$=\sum_{n=1}^{\infty}\left(\frac{1}{2n-1}-\frac{1}{2n}-\frac{1}{2n}+\frac{1}{2n+1}\right)$$

$$=\sum_{n=1}^{\infty}\left(\frac{1}{2n-1}-\frac{1}{2n}\right)+\sum_{n=1}^{\infty}\left(-\frac{1}{2n}+\frac{1}{2n+1}\right)$$

$$=\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+...\right)+\left(-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+...\right)$$

$$=\log_e 2+\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+...\right)-1$$

$$=\log_e 2+\log_e 2-1$$

$$=\log_e 4-\log_e e=\log_e\left(\frac{4}{e}\right)$$

$$\therefore \frac{1}{1\cdot 3}+\frac{1}{2}\left(\frac{1}{3\cdot 5}\right)+\frac{1}{3}\left(\frac{1}{5\cdot 7}\right)+...=\log_e\left(\frac{4}{e}\right)$$

17. Show that, if x > 0 $\log_e x=\frac{x-1}{x+1}+\frac{1}{2}\frac{x^2-1}{(x+1)^2}+\frac{1}{3}\frac{x^3-1}{(x+1)^3}+...$

Sol. $x > 0$

$$\text{RHS}=\frac{x-1}{x+1}+\frac{1}{2}\frac{x^2-1}{(x+1)^2}+\frac{1}{3}\frac{x^3-1}{(x+1)^3}+...$$

$$=\left[\frac{x}{x+1}+\frac{1}{2}\left(\frac{x}{x+1}\right)^2+\frac{1}{3}\left(\frac{x}{x+1}\right)^3+...\right]-\left[\frac{1}{x+1}+\frac{1}{2}\left(\frac{1}{x+1}\right)^2+\frac{1}{3}\left(\frac{1}{x+1}\right)^3+...\right]$$

$$=-\log_e\left(1-\frac{x}{x+1}\right)+\log_e\left(1-\frac{1}{x+1}\right)$$

$$\because 0<\frac{x}{x+1}<1 \text{ and } 0<\frac{1}{x+1}<1, \text{ if } x>0$$

$$=-\log_e\left(\frac{1}{x+1}\right)+\log_e\left(\frac{x}{x+1}\right)$$

$$= \log_e\left(\frac{x}{x+1}\right) - \log_e\left(\frac{1}{x+1}\right)$$

$$= \log_e\left\{\frac{x}{x+1} + \frac{1}{x+1}\right\} = \log_e(x) = \text{LHS}$$

18. Show that

$$\frac{1}{2}\cdot x^2 + \frac{2}{3}\cdot x^3 + \frac{3}{4}\cdot x^4 + \frac{4}{5}\cdot x^5 + \ldots = \frac{x}{1-x} - \log_e(1-x), \text{ if } |x| < 1.$$

Sol. N^{th} term of the given series

$$T_n = \left(\frac{n}{n+1}\right)x^{n+1} = \left(\frac{n+1-1}{n+1}\right)x^{n+1}$$

$$= \left(1 - \frac{1}{n+1}\right)x^{n+1} = x^{n+1} - \frac{x^{n+1}}{n+1}$$

From $T_n = x^{n+1} - \frac{x^{n+1}}{n+1}$,

$$T_1 = x^2 - \frac{x^2}{2}, T_2 = x^3 - \frac{x^3}{3}, T_3 = x^4 - \frac{x^4}{4}$$

Adding we get

$$\left(x^2 - \frac{x^2}{2}\right) + \left(x^3 - \frac{x^3}{3}\right) + \left(x^4 - \frac{x^4}{4}\right) + \ldots$$

$$= (x^2 + x^3 + x^4 + \ldots) - \left(\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots\right)$$

$$= x^2(1 + x + x^2 + \ldots) + x - x - \left(\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots\right)$$

$$= x^2\left(\frac{1}{1-x}\right) + x - \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots\right), |x| < 1$$

$$= \frac{x^2}{1-x} + x + \log_e(1-x), \text{ if } |x| < 1$$

$$= \frac{x^2 + x - x^2}{1-x} + \log_e(1-x)$$

$$= \frac{x}{1-x} + \log_e(1-x)$$

19. Find the sum of the series $\frac{1}{1\cdot 2}+\frac{x}{3\cdot 4}+\frac{x^2}{5\cdot 6}+\frac{x^3}{7\cdot 8}+\ldots$

Sol. N^{th} term of the given series

$$T_n=\frac{x^{n-1}}{(2n-1)(2n)}=\left(\frac{1}{2n-1}-\frac{1}{2n}\right)x^{n-1}$$

Sum of the series

$$\sum_{n=1}^{\infty}T_n=\sum_{n=1}^{\infty}\left[\frac{1}{2n-1}-\frac{1}{2n}\right]\cdot x^{n-1}$$

$$=\sum_{n=1}^{\infty}\left(\frac{1}{2n-1}\right)\cdot x^{n-1}-\sum_{n=1}^{\infty}\left(\frac{1}{2n}\right)\cdot x^{n-1}$$

$$=\left(\frac{1}{1}+\frac{x}{3}+\frac{x^2}{5}+\frac{x^3}{7}+\ldots\right)-\frac{1}{2}\left(\frac{1}{1}+\frac{x}{2}+\frac{x^2}{3}+\frac{x^4}{4}+\ldots\right)$$

$$=\frac{1}{\sqrt{x}}\left[\frac{\sqrt{x}}{1}+\frac{x\sqrt{x}}{3}+\frac{x^2\sqrt{x}}{5}+\ldots\right]-\frac{1}{2x}\left[x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\ldots\right]$$

$$=\frac{1}{\sqrt{x}}\left[\sqrt{x}+\frac{(\sqrt{x})^3}{3}+\frac{(\sqrt{x})^5}{5}+\ldots\right]-\frac{1}{2x}\left[x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\ldots\right]$$

$$=\frac{1}{\sqrt{x}}\frac{1}{2}\log_e\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)+\frac{1}{2x}\log_e(1-x)$$

$$=\frac{1}{2\sqrt{x}}\log_e\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)+\frac{1}{2x}\log_e(1-x)$$

20. Show that $\log_e\sqrt{12}=1+\left(\frac{1}{2}+\frac{1}{3}\right)\frac{1}{4}+\left(\frac{1}{4}+\frac{1}{5}\right)\frac{1}{4^2}+\left(\frac{1}{6}+\frac{1}{7}\right)\frac{1}{4^3}+\ldots$

Sol. $1+\left(\frac{1}{2}+\frac{1}{3}\right)\frac{1}{4}+\left(\frac{1}{4}+\frac{1}{5}\right)\frac{1}{4^2}+\left(\frac{1}{6}+\frac{1}{7}\right)\frac{1}{4^3}+\ldots$

$$=1+\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{3}\cdot\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{4^2}+\frac{1}{5}\cdot\frac{1}{4^2}+\frac{1}{6}\cdot\frac{1}{4^3}+\frac{1}{7}\cdot\frac{1}{4^3}+\ldots$$

$$=\left(\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{4^2}+\frac{1}{6}\cdot\frac{1}{4^3}+\ldots\right)+\left(1+\frac{1}{3}\cdot\frac{1}{4}+\frac{1}{5}\cdot\frac{1}{4^2}+\frac{1}{7}\cdot\frac{1}{4^3}+\ldots\right)$$

$$=\frac{1}{2}\left[\frac{1}{4}+\frac{1}{2}\left(\frac{1}{4}\right)^2+\frac{1}{3}\left(\frac{1}{4}\right)^3+\ldots\right]+\left[1+\frac{1}{3}\left(\frac{1}{2}\right)^2+\frac{1}{5}\left(\frac{1}{2}\right)^4+\frac{1}{7}\left(\frac{1}{2}\right)^6+\ldots\right]$$

$$=\frac{1}{2}\left[\frac{1}{4}+\frac{1}{2}\left(\frac{1}{4}\right)^2+\frac{1}{3}\left(\frac{1}{4}\right)^3+\ldots\right]+2\left[\frac{1}{2}+\frac{1}{3}\left(\frac{1}{2}\right)^3+\frac{1}{5}\left(\frac{1}{2}\right)^5+\frac{1}{7}\left(\frac{1}{2}\right)^7+\ldots\right]$$

$$=-\frac{1}{2}\log_e\left(1-\frac{1}{4}\right)+\log_e\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right)$$

$$= -\frac{1}{2}\log_e\left(\frac{3}{4}\right) + \log_e\left(\frac{3/2}{1/2}\right)$$

$$= \frac{1}{2}\left[2\log_e 3 - \log_e\left(\frac{3}{4}\right)\right]$$

$$= \frac{1}{2}\left[\log_e 9 - \log_e\left(\frac{3}{4}\right)\right]$$

$$= \frac{1}{2}\log_e\left(9\times\frac{4}{3}\right) = \frac{1}{2}\log_e 12 = \log_e\sqrt{12}$$

$$\therefore \log_e\sqrt{12} = 1 + \left(\frac{1}{2}+\frac{1}{3}\right)\frac{1}{4} + \left(\frac{1}{4}+\frac{1}{5}\right)\frac{1}{4^2} + \left(\frac{1}{6}+\frac{1}{7}\right)\frac{1}{4^3} + \ldots$$

21. Sum the series $\log_3^e - \log_9^e + \log_{27}^e - \log_{81}^e + \ldots$

Sol. $\log_3^e - \log_9^e + \log_{27}^e - \log_{81}^e + \ldots$

$$= \log_3^e - \log_{3^2}^e + \log_{3^3}^e - \log_{3^4}^e + \ldots$$

$$= \log_3^e - \frac{1}{2}\log_3^e + \frac{1}{3}\log_3^e - \frac{1}{4}\log_3^e + \ldots$$

$$= \log_3^e\left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\right]$$

$$= \log_3^e \cdot \log_e^2 = \log_3^2 \text{ or } \frac{\log_e^2}{\log_e^3}$$

$$\therefore \log_3^e - \log_9^e + \log_{27}^e - \log_{81}^e + \ldots = \log_3^2 = \frac{\log_e^2}{\log_e^3}$$

22. Let –1 < x < 1 and $y = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots$**, then show that** $x = y + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots$**.**

Sol. Given $y = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots$

$$\Rightarrow y = \log_e(1+x)$$

$$\Rightarrow 1 + x = e^y$$

$$\Rightarrow x = e^y - 1 = \left(1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots\right) - 1$$

$$= y + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots$$

$$\therefore x = y + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots$$

23. Expand $\log_e(1 + x^2) - \log_e(1 + x) - \log_e(1 - x)$ in a series of ascending powers of x, find the sum when x is very small and neglecting x^3 and higher powers of x.

Sol. $\log_e(1+x^2) - \log_e(1+x) - \log_e(1-x)$

$$= \left(x^2 - \frac{x^4}{2} + ...\right) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - .....\right) + \left(x + \frac{x^2}{2} + \frac{x^3}{3} + ...\right)$$

$$= x^2 - x + \frac{x^2}{2} + x + \frac{x^2}{2}$$

(Neglecting x^3 and higher powers of x) = $2x^2$

24. If $\log_e\left(\frac{1}{1-x-x^2+x^3}\right)$ be expanded in a series of ascending powers of x, show that coefficient of x^n will be $\frac{1}{n}$ or $\frac{3}{n}$ according as n is odd or even.

Sol. $\log_e\left(\frac{1}{1-x-x^2+x^3}\right)$

$$= \log_e \frac{1}{(1-x)(1-x^2)}$$

$$= -\log_e(1-x) - \log_e(1-x^2)$$

$$= \left(x + \frac{x^2}{2} + \frac{x^3}{3} + ...\right) + \left(x^2 + \frac{x^4}{2} + \frac{x^6}{3} + ...\right)$$

Case (i) : n is odd

The term containing x^n will occur only in the first series.

$\therefore$ The coefficient of x^n is $\frac{1}{n}$.

Case (ii) : n is even

The term containing x^n will occur in both the series. Let n = 2r, the coefficient of

$$x^{2r} = \frac{1}{2r} + \frac{1}{r} = \frac{3}{2r}$$

$\therefore$ Coefficient of x^n is $\frac{3}{n}$, when n is even.

$\therefore$ Coefficient of x^n will be $\frac{1}{n}$ or $\frac{3}{n}$ according as n is odd or n is even in the expansion of $\log_e\left(\frac{1}{1-x-x^2+x^3}\right)$.