PARTIAL FRACTIONS

1. Polynomial : A function of the form $a_0 + a_1x + a_2x^2 + ...a_nx^n$

Where $(a_n \neq 0)$ and $a_0, a_1, ...a_n$ are complex numbers

2. Degree of the polynomial is denoted by deg of f(x)

3. Division alogorithm: f(x), g(x) are two polynomials then there exist two polynomials q(x) and r(x) such that

F(x) = g(x) q(x) + r(x)

q (x) is called quotient

r(x) is called remainder

deg r(x) < deg g(x0

4. Remainder theorem : if f(x) is a polynomial divided by x-a then f(a) is the remainder

5. Factor theorem : If f(x) is a polynomial and f(a) = 0 then x- a is a factor of f(x) = 0

6. Rational function : A function of the form $\frac{f(x)}{g(x)}$ $g(x) \neq 0$ is called rational function

7. Proper fraction : If deg f(x) < deg g(x) then $\frac{f(x)}{g(x)}$ is called proper fraction

8. Improper fraction: If deg $f(x) \geq \deg g(x)$ then $\frac{f(x)}{g(x)}$ is called improper fraction

Type -1 If the denominator contain the non-repeated linear fraction of the form (ax + b) then partial fraction are of the form $\frac{A}{ax+b}$

Type – 2: If the denominator contains a repeated linear factors of the form $(ax+b)^n$ then partial fraction are of the form $\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + ... + \frac{A_n}{(ax+b)^n}$

Type -3: If the denominator contains irreducible quadratic factor of the form $ax^2 + bx + c$ then the partial fraction is of the form $\frac{Ax+B}{ax^2+bx+c}$

Type – 4: If the denominator contains repeated irreducible quadratic factor of the form $(ax^2 + bx + c)^n$ then the partial fraction are of the form $\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + ... + \frac{A_nx+B_n}{(ax^2+bx+c)^n}$

Type -5: If the given fraction $\frac{f(x)}{f(g)}$ is an improper fraction then write $\frac{f(x)}{g(x)} = q(x) = \frac{r(x)}{g(x)}$

$\frac{r(x)}{g(x)}$ is proper fraction apply one of the above four types to get partial fractions

EXERCISE – 6(a)

I. Resolve the following into partial fractions.

1. $\frac{2x+3}{(x+1)(x-3)}$

Sol. Let $\frac{2x+3}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}$

Multiplying with (x+1)(x–3)

$2x+3 = A(x-3) + B(x+1)$

$x = -1 \Rightarrow 1 = A(-4) \Rightarrow A = -\frac{1}{4}$

$x = 3 \Rightarrow 9 = B(4) \Rightarrow B = \frac{9}{4}$

$\frac{2x+3}{(x+1)(x-3)} = \frac{-1}{4(x+1)} + \frac{9}{4(x-3)}$

2. $\frac{5x+6}{(2+x)(1-x)}$

Sol. Let $\frac{5x+6}{(2+x)(1-x)} = \frac{A}{2+x} + \frac{B}{1-x}$

Multiplying with (2 + x) (1 – x)

$5x + 6 = A(1 - x) + B(2 + x)$

Put x = –2, $-10 + 6 = A(1 + 2) \Rightarrow A = -\frac{4}{3}$

Put x = 1, $5 + 6 = B(2 + 1) \Rightarrow B = \frac{11}{3}$

$\therefore \frac{5x+6}{(2+x)(1-x)} = -\frac{4}{3(2+x)} + \frac{11}{3(1-x)}$

II.

1. $\frac{3x}{x^2-3x+2}$

Sol. $\frac{3x}{x^2-3x+2} = \frac{3x+7}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$

Multiplying with $x^2 - 3x + 2$

$3x + 7 = A(x - 2) + B(x - 1)$

$x = 1 \Rightarrow 10 = -A \Rightarrow A = -10$

$x = 2 \Rightarrow 13 = B$

$\therefore \frac{3x}{x^2-3x+2} = \frac{-10}{x-1} + \frac{13}{x-2}$

2. $\dfrac{x+4}{(x^2-4)(x+1)}$

Sol. $\dfrac{x+4}{(x^2-4)(x+1)} = \dfrac{A}{x+1} + \dfrac{B}{x+2} + \dfrac{C}{x-2}$

Multiplying with $(x^2 - 4)(x + 1)$

$x+4 = A(x^2-4) + B(x+1)(x-2) + C(x+1)(x+2)$

$x = -1 \Rightarrow 3 = A(1-4) = -3A \Rightarrow A = -1$

$x = -2 \Rightarrow 2 = B(-2+1)(-2-2) = 4B$

$\Rightarrow B = \dfrac{2}{4} = \dfrac{1}{2}$

$x = 2 \Rightarrow 6 = C(2+1)(2+2) = 12C \Rightarrow C = \dfrac{1}{2}$

$\therefore \dfrac{x+4}{(x^2-4)(x+1)} = -\dfrac{1}{x+1} + \dfrac{1}{2(x+2)} + \dfrac{1}{2(x-2)}$

3. $\dfrac{2x^2+2x+1}{x^3+x^2}$

Sol. Let $\dfrac{2x^2+2x+1}{x^3+x^2} = \dfrac{2x^2+2x+1}{x^2(x+1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+1}$

Multiplying with $x^2(x + 1)$

$2x^2 + 2x + 1 = A\,x(x + 1) + B(x + 1) + Cx^2$

Put $x = 0$, $1 = B$

Put $x = -1$, $2 - 2 + 1 = C(1) \Rightarrow C = 1$

Equating the coefficients of x^2,

$2 = A + C \Rightarrow A = 2 - C = 2 - 1 = 1$

$\therefore \dfrac{2x^2+2x+1}{x^3+x^2} = \dfrac{1}{x} + \dfrac{1}{x^2} + \dfrac{1}{x+1}$

4. $\dfrac{2x+3}{(x-1)^3}$

Sol. Put $x - 1 = y \Rightarrow x = y + 1$

$\Rightarrow \dfrac{2x+3}{(x-1)^3} = \dfrac{2(y+1)+3}{y^3} = \dfrac{2y+5}{y^3}$

$= \dfrac{2}{y^2} + \dfrac{5}{y^3} = \dfrac{2}{(x-1)^2} + \dfrac{5}{(x-1)^3}$

$\therefore \dfrac{2x+3}{(x-1)^3} = \dfrac{2}{(x-1)^2} + \dfrac{5}{(x-1)^3}$

III.

1. $\dfrac{x^2-x+1}{(x+1)(x-1)^2}$

Sol. Let $\dfrac{x^2-x+1}{(x+1)(x-1)^2} = \dfrac{A}{x+1} + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2}$

Multiplying with $(x + 1)(x - 1)^2$

$x^2 - x + 1 = A(x - 1)^2 + B(x + 1)(x - 1) + C(x + 1)$

Put $x = -1$, $1 + 1 + 1 = A(4) \Rightarrow A = \dfrac{3}{4}$

Put x = 1, $1 - 1 + 1 = C(2) \Rightarrow C = \frac{1}{2}$

Equating the coefficients of x^2

$A + B = 1 \Rightarrow B = 1 - A = 1 - \frac{3}{4} = \frac{1}{4}$

$$\therefore \frac{x^2 - x + 1}{(x+1)(x-1)^2} = \frac{3}{4(x+1)} + \frac{1}{4(x-1)} + \frac{1}{2(x-1)^2}$$

2. $\frac{9}{(x-1)(x+2)^2}$

Sol. Let $\frac{9}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$

Multiplying with $(x - 1)(x + 2)^2$

$9 = A(x + 2)^2 + B(x - 1)(x + 2) + C(x - 1)$

$x = 1 \Rightarrow 9 = 9A \Rightarrow A = 1$

$x = -2 \Rightarrow 9 = -3C \Rightarrow C = -3$

Equating the coefficients of x^2

$A + B = 0 \Rightarrow B = -A = -1$

$$\therefore \frac{9}{(x-1)(x+2)^2} = \frac{1}{x-1} - \frac{1}{x+2} - \frac{3}{(x+2)^2}$$

3. $\frac{x-1}{(x+1)(x-2)^2}$

Sol. Let $\frac{x-1}{(x+1)(x-2)^2} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$

$\Rightarrow x - 1 =$

$A(x - 2)^2 + B(x - 2)(x + 1) + C(x + 1) \ldots(1)$

Put x = 2 in (1)

$2 - 1 = A(0) + B(0) + C(2 + 1)$

$\Rightarrow 1 = 3C \Rightarrow C = \frac{1}{3}$

Put x = –1 in (1)

$-1 - 1 = A(-1-2)^2 + B(0) + C(0)$

$\Rightarrow 9A = -2 \Rightarrow A = \frac{-2}{9}$

Equating the coefficients of x^2 in (1)

$0 = A + B \Rightarrow B = -A = \frac{2}{9}$

$$\therefore \frac{x-1}{(x+1)(x-2)^2} = \frac{\frac{-2}{9}}{x+1} + \frac{\frac{2}{9}}{x-2} + \frac{\frac{1}{3}}{(x-2)^2}$$

$$\frac{x-1}{(x+1)(x-2)^2} = \frac{-2}{9(x+1)} + \frac{2}{9(x-2)} + \frac{1}{3(x-2)^2}$$

4. $\dfrac{1}{(1-2x)^2(1-3x)}$

Sol. Let

$$\frac{1}{(1-2x)^2(1-3x)}=\frac{A}{1-3x}+\frac{B}{1-2x}+\frac{C}{(1-2x)^2}$$

Multiplying with $(1-2x)^2(1-3x)$

$$1=A(1-2x)^2+B(1-3x)(1-2x)+C(1-3x)$$

$$x=\frac{1}{3}\Rightarrow 1=A\left(1-\frac{2}{3}\right)^2=\frac{A}{9}\Rightarrow A=9$$

$$x=\frac{1}{2}\Rightarrow 1=C\left(1-\frac{3}{2}\right)=-\frac{C}{2}\Rightarrow C=-2$$

Equating the coefficients of x^2

$0 = 4A + 6B$

$6B = -4A = -36$

$B = -6$

$$\frac{1}{(1-2x)^2(1-3x)}=\frac{9}{1-3x}-\frac{6}{1-2x}-\frac{2}{(1-2x)^2}$$

5. $\dfrac{1}{x^3(x+a)}$

Sol. Let $\dfrac{1}{x^3(x+a)}=\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x^3}+\dfrac{D}{x+a}$

$$=\frac{A\cdot x^2(x+a)+B(a)(x+a)+C(x+a)+Dx^3}{x^3(x+a)}$$

$$\therefore 1=A(x^2)(x+a)+Bx(x+a)+C(x+a)+Dx^3\ldots(1)$$

Put $x = 0$ in (1)

$1 = A(0) + B(0) + C(0 + a) + D(0)$

$\Rightarrow 1 = C(a) \Rightarrow C = \dfrac{1}{a}$

Put $x = -a$ in (1)

$1 = A(0) + B(0) + C(0) + D(-a)^3$

$\Rightarrow 1 = -Da^2 \Rightarrow D = -\dfrac{1}{a^3}$

Equating the coefficients of x^3 in (1)

$0 = A + D$

$\Rightarrow A = -D = \dfrac{1}{a^3}$, $A = \dfrac{1}{a^3}$

Equating the coefficients of x^2 in (1)

$0 = Aa + B$

$$\Rightarrow B=-aA=-a\left(\frac{1}{a^3}\right)=-\frac{1}{a^2}$$

$$\therefore B=-\frac{1}{a^2}$$

$$\frac{1}{x^3(x+a)}=\frac{\left(\frac{1}{a^3}\right)}{x}+\frac{\left(-\frac{1}{a^2}\right)}{x^2}+\frac{\left(\frac{1}{a}\right)}{x^3}+\frac{\left(-\frac{1}{a^3}\right)}{x+a}$$

$$\Rightarrow \frac{1}{x^3(x+a)}=\frac{1}{a^3x}-\frac{1}{a^2x^2}+\frac{1}{ax^3}-\frac{1}{a^3(x+a)}$$

EXERCISE – 6(B)

I. Resolve the following into partial fractions.

1. $\frac{2x^2+3x+4}{(x-1)(x^2+2)}$

Sol. Let $\frac{2x^2+3x+4}{(x-1)(x^2+2)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+2}$

Multiplying with $(x-1)(x^2+2)$

$2x^2+3x+4=A(x^2+2)+(Bx+C)(x-1)$

$x=1\Rightarrow 2+3+4=A(1+2)$

$9=3A\Rightarrow A=3$

Equating the coefficients of x^2

$2 = A + B \Rightarrow B = 2 - A = 2 - 3 = -1$

Equating constants

$4 = 2A - C \Rightarrow C = 2A - 4 = 6 - 4 = 2$

$$\frac{2x^2+3x+4}{(x-1)(x^2+2)}=\frac{3}{x-1}+\frac{-x+2}{x^2+2}$$

2. $\frac{3x-1}{(1-x+x^2)(x+2)}$

Sol. Let $\frac{3x-1}{(1-x+x^2)(x+2)}=\frac{A}{2+x}+\frac{Bx+C}{1-x+x^2}$

Multiplying with $(2 + x)(1 - x + x^2)$

$3x - 1 = A(1 - x + x^2) (Bx + C)(2 + x)$

$x = -2 \Rightarrow -7 = A(1+2+4) = 7A \Rightarrow A = -1$

Equating the coefficients of x^2

$0 = A + B \Rightarrow B = -A = 1$

Equating the constants $-1 = A + 2C$

$2C = -1 - A = -1 + 1 = 0 \Rightarrow C = 0$

$$\frac{3x-1}{(1-x+x^2)(x+2)}=-\frac{1}{2+x}+\frac{x}{1-x+x^2}$$

3. $\frac{x^2-3}{(x+2)(x^2+1)}$

Sol. Let $\frac{x^2-3}{(x+2)(x^2+1)}=\frac{A}{x+2}+\frac{Bx+C}{x^2+1}$

Multiplying with $(x + 2)(x^2 + 1)$

$x^2-3=A(x^2+1)+(Bx+C)(x+2)$

$x=-2\Rightarrow 4-3=A(4+1)$

$1=5A\Rightarrow A=\frac{1}{5}$

Equating the coefficients of x^2

$1 = A + B \Rightarrow B = 1 - A = 1 - \frac{1}{5} = \frac{4}{5}$

Equating the constants –3 = A + 2C

$2C = -3 - A = -3 - \frac{1}{5} = -\frac{16}{5} \Rightarrow C = -\frac{8}{5}$

$$\frac{x^2 - 3}{(x+2)(x^2+1)} = \frac{1}{5(x+2)} + \frac{4x-8}{5(x^2+1)}$$

4. $\frac{x^2+1}{(x^2+x+1)^2}$

Sol. Let $\frac{x^2+1}{(x^2+x+1)^2} = \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{(x^2+x+1)^2}$

Multiplying with $(x^2 + x + 1)^2$

$x^2 + 1 = (Ax + B)(x^2 + x + 1) + (Cx + D)$

Equating the coefficients of x^3, A = 0

Equating the coefficients of x^2,

A + B = 1 $\Rightarrow$ B = 1

Equating the coefficients of x,

A + B + C = 0

$\Rightarrow 1 + C = 0 \Rightarrow C = -1$

Equating the constant, B + D = 1

$\Rightarrow D = 1 - B = 1 - 1 = 0$

$\therefore$ Ax + B = 1, Cx + D = –x

$$\therefore \frac{x^2+1}{(x^2+x+1)^2} = \frac{1}{x^2+x+1} - \frac{x}{(x^2+x+1)^2}$$

5. $\frac{x+3}{(1-x)^2(1+x)^2}$

Sol. Let $\frac{x+3}{(1-x)^2(1+x)^2} = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{Cx+D}{1+x^2}$

Multiplying with $(1 – x)^2 (1 + x)^2$

$$x + 3 = A(1-x)(1+x)^2 + B(1+x^2) + (Cx+D)(1-x)^2$$

$$= A(1 - x + x^2 - x^3) + B(1 + x^2) + (Cx + D)(1 - 2x + x^2)$$

Put x = 1, 4 = B(1+1) $\Rightarrow$ B = 2

Equating the coefficients of x^3,

–A + C = 0 $\Rightarrow$ C = A

Equating the coefficients of x,

$-A + C - 2D = 1 \Rightarrow D = -\frac{1}{2}$

Equating the coefficients of x^2,

A + B – 2C + D = 0

$\Rightarrow A + 2 - 2A - \frac{1}{2} = 0 \Rightarrow A = \frac{3}{2}$

$\therefore C = \frac{3}{2}, CX + D = \frac{3}{2}x - \frac{1}{2} = \frac{3x-1}{2}$

$$\therefore \frac{x+3}{(1-x)^2(1+x)^2} = \frac{3}{2(1-x)} + \frac{2}{(1-x)^2} + \frac{3x+1}{2(1+x^2)}$$

6. $\dfrac{x^3+x^2+1}{(x-1)(x^3-1)}$

Sol. Let $\dfrac{x^3+x^2+1}{(x-1)(x^3-1)}=\dfrac{x^3+x^2+1}{(x-1)(x-1)(x^2+x+1)}$

Let $\dfrac{x^3+x^2+1}{(x-1)(x-1)(x^2+x+1)}=$

$$\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{Cx+D}{x^2+x+1}...(1)$$

$$=\frac{A(x-1)(x^2+x+1)+B(x^2+x+1)+(Cx+D)(x-1)^2}{(x-1)^2(x^2+x+1)}$$

$$\therefore x^3+x^2+1=A(x-1)(x^2+x+1)$$

$$+B(x^2+x+1)+(Cx+D)(x-1)^2...(2)$$

Put $x=1$ in (2)

$$1+1+1=A(0)+B(1+1+1)+(C(1)+D)(0)$$

$$\Rightarrow 3B=3\Rightarrow B=1$$

Equating the coefficients of x^3 in (2)

$1 = A + C$ …(3)

Equating the coefficients of x^2 in (2)

$1 = A(1 – 1) + B(1) + C(–2) + D(1)$

$\Rightarrow 1 = B – 2C + D$

$\because B=1\Rightarrow 1=1-2C+D\Rightarrow 2C=D$ …(4)

Put x = 0 in (2)

$1 = A(–1)(1) + B(1) + D(–1)^2$

$\Rightarrow –A + B + D = 1 \Rightarrow –A + 1 + D = 1$

$\Rightarrow A = D$ …(5)

From (3), (4) and (5)

$$1=D+\frac{D}{2}\Rightarrow\frac{3D}{2}=1\Rightarrow D=\frac{2}{3}$$

From (5) $A=\dfrac{2}{3}$, from (4) $C=\dfrac{D}{2}=\dfrac{(2/3)}{2}=\dfrac{1}{3}$

$$\therefore\frac{x^3+x^2+1}{(x-1)(x^3-1)}=\frac{(2/3)}{x-1}+\frac{1}{(x-1)^2}+\frac{\left(\frac{1}{3}x+\frac{2}{3}\right)}{x^2+x+1}$$

$$\Rightarrow\frac{x^3+x^2+1}{(x-1)(x^3-1)}=\frac{2}{3(x-1)}+\frac{1}{(x-1)^2}+\frac{x+2}{3(x^2+x+1)}$$

EXERCISE – 6(C)

I. Resolve the following into partial fractions.

1. $\frac{x^2}{(x-1)(x-2)}$

Sol. Let $\frac{x^2}{(x-1)(x-2)} = 1 + \frac{A}{x-1} + \frac{B}{x-2}$

Multiplying with $(x-1)(x-2)$

$x^2 = (x-1)(x-2) + A(x-2) + B(x-1)$

Put $x = 1$, $1 = A(-1) \Rightarrow A = -1$

Put $x = 2$, $4 = B(1) \Rightarrow B = 4$

$\therefore \frac{x^2}{(x-1)(x-2)} = 1 - \frac{1}{x-1} + \frac{4}{x-2}$

2. $\frac{x^3}{(x-1)(x+2)}$

Sol. Let $\frac{x^3}{(x-1)(x+2)} = \frac{x^3}{x^2+x-2}$

$= \frac{x(x^2+x-2) - 1(x^2+x-2) + 3x - 2}{x^2+x-2}$

$= x - 1 + \frac{3x-2}{x^2+x-2} = x - 1 + \frac{3x-2}{(x-1)(x+2)}$

Let $\frac{3x-2}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$

Multiplying with $(x-1)(x+2)$

$3x - 2 = A(x+2) + B(x-1)$

Put $x = 1$, $1 = A(3) \Rightarrow A = \frac{1}{3}$

Put $x = -2$, $-8 = B(-3) \Rightarrow B = \frac{8}{3}$

$\therefore \frac{x^3}{(x-1)(x+2)} = x - 1 + \frac{1}{3(x-1)} + \frac{8}{3(x+2)}$

3. $\frac{x^3}{(2x-1)(x-1)^2}$

Sol. Let

$\frac{x^3}{(2x-1)(x-1)^2} = \frac{1}{2} + \frac{A}{2x-1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$

Multiplying with $2(2x-1)(x-1)^2$

$2x^3 = (2x-1)(x-1)^2 + 2A(x-1)^2 + 2B(2x-1)(x-1) + 2C(2x-1)$

Put $x = \frac{1}{2}, 2\left(\frac{1}{8}\right) = 2A\left(\frac{1}{4}\right) \Rightarrow A = \frac{1}{2}$

Put $x = 1$, $2(1) = 2C(1) \Rightarrow C = 1$

Put $x = 0$,

$0 = (-1)(1) + 2A(1) + 2B(-1)(-1) + 2C(-1)$

$\Rightarrow 2A + 2B - 2C = 1$

$\Rightarrow 2B = 1 + 2C - 2A = 1 + 2 - 1 = 2 \Rightarrow B = 1$

$$\therefore \frac{x^3}{(2x-1)(x-1)^2} = \frac{1}{2} + \frac{1}{2(2x-1)} + \frac{1}{(x-1)} + \frac{1}{(x-1)^2}$$

4. $\dfrac{x^3}{(x-a)(x-b)(x-c)}$

Sol. Let $\dfrac{x^3}{(x-a)(x-b)(x-c)} =$

$$1 + \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$$

Multiplying with $(x - a)(x - b)(x - c)$

$x^3 = (x-a)(x-b)(x-c) + A(x-b)(x-c) + B(x-a)(x-c) + C(x-a)(x-b)$

Put $x = a$, $a^3 = A(a - b)(a - c)$

$$\Rightarrow A = \frac{a^3}{(a-b)(a-c)}$$

Put $x = b$, $b^3 = B(b - a)(b - c)$

$$\Rightarrow B = \frac{b^3}{(b-a)(b-c)}$$

Put $x = c$, $c^3 = C(c - a)(c - b)$

$$\Rightarrow C = \frac{c^3}{(c-a)(c-b)}$$

$$\frac{x^3}{(x-a)(x-b)(x-c)} = 1 + \frac{a^3}{(a-b)(a-c)(x-a)} + \frac{b^3}{(b-a)(b-c)(x-b)} + \frac{c^3}{(c-a)(c-b)(x-c)}$$

EXERCISE – 6(d)

1. Find the coefficients of x^3 in the expansion of $\dfrac{5x+6}{(2+x)(1-x)}$.

Sol. Let $\dfrac{5x+6}{(2+x)(1-x)} = \dfrac{A}{2+x} + \dfrac{B}{1-x}$

Multiplying with $(2 + x)(1 - x)$

$5x + 6 = A(1 - x) + B(2 + x)$

$$x = 1 \Rightarrow 11 = B(2+1) = 3B \Rightarrow B = \frac{11}{3}$$

$$x = -2 \Rightarrow -4 = A(1+2) = 3A \Rightarrow A = \frac{-4}{3}$$

$$\frac{5x+6}{(2+x)(1-x)} = \frac{-4}{3(2+x)} + \frac{11}{3(1-x)}$$

$$= \frac{-4}{3 \cdot 2\left(1+\frac{x}{2}\right)} + \frac{11}{3(1-x)}$$

$$= -\frac{2}{3}\left(1+\frac{x}{2}\right)^{-1} + \frac{11}{3}(1-x)^{-1}$$

$$= -\frac{2}{3}\left(1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \ldots\right) + \frac{11}{3}\left(1 + x + x^2 + x^3 \ldots\right)$$

$$\therefore \text{ Coefficient of } x^3 = -\frac{2}{3}\left(-\frac{1}{8}\right) + \frac{11}{3}(1)$$

$$= \frac{2+88}{24} = \frac{90}{24} = \frac{15}{4}$$

2. What is the coefficient of x^4 in the expansion of $\frac{3x^2+2x}{(x^2+2)(x-3)}$.

Sol. Let $\frac{3x^2+2x}{(x^2+2)(x-3)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+2}$

Multiplying with $(x^2 + 2)(x - 3)$

$3x^2 + 2x = A(x^2 + 2) + (Bx + C)(x - 3)$

$x = 3 \Rightarrow 27 + 6 = A(9 + 2)$

$33 = 11A \Rightarrow A = 3$

Equating the coefficients of x^2

$3 = A + B \Rightarrow B = 3 - A = 3 - 3 = 0$

Equating the constants,

$2A - 3C = 0 \Rightarrow 3C = 2A = 6 \Rightarrow C = 2$

$$\frac{3x^2+2x}{(x^2+2)(x-3)} = \frac{3}{x-3} + \frac{2}{x^2+2}$$

$$= \frac{3}{-3\left(1-\frac{x}{3}\right)} + \frac{2}{2\left(1+\frac{x^2}{2}\right)}$$

$$= -\left(1-\frac{x}{3}\right)^{-1} + \left(1+\frac{x^2}{2}\right)^{-1}$$

$$= -\left(1+\frac{x}{3}+\frac{x^2}{9}+\frac{x^3}{27}+\frac{x^4}{81}+\ldots\right)+\left(1-\frac{x^2}{2}+\frac{x^4}{4}-\ldots\right)$$

$\therefore$ Coefficients of $x^4 = -\frac{1}{81} + \frac{1}{4}$

$$= \frac{-4+81}{324} = \frac{77}{324}$$

3. Find the coefficient of x^n in the expansion of $\frac{x-4}{x^2-5x+6}$.

Sol. Let $\frac{x-4}{x^2-5x+6} = \frac{A}{x-2} + \frac{B}{x-3}$

Multiplying with $(x - 2)(x - 3)$

$x - 4 = A(x - 3) + B(x - 2)$

$x = 2 \Rightarrow -2 = A(2 - 3) = -A \Rightarrow A = 2$

$x = 3 \Rightarrow -1 = B(3 - 2) = B \Rightarrow B = -1$

$$\frac{x-4}{x^2-5x+6}=\frac{2}{x-2}-\frac{1}{x-3}$$

$$=\frac{2}{-2\left(1-\frac{x}{2}\right)}+\frac{1}{3\left(1-\frac{x}{3}\right)}$$

$$=-\left(1-\frac{x}{2}\right)^{-1}+\frac{1}{3}\left(1-\frac{x}{3}\right)^{-1}$$

$$=-\left(1+\frac{x}{2}+\frac{x^2}{4}+...+\frac{x^n}{2^n}+...\right)+\frac{1}{3}\left(1+\frac{x}{3}+\frac{x^2}{9}+...+\frac{x^n}{3^n}+...\right)$$

Coefficient of $x^n=\frac{1}{3^{n+1}}-\frac{1}{2^n}$

SOLVED PROBLEMS

1. Resolve $\frac{5x+1}{(x+2)(x-1)}$ into Partial fractions.

Sol. Let $\frac{5x+1}{(x+2)(x-1)}=\frac{A}{x+2}+\frac{B}{x-1}$, then

$$\frac{5x+1}{(x+2)(x-1)}=\frac{A(x-1)+B(x-2)}{(x+2)(x-1)}$$

$\Rightarrow 5x+1=A(x-1)+B(x-2)...(1)$

Put x = 1 in (1)

5(1) + 1 = A(0) + B(1 + 2)

$\Rightarrow 3B = 6 \Rightarrow B = 2$

Put x = –2 in (1)

5(–2) + 1 = A(–2–1) + B(0)

$\Rightarrow -9 = -3A \Rightarrow A = 3$

$\therefore \frac{5x+1}{(x+2)(x-1)}=\frac{3}{x+2}+\frac{2}{x-1}$

2. Resolve $\frac{2x+3}{5(x+2)(2x+1)}$ into Partial fractions.

Sol. Let $\frac{2x+3}{(x+2)(2x+1)}=\frac{A}{x+2}+\frac{B}{2x+1}$

$$\Rightarrow \frac{2x+3}{(x+2)(2x+1)}=\frac{A(2x+1)+B(x+2)}{(x+2)(2x+1)}$$

$\Rightarrow 2x + 3 = A(2x + 1) + B(x + 2)$ …(1)

Put x = –2 in (1)

2(–2) + 3 = A[2(–2) + 1] + B(0)

$\Rightarrow -1 = A(-3) \Rightarrow A = \frac{1}{3}$

Put $x=-\frac{1}{2}$ in (1)

$$2\left(-\frac{1}{2}\right)+3=A(-1+1)+B\left(-\frac{1}{2}+2\right)$$

$$\Rightarrow 2=B\left(\frac{3}{2}\right)\Rightarrow B=\frac{4}{3}$$

$$\therefore \frac{2x+3}{(x+2)(2x+1)}=\frac{1/3}{x+2}+\frac{4/3}{2x+1}$$

$$\therefore \frac{2x+3}{(x+2)(2x+1)}=\frac{1}{3(x+2)}+\frac{4}{3(2x+1)}$$

Hence $\frac{2x+3}{5(x+2)(2x+1)}=\frac{1}{15(x+2)}+\frac{4}{15(2x+1)}$

3. Resolve $\frac{x^2+5x+7}{(x-3)^3}$ into partial fractions.

Sol. Let $x - 3 = y \Rightarrow x = y + 3$

$$\frac{x^2+5x+7}{(x-3)^3}=\frac{(y+3)^2+5(y+3)+7}{y^3}$$

$$=\frac{y^2+6y+9+5y+15+7}{y^3}$$

$$=\frac{y^2+11y+31}{y^3}=\frac{1}{y}+\frac{11}{y^2}+\frac{31}{y^3}$$

$$\therefore \frac{x^2+5x+7}{(x-3)^3}=\frac{1}{x-3}+\frac{11}{(x-3)^2}+\frac{31}{(x-3)^3}$$

4. Write $\frac{x^2+13x+15}{(2x+3)(x+3)^2}$ as a sum of partial fractions.

Sol. Let $\frac{x^2+13x+15}{(2x+3)(x+3)^2}=\frac{A}{2x+3}+\frac{B}{x+3}+\frac{C}{(x+3)^2}$

Multiplying $(2x + 3)(x + 3)^2$

$$x^2+13x+15=A(x+3)^2+B(x+3)(2x+3)+C(2x+3)$$

$$\text{Put } x=-\frac{3}{2},\frac{9}{4}-\frac{39}{2}+15=A\left(\frac{9}{4}\right)$$

$$\Rightarrow \frac{9A}{4}=\frac{9-78+60}{4}=-\frac{9}{4}\Rightarrow A=-1$$

Put $x = -3$, $9 - 39 + 15 = C(-3)$

$\Rightarrow -3C = -15 \Rightarrow C = 5$

Equating the coefficients of x^2,

$A + 2B = 1 \Rightarrow 2B = 1 - A = 1 + 1 = 2 \Rightarrow B = 1$

$$\therefore \frac{x^2+13x+15}{(2x+3)(x+3)^2}=\frac{-1}{2x+3}+\frac{1}{x+3}+\frac{5}{(x+3)^2}$$

5. Resolve $\frac{1}{(x-1)^2(x-2)}$ into partial fractions.

Sol. Let $\frac{1}{(x-1)^2(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}$

$\Rightarrow \frac{1}{(x-1)^2(x-2)}$

$= \frac{A(x-1)(x-2) + B(x-2) + C(x-1)^2}{(x-1)^2(x-2)}$

$\Rightarrow 1 = A(x-1)(x-2) + B(x-2) + C(x-1)^2 ...(1)$

Put x = 1 in (1)

$1 = A(0) + B(1-2) + C(0)$

$\Rightarrow -B = 1 \Rightarrow B = -1$

Put x = 2 in (1)

$1 = A(0) + B(0) + C(2-1)^2 \Rightarrow C = 1$

Equating the coefficients of x^2 in (1)

$0 = A + C \Rightarrow A = -C = -1 \Rightarrow A = -1$

$\therefore \frac{1}{(x-1)^2(x-2)} = \frac{-1}{x-1} - \frac{1}{(x-1)^2} + \frac{1}{x-2}$

6. Resolve $\frac{3x-18}{x^3(x+3)}$ into partial fractions.

Sol. Let $\frac{3x-18}{x^3(x+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+3}$

$\therefore \frac{3x-18}{x^3(x+3)} = \frac{Ax^2(x+3) + Bx(x+3) + C(x+3) + Dx^3}{x^3(x+3)}$

$\Rightarrow 3x-18 = Ax^2(x+3) + Bx(x+3) + C(x+3) + Dx^3 ...(1)$

Put x = –3 in (1)

$3(-3) - 18 = A(0) + B(0) + C(0) + D(-3)^3$

$\Rightarrow -27D = -27 \Rightarrow D = 1$

Put x = 0 in (1)

$3(0) - 18 = A(0) + B(0) + C(0 + 3) + D(0)$

$\Rightarrow 3C = -18 \Rightarrow C = -6$

Equating the coefficients of x^3 in (1)

$0 = A + D \Rightarrow A = -D = -1 \Rightarrow A = -1$

Equating the coefficients of x^2 in (1)

$0 = 3A + B \Rightarrow B = -3A = -3(-1) = 3 \Rightarrow B = 3$

$\frac{3x-18}{x^3(x+3)} = \frac{-1}{x} + \frac{3}{x^2} - \frac{6}{x^3} + \frac{1}{x+3}$

7. Resolve $\frac{2x^2+1}{x^3-1}$ into partial fractions.

Sol. $\frac{2x^2+1}{x^3-1}=\frac{2x^2+1}{(x-1)(x^2+x+1)}$ ...(1)

$$\frac{2x^2+1}{(x-1)(x^2+x+1)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}$$

$$\Rightarrow \frac{2x^2+1}{(x-1)(x^2+x+1)}=\frac{A(x^2+x+1)+(Bx+C)(x-1)}{(x-1)(x^2+x+1)}$$

$\therefore 2x^2+1=A(x^2+x+1)+(Bx+C)(x-1)...(1)$

Put x = 1 in (1)

$2(1) + 1= A(1 + 1 + 1) + (B + C)(0)$

$\Rightarrow 3A = 3 \Rightarrow A = 1$

Put x = 0 in (1)

$0 + 1 = A(1) + (0 + C)(0 – 1)$

$\Rightarrow 1 = A – C \Rightarrow C = 0$

Equating the coefficients of x^2 in (1)

$2 = A + B \Rightarrow 2 = 1 + B \Rightarrow B = 1$

$$\therefore \frac{2x^2+1}{x^3-1}=\frac{1}{x-1}+\frac{(1)(x)+0}{x^2+x+1}=\frac{1}{x-1}+\frac{x}{x^2+x+1}$$

8. Resolve $\frac{x^3+x^2+1}{(x^2+2)(x^2+3)}$ into partial fractions.

Sol. Let $\frac{x^3+x^2+1}{(x^2+2)(x^2+3)}=\frac{Ax+B}{x^2+2}+\frac{Cx+D}{x^2+3}$

$$=\frac{(Ax+B)(x^2+3)+(Cx+D)(x^2+2)}{(x^2+2)(x^2+3)}$$

$\therefore x^3+x^2+1=$

$(Ax+B)(x^2+3)+(Cx+D)(x^2+2)$ ...(1)

$\Rightarrow x^3+x^2+1=(A+C)x^3+(B+D)x^2+$

$(3A+2C)x+(3B+2D)$

Comparing the coefficients of x^3, x^2, x and constant terms

A + C = 1, B + D = 1,

3A + 2C = 0, 3B + 2D = 1

Solve A = –2, C = 3, B = –1, D = 2

$$\therefore \frac{x^3+x^2+1}{(x^2+2)(x^2+3)}=\frac{-2x-1}{x^2+2}+\frac{3x+2}{x^2+3}$$

$$=\frac{3x+2}{x^2+3}-\frac{2x+1}{x^2+2}$$

9. Resolve $\frac{3x^3-2x^2-1}{x^4+x^2+1}$ into partial fractions.

Sol. $x^4+x^2+1=x^4+2x^2+1-x^2$

$=(x^2+1)^2-x^2=(x^2+x+1)(x^2-x+1)$

$$\frac{3x^3-2x^2-1}{x^4+x^2+1}=\frac{3x^3-2x^2-1}{(x^2+x+1)(x^2-x+1)}$$

$$=\frac{Ax+B}{x^2+x+1}+\frac{Cx+D}{x^2-x+1}$$

Multiplying with $x^4 + x^2 + 1$,

$3x^3 - 2x^2 - 1 =$

$(Ax + B)(x^2 - x + 1) + (Cx + D)(x^2 + x + 1)$

Equating the coefficients of like terms, we get

$A + C = 3$ …(1)

$\Rightarrow C = 3 - A$

$-A + B + C + D = -2$ …(2)

$A - B + C + D = 0$ …(3)

$B + D = -1$ …(4) $D = -1-B$

Substitute (C), (D) in (2)

$-A + B + 3 - A - 1 - B = -2$

$\Rightarrow -2A = -4 \Rightarrow A = 2$

Substitute C, D in (3)

$A - B + 3 - A - 1 - B = 0 \Rightarrow 2 = 2B \Rightarrow B = 1$

$\therefore C = 3 - 2 = 1, D = -1-1 = -2$

$Ax + B = 2x + 1, Cx + D = x - 2$

$$\therefore \frac{3x^3-2x^2-1}{x^4+x^2+1}=\frac{2x+1}{x^2+x+1}+\frac{x-2}{x^2-x+1}$$

10. Resolve $\frac{x^4+24x^2+28}{(x^2+1)^3}$ into partial fractions.

Sol. $\frac{x^4+24x^2+28}{(x^2+1)^3}$

Put $x^2 + 1 = y \Rightarrow x^2 = y - 1$

Now $\frac{x^4+24x^2+28}{(x^2+1)^3}=\frac{(y-1)^2+24(y-1)+28}{y^3}$

$$=\frac{y^2-2y+1+24y-24+28}{y^3}=\frac{y^2+22y+5}{y^3}$$

$$=\frac{y^2}{y^3}+\frac{22y}{y^3}+\frac{5}{y^3}=\frac{1}{y}+\frac{22}{y^2}+\frac{5}{y^3}$$

$$=\frac{1}{(x^2+1)}+\frac{22}{(x^2+1)^2}+\frac{5}{(x^2+1)^3}$$

$$\therefore \frac{x^4+24x^2+28}{(x^2+1)^3}=\frac{1}{(x^2+1)}+\frac{22}{(x^2+1)^2}+\frac{5}{(x^2+1)^3}$$

11. Resolve $\frac{x^3}{(2x-1)(x+2)(x-3)}$ into partial fractions.

Sol. $\frac{x^3}{(2x-1)(x+2)(x-3)} =$

$$\frac{1}{2}+\frac{A}{2x-1}+\frac{B}{x+2}+\frac{C}{x-3}$$

Multiplying with $2(2x-1)(x+2)(x-3)$

$2x^3 = (2x-1)(x+2)(x-3)+2A(x+2)$

$(x-3)+2B(2x-1)(x-3)+2C(2x-1)(x+2)$

Put $x=\frac{1}{2}, 2\left(\frac{1}{8}\right)=2A\left(\frac{5}{2}\right)\cdot\left(-\frac{5}{2}\right)\Rightarrow A=-\frac{1}{50}$

Put $x = -2$, $2(-8) = 2B(-5)(-5) \Rightarrow B = \frac{-8}{25}$

Put $x = 3$, $2(27) = 2C(5)(5) \Rightarrow C = \frac{27}{25}$

$$\therefore \frac{x^3}{(2x-1)(x+2)(x-3)}=\frac{1}{2}-\frac{1}{50(2x-1)}-\frac{8}{25(x+2)}+\frac{27}{25(x-3)}$$

12. Resolve $\frac{x^4}{(x-1)(x-2)}$ into partial fractions.

Sol. $\frac{x^4}{(x-1)(x-2)}=\frac{x^4}{x^2-3x+2}$

$$=\frac{x^2(x^2-3x+2)+3x(x^2-3x+2)+7(x^2-3x+2)+15x-14}{x^2-3x+2}$$

$$=x^2+3x+7+\frac{15x-14}{x^2-3x+2}$$

Let $\frac{15x-14}{(x-1)(x-2)}=\frac{A}{x-1}+\frac{B}{x-2}$

Equating the coefficients of $(x-1)(x-2)$

$15x-14 = A(x-2)+B(x-1)$

Put $x = 1$, $15 - 14 = A(-1) \Rightarrow A = -1$

Put $x = 2$, $30 - 14 = B(1) \Rightarrow B = 16$

$$\therefore \frac{x^4}{(x-1)(x-2)}=x^2+3x+7-\frac{1}{x-1}+\frac{16}{x-2}$$

13. Find the coefficient of x^4 in the expansion of $\frac{3x}{(x-2)(x+1)}$.

Sol. $\frac{3x}{(x-2)(x+1)}=\frac{A}{x-2}+\frac{B}{x+1}$

Multiplying with $(x-2)(x+1)$

$3x = A(x+1) + B(x-2)$

Put $x = -1, -3 = B(-3) \Rightarrow B = 1$

Put $x = 2$, $6 = A(3) \Rightarrow A = 2$

$$\therefore \frac{3x}{(x-2)(x+1)}=\frac{2}{x-2}+\frac{1}{x+1}$$

$$= \frac{2}{-2\left(1-\frac{x}{2}\right)} + \frac{1}{1+x} = -\left(1-\frac{x}{2}\right)^{-1} + (1+x)^{-1}$$

$$= -\left[1+\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{8}+\frac{x^4}{16}+\ldots\right]+\left(1-x+x^2-x^3+x^4\ldots\right)$$

$\therefore$ Coefficient of $x^4 = -\frac{1}{16}+1=\frac{15}{16}$

14. Find the coefficient of x^n in the expansion of $\frac{x}{(x-1)^2(x-2)}$.

Sol. $\frac{x}{(x-1)^2(x-2)} = \frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x-2}$

Multiplying with $(x-1)^2(x-2)$

$x = A(x-1)(x-2) + B(x-2) + C(x-1)^2$

Put $x = 1$, $1 = B(-1) \Rightarrow B = -1$

Put $x = 2$, $2 = C(1) \Rightarrow C = 2$

Put $x = 0$, $0 = 2A - 2B + C \Rightarrow 2A = 2B - C$

$= -2 - 2 = -4 \Rightarrow A = -2$

$$\therefore \frac{x}{(x-1)^2(x-2)} = \frac{-2}{x-1}-\frac{1}{(x-1)^2}+\frac{2}{x-2}$$

$$= \frac{2}{1-x}-\frac{1}{(1-x)^2}+\frac{2}{-2\left(1-\frac{x}{2}\right)}$$

$$= 2(1-x)^{-1}-(1-x)^{-2}-\left(1-\frac{x}{2}\right)^{-1}$$

$$= 2[1+x+x^2+\ldots+x^n+\ldots]-[1+2x+3x^2+\ldots+(n+1)x^n+\ldots]-\left[1+\frac{x^2}{2}+\frac{x^2}{4}+\ldots\frac{x^n}{2^n}+\ldots\right]$$

$\therefore$ Coefficient of $x^n = 2(1)-(n+1)-\left(\frac{1}{2^n}\right)$

$$= 2-n-1-\frac{1}{2^n} = 1-n-\frac{1}{2^n}.$$