SOLVED PROBLEMS

1. Find the largest binomial coefficients in the expansion of
(i) $(1 + x)^{19}$ (ii) $(1 + x)^{24}$

Sol. i) Here n = 19 is an odd integer. Hence the largest binomial coefficients are

$${}^{n}C_{\left(\frac{n-1}{2}\right)} \text{ and } {}^{n}C_{\left(\frac{n+1}{2}\right)}$$

i.e. ${}^{19}C_9$ and ${}^{19}C_{10}$ $\left({}^{19}C_9 = {}^{19}C_{10}\right)$

ii) Here n = 24 is an even integer. Hence the largest binomial coefficient is

$${}^{n}C_{\left(\frac{n}{2}\right)}$$ i.e. ${}^{24}C_{12}$

2. If $^{22}C_r$ is the largest binomial coefficient in the expansion of $(1 + x)^{22}$, find the value of $^{13}C_r$.

Sol. Here n = 22 is an even integer. There is only one largest binomial coefficient and it is

$${}^{n}C_{(n/2)} = {}^{22}C_{11} = {}^{22}C_r \Rightarrow r = 11$$

$$\therefore {}^{13}C_r = {}^{13}C_{11} = {}^{13}C_2 = \frac{13\times12}{1\times2} = 78$$

3. Find the 7th term in the expansion of $\left(\frac{4}{x^3}+\frac{x^2}{2}\right)^{14}$.

Sol. The general term in the expansion of $(X+ a)^n$ is

$$T_{r+1} = {}^{n}C_r (X)^{n-r} a^r$$

Put $X = \frac{4}{x^3}, a = \frac{x^2}{2}, n = 14, r = 6$

$$T_7 \text{ in } \left(\frac{4}{x^3}+\frac{x^2}{2}\right)^{14} \text{ is } = {}^{14}C_6 \left(\frac{4}{x^3}\right)^{14-6} \left(\frac{x^2}{2}\right)^6$$

$$= {}^{14}C_6 \frac{4^8}{2^6} \cdot \frac{x^{12}}{x^{24}} = {}^{14}C_6 \cdot 4^5 \cdot \frac{1}{x^{12}}$$

4. Find the 3rd term from the end in the expansion of $\left(x^{-2/3}-\frac{3}{x^2}\right)^8$.

Sol. Comparing with $(X + a)^n$, we get

$$X = x^{-2/3},\ a = \frac{-3}{x^2}, n = 8$$

In the given expansion $\left(x^{-2/3}-\frac{3}{x^2}\right)^8$, we have n + 1 = 8 + 1 = 9 terms.

Hence the 3rd term from the end is 7th term from the beginning.

$$\therefore\ T_7 = {}^{n}C_6 (X)^{n-6} (a^6)$$

$$= {}^{8}C_6 (x^{-2/3})^{8-6} \left(\frac{-3}{x^2}\right)^6 = {}^{8}C_6 x^{-4/3} \cdot \frac{3^6}{x^{12}}$$

$$= \frac{8\times7}{1\times2} \cdot 3^6 \cdot x^{-4/3-12} = 28 \cdot 3^6 \cdot x^{-40/3}$$

5. Find the coefficient of x^9 and x^{10} in the expansion of $\left(2x^2-\frac{1}{x}\right)^{20}$.

Sol. If we write $X = 2x^2$ and $a = -\frac{1}{x}$, then the general term in the expansion of

$\left(2x^2-\frac{1}{x}\right)^{20} = (X+a)^{20}$ is

$$T_{r+1} = {}^nC_r X^{n-r} a^r = {}^{20}C_r (2x^2)^{20-r}\left(-\frac{1}{x}\right)^r$$

$$= (-1)^r\ {}^{20}C_r\ 2^{20-r}\ x^{40-3r}$$

Now x^9 coefficient is x^{40-3r}

$\Rightarrow x^9 = 40-3r = 9$

$\Rightarrow 3r = 31 \Rightarrow r = \frac{31}{3}$

Since r = 31/3 which is impossible since r must be a positive integer. Thus there is no term containing x^9 in the expansion of the given expression. In otherwords the coefficient of x^9 is 0.

Now to find the coefficient of x^{10}.

Put $40 - 3r = 10 \Rightarrow r = 10$

$T_{10+1} = (-1)^{10}\ {}^{20}C_{10}\ 2^{20-10}\ x^{40-30} = {}^{20}C_{10}\ 2^{10}\ x^{10}$

$\therefore$ The coefficient of x^{10} is ${}^{20}C_{10}\ 2^{10}$.

6. Find the term independent of x (that is the constant term) in the expansion of $\left(\frac{\sqrt{x}}{3}+\frac{3}{2x^2}\right)^{10}$.

Sol. $T_{r+1} = {}^{10}C_r\left(\sqrt{\frac{x}{3}}\right)^{10-r}\left(\frac{3}{2x^2}\right)^r$

$$= \frac{{}^{10}C_r \cdot 3^{\frac{3r-10}{2}}}{2^r}\cdot x^{\frac{10-5r}{2}}$$

To find the term independent of x, put

$$\frac{10-5r}{2} = 10 \Rightarrow r = 2$$

$$\therefore T_3 = \frac{{}^{10}C_2\, 3^{\frac{6-10}{2}}}{2^2}\cdot x^{\frac{10-10}{2}} = \frac{{}^{10}C_2\, 3^{-2}\, x^0}{2^2} = \frac{5}{4}$$

7. If the coefficients of x^{10} in the expansion of $\left(ax^2+\frac{1}{bx}\right)^{11}$ is equal to the coefficient of x^{-10} in the expansion of $\left(ax-\frac{1}{bx^2}\right)^{11}$; find the relation between a and b where a and b are real numbers.

Sol. The general term in the expansion of $\left(ax^2+\frac{1}{bx}\right)^{11}$ is

$$T_{r+1}={}^{11}C_r(ax^2)^{11-r}\left(\frac{1}{bx}\right)^r$$

$$={}^{11}C_r a^{11-b}\left(\frac{1}{b}\right)^r x^{22-2r-r}$$

To find the coefficient of x^{10}, put

$22 - 3r = 10 \Rightarrow 3r = 12 \Rightarrow r = 4$

Hence the coefficient of x^{10} in $\left(ax^2+\frac{1}{bx}\right)^{11}$ is

$$={}^{11}C_7\cdot a^7\left(\frac{1}{b}\right)^4={}^{11}C_7\frac{a^7}{b^4} \qquad ...(1)$$

The general term in the expansion of $\left(ax-\frac{1}{bx^2}\right)^{11}$ is

$$T_{r+1}={}^{11}C_r(ax)^{11-r}\left(\frac{-1}{bx^2}\right)^r$$

$$=(-1)^r\,{}^{11}C_r a^{11-r}\left(\frac{1}{b}\right)^r x^{11-r-2r}$$

For the coefficient of x^{-10} put

$11 - 3r = -10 \Rightarrow 3r = 21 \Rightarrow r = 7$

$\therefore$ The coefficient of x^{-10} in $\left(ax-\frac{1}{bx^2}\right)^{11}$ is

$$=(-1)^7\cdot{}^{11}C_7(a)^4\left(\frac{1}{b}\right)^7=(-1)\,{}^{11}C_7\frac{(a^4)}{b^7}\ldots(2)$$

Given that the coefficients are equal.

Hence from (1) and (2), we get

$${}^{11}C_7\cdot\frac{a^7}{a^4}=-{}^{11}C_7\cdot\frac{a^4}{b^7}$$

$$\Rightarrow a^3=\frac{-1}{b^3}\Rightarrow a^3b^3=-1\Rightarrow ab=-1$$

8. If the k^{th} term is the middle term in the expansion of $\left(x^2-\frac{1}{2x}\right)^{20}$, find T_k and T_{k+3}.

Sol. The general term in the expansion of $\left(x^2-\frac{1}{2x}\right)^{20}$ is

$$T_{r+1}={}^{20}C_r(x^2)^{20-r}\left(\frac{-1}{2x}\right)^r \qquad \ldots(1)$$

$\because$ The given expansion has (20 + 1) = 21 times, $\left(\frac{n}{2}+1\right)^{th}$ term, i.e. $\left(\frac{20}{2}+1\right)=11^{th}$ term is the only middle term.

$\therefore$ k = 11

Put r = 10 in eq.(1)

$$T_{13+1} = {}^{20}C_{13}(x^2)^7\left(\frac{-1}{2x}\right)^{13} = (-1)\,{}^{20}C_{13}\frac{1}{2^{13}}x$$

9. If the coefficients of $(2r + 4)^{th}$ and $(r – 2)$nd terms in the expansion of $(1 + x)^{18}$ are equal, find r.

Sol. T_{2r+4} term of $(1 + x)^{18}$ is

$$T_{2r+4} = {}^{18}C_{2r+3}(x)^{2r+3}$$

T_{r-2} term of $(1+x)^{18}$

$$T_{r-2} = {}^{18}C_{r-3}(x)^{r-3}$$

Given that the coefficients of $(2r + 4)^{th}$ term = The coefficient of $(r – 2)$nd term.

$$\Rightarrow {}^{18}C_{2r+3} = {}^{18}C_{r-3}$$

$$\Rightarrow 2r+3 = r-3 \text{ (or) } (2r+3)+(r-3)=18$$

$$\Rightarrow r = -6 \text{ (or) } 3r = 18 \Rightarrow r = 6$$

$(\because$ r is a positive integer$)$

10. Prove that $2\cdot C_0 + 7\cdot C_1 + 12\cdot C_2 + ... + (5n+2)C_n = (5n+4)2^{n-1}$

Sol. First method :

The coefficients of $C_0, C_1, C_2, \ldots\ldots C_n$ are in A.P. with first term a = 2, C.d. (d) = 5

$$\because a\cdot C_0 + (a+d)C_1 + (a+2d)C_2 + ...$$

$$+(a+(n-1)d)C_{n-1} + (a+nd)C_n$$

$$= (2a+nd)2^{n-1}$$

$$= (2\times 2 + n\cdot 5)\cdot 2^{n-1} = (4+5n)2^{n-1}$$

Second method :

General term in L.H.S.

i.e. $T_{r+1} = (5r+2)C_n$

$$\therefore 2\cdot C_0 + 7\cdot C_1 + 12\cdot C_2 + ... + (5n+2)C_n$$

$$= \sum_{r=0}^{n}(5r+2)C_r$$

$$= 5\sum_{r=0}^{n} r\cdot C_r + 2\sum_{r=0}^{n} C_r$$

$$= 5(n)\cdot 2^{n-1} + 2\cdot 2^n = (5n+4)2^{n-1}$$

11. Prove that

i) $C_0 + 3C_1 + 3^2C_2 + ... + 3^nC_n = 4^n$

ii) $\frac{C_1}{C_0} + 2\cdot\frac{C_2}{C_1} + 3\cdot\frac{C_3}{C_2} + ... + n\frac{C_n}{C_{n-1}} = \frac{n(n+1)}{2}$

Sol. (i) We have

$(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$

Put x = 3, we get

$(1+3)^n = C_0 + C_1 \cdot 3 + C_2 3^2 + ... + C_n 3^n$

$\therefore C_0 + 3C_1 + 3^2 C_2 + ... + 3^n C_n = 4^n$

(ii) $\frac{C_1}{C_0} + 2 \cdot \frac{C_2}{C_1} + 3 \cdot \frac{C_3}{C_2} + ... + n\frac{C_n}{C_{n-1}}$

$= \frac{{}^nC_1}{{}^nC_0} + 2\left(\frac{{}^nC_2}{{}^nC_1}\right) + 3\left(\frac{{}^nC_3}{{}^nC_2}\right) + ... + n\left(\frac{{}^nC_n}{{}^nC_1}\right)$

$= \frac{n}{1} + 2\frac{n-1}{2} + 3\frac{n-2}{3} + ... + n\frac{1}{n}$

$= n + (n-1) + (n-2) + ... + 3 + 2 + 1$

$= 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$

12. For n = 0, 1, 2, 3, … n, prove that $C_0 \cdot C_r + C_1 \cdot C_{r+1} + C_2 \cdot C_{r+2} + ... + C_{n-r} \cdot C_n = {}^{2n}C_{n+r}$ **and hence deduce that**

i) $C_0^2 + C_1^2 + C_2^2 + ... + C_n^2 = {}^{2n}C_n$

ii) $C_0 \cdot C_1 + C_1 \cdot C_2 + C_2 \cdot C_3 + ... + C_{n-1}C_n = {}^{2n}C_{n+1}$

Sol. We know that

$(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$ …(1)

On replacing x by 1/x in the above equation,

$\left(1+\frac{1}{x}\right)^n = C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + ... + \frac{C_n}{x^n}$...(2)

From (1) and (2)

$\left(1+\frac{1}{x}\right)^n (1+x)^n = \left(C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + ... + \frac{C_n}{x^n}\right)$

$(C_0 + C_1x + C_2x^2 + ... + C_nx^n)$ ...(3)

The coefficient of x^r in R.H.S. of (3)

$= C_0C_r + C_1C_{r+1} + C_2C_{r+2} + ... + C_{n-r}C_n$

The coefficient of x^r in L.H.S. of (3)

= the coefficient of x^r in $\frac{(1+x)^{2n}}{x^n}$

= the coefficient of x^{n+r} is $(1 + x)^{2n}$

= ${}^{2n}C_{n+r}$

From (3) and (4), we get

$C_0 \cdot C_1 + C_1 \cdot C_2 + C_2 \cdot C_3 + ... + C_{n-1}C_n = {}^{2n}C_{n+1}$

i) On putting r = 0 in (i), we get

$C_0^2 + C_1^2 + C_2^2 + ... + C_n^2 = {}^{2n}C_n$

ii) On substituting r = 1 in (i) we get

$C_0 \cdot C_1 + C_1 \cdot C_2 + C_2 \cdot C_3 + ... + C_{n-1}C_n = {}^{2n}C_{n+1}$

13.Prove that

$$3\cdot C_0^2 + 7\cdot C_1^2 + 11\cdot C_2^2 + \ldots + (4n+3)C_n^2$$

$$= (2n+3)\,{}^{2n}C_n$$

Sol. Let $S = 3\cdot C_0^2 + 7\cdot C_1^2 + 11\cdot C_2^2 + \ldots + (4n-1)C_{n-1}^2 + (4n+3)C_n^2$ ...(1)

$\because$ $C_0 = C_n$, $C_1 = C_{n-1}$ etc., on writing the terms of R.H.S. of (1) in the reverse order, we get

$S = (4n+3)C_0^2 + (4n-1)C_1^2 + \ldots + 7C_{n-1}^2 + 3C_n^2 \ldots\ldots(2)$

Add (1) and (2)

$2S = (4n+6)C_0^2 + (4n+6)C_1^2 + \ldots + (4n+6)C_n^2$

$\Rightarrow 2S = (4n+6)(C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2)$

$= 2(2n+3)\,{}^{2n}C_n$

$\therefore S = (2n+3)\,{}^{2n}C_n$

14. Find the numerically greatest term(s) in the expansion of

i) $(2 + 3x)^{10}$ when x = $\frac{11}{8}$

Sol. Write $(2+3x)^{10} = \left[2\left(1+\frac{3}{2}x\right)^{10}\right] = 2^{10}\left(1+\frac{3x}{2}\right)^{10}$

First find N.G. term in $\left(1+\frac{3x}{2}\right)^{10}$

Let $X = \frac{3x}{2} = \frac{3\times\frac{11}{8}}{2} = \frac{33}{16}$

Now consider

$$\frac{(n+1)|x|}{1+|x|} = \frac{(10+1)\left(\frac{33}{16}\right)}{\frac{33}{16}+1} = \frac{11\times 33}{48} = \frac{363}{48}$$

Its integral part $m = \left[\frac{363}{48}\right] = 7$

$\therefore$ T_{m+1} is the numerically greatest term in

$$\left(1+\frac{3x}{2}\right)^{10}$$

i.e. $T_{7+1} = T_8 = {}^{10}C_7\left(\frac{3x}{2}\right)^7$

$= {}^{10}C_7\left(\frac{3}{2}\times\frac{11}{8}\right)^7 = {}^{10}C_7\left(\frac{33}{16}\right)^7$

$\therefore$ N.G. term in the expansion of $(2 + 3x)^{10}$ is = $2^{10}\cdot{}^{10}C_7\left(\frac{33}{16}\right)^7$.

ii) $(3x - 4y)^{14}$ when x= 8, y = 3.

Sol. $(3x-4y)^{14} = \left(3x\left(1-\frac{4y}{3x}\right)\right)^{14}$

$= (3x)^{14}\left(1-\frac{4y}{3x}\right)^{14}$

Write $X = \frac{-4y}{3x} = -\left(\frac{4\times 3}{3\times 8}\right) = -\frac{1}{2}$

$|X| = \frac{1}{2}$

Now $\frac{(n+1)|X|}{1+|X|} = \frac{(14+1)\frac{1}{2}}{1+\frac{1}{2}} = 5$, an integer.

Here $|T_5| = |T_6|$ are N.G. terms.

T_5 in the expansion of $\left(1-\frac{4y}{3x}\right)^{14}$ is

$T_5 = {}^{14}C_4\left(\frac{-4y}{3x}\right)^4 = {}^{14}C_4\left(\frac{1}{2}\right)^4$

and $T_6 = {}^{14}C_5\left(\frac{-4y}{3x}\right)^5 = -{}^{14}C_5\left(\frac{1}{2}\right)^5$

Here N.G. terms are T_5 and T_6. They are

$T_5 = {}^{14}C_4\left(\frac{1}{2}\right)^4 (24)^{14}$

$T_6 = -{}^{14}C_5\left(\frac{1}{2}\right)^5 (24)^{14}$

But $|T_5| = |T_6|$

15. Prove that $6^{2n} - 35n - 1$ is divisible by 1225 for all natural numbers of n.

Sol. $6^{2n} - 35n - 1 = (36)^n - 35n - 1$

$= (35+1)^n - 35n - 1$

$= \left[(35)^n + {}^nC_1(35)^{n-1} + {}^nC_2(35)^{n-2} + ...+ {}^nC_{n-2}(35)^2 + {}^nC_{n-1}(35)^1 + {}^nC_n\right] - 35n - 1$

$= (35)^n + {}^nC_1(35)^{n-1} + {}^nC_2(35)^{n-2} + ... + {}^nC_{n-2}(35)^2$

$= (35)^2\left[(35)^{n-2} + {}^nC_1(35)^{n-3} + {}^nC_2(35)^{n-4} + ... + {}^nC_{n-2}\right]$

= 1225 (k), for same integer k.

Hence $6^{2n} - 35n - 1$ is divisible by 1225 for all integral values of n.

16. Suppose that n is a natural number and I, F are respectively the integral part and fractional part of $(7+4\sqrt{3})^n$, then show that (i) I is an odd integer, (ii) (I + F)(I – F) = 1.

Sol. Given that $(7+4\sqrt{3})^n = I + F$ where I is an integer and $0 < F < 1$.

Write $f = (7-4\sqrt{3})^n$

Now $36 < 48 < 49$

$6 < \sqrt{48} < 7$

i.e. $-7 < -\sqrt{48} < -6$

i.e. $0 < 7-4\sqrt{3} < 1$

i.e. $0 < (7-4\sqrt{3})^n < 1$

$\therefore\ 0 < f < 1$

Now $I + F + f = (7+4\sqrt{3})^n + (7-4\sqrt{3})^n =$

$$= \left({}^nC_0 \cdot 7^n + {}^nC_1(7)^{n-1}(4\sqrt{3}) + {}^nC_2(7)^{n-2}(4\sqrt{3})^2 + ...\right) +$$

$$\left({}^nC_0 \cdot 7^n - {}^nC_1(7)^{n-1}(4\sqrt{3}) + {}^nC_2(7)^{n-2}(4\sqrt{3})^2 - ...\right)$$

$$= 2\left[7^n + {}^nC_2 7^{n-2}(4\sqrt{3})^2 + {}^nC_4 7^{n-4}(4\sqrt{3})^4 + ...\right]$$

= 2k, where k is a positive integer …(1)

Thus I + F + f is an even integer.

Since I is an integer, we get that F + f is an integer. Also since $0 < F < 1$ and $0 < f < 1$

$\Rightarrow 0 < F + f < 2$

$\because$ F + f is an integer

We get F + f = 1

(i.e.) I – F = f …(2)

(i) From (1) I + F + f = 2k

$\Rightarrow$ f = 2k – 1, an odd integer.

(ii) (I + F)(I – F) = (I + F)f

$(7+4\sqrt{3})^n(7-4\sqrt{3})^n = (49-49)^n = 1$

17. Find the coefficient of x^6 in $(3 + 2x + x^2)^6$.

Sol. $(3+2x+x^2) = [(3+2x)+x^2)]^6$

$$= {}^6C_0(3+2x)^6 + {}^6C_1(3+2x)^5(x^2) + {}^6C_2(3+2x)^4(x^2)^2 + {}^6C_3(3+2x)^3(x^2)^3 + ...$$

$$= (3+2x)^6 + 6(3+2x)^5(x^2) + 15x^4(3+2x)^4x^4 + 20x^6(3+2x)^3 + ...$$

$$= \left[\sum_{r=0}^{6} {}^6C_r \cdot 3^{6-r}(2x)^r\right] + 6x^2\left[\sum_{r=0}^{5} {}^5C_r \cdot 3^{5-r}(2x)^r\right] + 15x^4\left[\sum_{r=0}^{4} {}^4C_r \cdot 3^{4-r}(2x)^r\right] + 20x^6\left[\sum_{r=0}^{3} {}^3C_r \cdot 3^{3-r}(2x)^r\right] + ...$$

$\therefore$ The coefficient of x^6 in $(3 + 2x + x^2)^6$ is

$$= {}^6C_6 \cdot 3^0 \cdot 2^6 + 6({}^5C_4 \cdot 3^1 \cdot 2^4) + 15({}^4C_2 \cdot 3^2 \cdot 2^2) + 20({}^3C_0 \cdot 3^3 \cdot 2^0)$$

$$= 64 + 1440 + 3240 + 540 = 5284$$

18. If n is a positive integer, then prove that $C_0+\frac{C_1}{2}+\frac{C_2}{3}+...+\frac{C_n}{n+1}=\frac{2^{n+1}-1}{n+1}$.

Sol. Write $S=C_0+\frac{C_1}{2}+\frac{C_2}{3}+...+\frac{C_n}{n+1}$ then

$$S={}^nC_0+\frac{1}{2}\cdot{}^nC_1+\frac{1}{3}\cdot{}^nC_2+...+\frac{1}{n+1}\cdot{}^nC_n$$

$$\therefore (n+1)S=\frac{n+1}{1}\cdot{}^nC_0+\frac{n+1}{2}\cdot{}^nC_1$$

$$+\frac{n+1}{3}\cdot{}^nC_2+...+\frac{n+1}{n+1}\cdot{}^nC_n$$

$$\therefore (n+1)S={}^{(n+1)}C_1+{}^{(n+1)}C_2+{}^{(n+1)}C_3+...+{}^{(n+1)}C_{n+1}$$

$$\left(\text{since } \frac{n+1}{r+1}\cdot{}^nC_r={}^{n+1}C_{r+1}\right)$$

$$=2^{n+1}-1$$

Hence $S=\frac{2^{n+1}-1}{n+1}$

$$\therefore\ C_0+\frac{C_1}{2}+\frac{C_2}{3}+...+\frac{C_n}{n+1}=\frac{2^{n+1}-1}{n+1}$$

19. If n is a positive integer and x is any non-zero real number, then prove that

$$C_0+C_1\frac{x}{2}+C_2\cdot\frac{x^2}{3}+C_3\cdot\frac{x^3}{4}+...+C_n\cdot\frac{x^n}{n+1}$$

$$=\frac{(1+x)^{n+1}-1}{(n+1)x}$$

Sol. $C_0+C_1\frac{x}{2}+C_2\cdot\frac{x^2}{3}+C_3\cdot\frac{x^3}{4}+...+C_n\cdot\frac{x^n}{n+1}$

$$={}^nC_0+\frac{1}{2}{}^nC_1x+\frac{1}{3}{}^nC_2x^2+...+\frac{1}{n+1}{}^nC_nx^n$$

$$=1+\frac{n}{1!}\frac{x}{2}+\frac{n(n-1)}{2!}\frac{x^2}{3}+.....$$

$$=1+\frac{n}{2!}x^1+\frac{n(n-1)}{3!}x^2+.....$$

$$= \frac{1}{(n+1)x}\left[\begin{array}{l}\dfrac{(n+1)x^1}{1!}+\dfrac{(n+1)n}{2!}x^2 \\ \qquad +\dfrac{(n+1)n(n-1)}{3!}x^3+\ldots\end{array}\right]$$

$$= \frac{1}{(n+1)x}\left[\begin{array}{l}{}^{(n+1)}C_1x+{}^{(n+1)}C_2x^2+ \\ \qquad {}^{(n+1)}C_3x^3+\ldots\end{array}\right]$$

$$= \frac{1}{(n+1)x}\left[\begin{array}{l}1+{}^{n+1}C_1x+{}^{n+1}C_2x^2 \\ \qquad +\ldots+{}^{n+1}C_{n+1}x^{n+1}-1\end{array}\right]$$

$$= \frac{1}{(n+1)x}\left[(1+x)^{n+1}-1\right]$$

$$C_0+C_1\frac{x}{2}+C_2\cdot\frac{x^2}{3}+C_3\cdot\frac{x^3}{4}+\ldots+C_n\cdot\frac{x^n}{n+1}$$

$$= \frac{(1+x)^{n+1}-1}{(n+1)x}$$

20. Prove that $C_0^2-C_1^2+C_2^2-C_3^2+\ldots+(-1)^nC_n^2$

$$= \begin{cases}(-1)^{n/2}\,{}^nC_{n/2}, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd}\end{cases}$$

Sol. Take $(1-x)^n\left(1+\dfrac{1}{x}\right)^n$

$$= (C_0-C_1x+C_2x^2-C_3x^3+\ldots+(-1)^n\cdot C_nx^n)$$

$$\left(C_0+\frac{C_1}{x}+\frac{C_2}{x^2}+\ldots+\frac{C_n}{x^n}\right) \qquad \ldots(1)$$

The term independent of x in R.H.S. of (1) is

$= C_0^2-C_1^2+C_2^2-C_3^2+\ldots+(-1)^nC_n^2$

Now we can find the term independent of in the L.H.S. of (1).

L.H.S. of (1) = $(1-x)^n\left(1+\dfrac{1}{x}\right)^n$

$$= (1-x)^n\left(\frac{1+x}{x}\right)^n = \frac{(1-x^2)^n}{x^n}$$

$$= \sum_{r=0}^{n}{}^nC_r(-x^2)^r \qquad \ldots(2)$$

Suppose n is an even integer, say n = 2k.

Then from (2),

$$(1-x)^n\left(1+\frac{1}{x}\right)^n = \frac{\sum_{r=0}^{n}{}^nC_r(-x^2)^r}{x^n}$$

$$= \frac{\sum_{r=0}^{2k}{}^{2k}C_r(-x^2)^r}{x^{2k}} = \sum_{r=0}^{2k}{}^{2k}C_r(-1)^rx^{2r-2k} \ldots(3)$$

To set term independent of x in (3), put

$2r - 2k = 0 \Rightarrow r = k$

Hence the term indep. of x in

$(1-x)^n\left(1+\frac{1}{x}\right)^n$ is ${}^{2k}C_k(-1)^k = {}^nC_{(n/2)}(-1)^{n/2}$

When n is odd :

Observe that the expansion in the numerator of (2) contains only even powers of x.

$\therefore$ If n is odd, then there is no constant term in (2) (i.e.) the term independent of x in $(1-x)^n\left(1+\frac{1}{x}\right)^n$ is zero.

$\therefore$ From (1), we get

$$C_0^2 - C_1^2 + C_2^2 - C_3^2 + ... + (-1)^n C_n^2 = \begin{cases} (-1)^{n/2}\, {}^nC_{n/2}, & \text{if n is even} \\ 0, & \text{if n is odd} \end{cases}$$

21. Find the set E of x for which the binomial expansions for the following are valid

(i) $(3-4x)^{3/4}$ (ii) $(2+5x)^{-1/2}$

(iii) $(7-4x)^{-5}$ (iv) $(4+9x)^{-2/3}$

(v) $(a+bx)^r$

Sol. i) $(3-4x)^{3/4} = 3^{3/4}\left(1-\frac{4x}{3}\right)^{3/4}$

The binomial expansion of $(3-4x)^{3/4}$ is valid, when $\left|\frac{4x}{3}\right| < 1$.

i.e. $|x| < \frac{3}{4}$

i.e. $E = \left(\frac{-3}{4}, \frac{3}{4}\right)$

ii) $(2+5x)^{-1/2} = 2^{-1/2}\left(1+\frac{5x}{2}\right)^{-1/2}$

The binomial expansion of $(2+5x)^{-1/2}$ is valid when $\left|\frac{5x}{3}\right| < 1 \Rightarrow |x| < \frac{2}{5}$

i.e. $E = \left(\frac{-2}{5}, \frac{2}{5}\right)$

iii) $(7-4x)^{-5} = 7^{-5}\left(1-\frac{4x}{7}\right)^{-5}$

The binomial expansion of $(7-4x)^{-5}$ is valid when $\left|\frac{4x}{7}\right| < 1 \Rightarrow |x| < \frac{7}{4}$

i.e. $E = \left(\frac{-7}{4}, \frac{7}{4}\right)$

iv) $(4+9x)^{-2/3} = 4^{-2/3}\left(1+\frac{9x}{4}\right)^{-2/3}$

The binomial expansion of $(4+9x)^{-2/3}$ is valid when $\left|\frac{9x}{4}\right| < 1 \Rightarrow |x| < \frac{4}{9}$

$\Rightarrow x \in \left(\frac{-4}{9}, \frac{4}{9}\right)$

i.e. E = $\left(\frac{-4}{9},\frac{4}{9}\right)$

v) For any non zero reals a and b, the set of x for which the binomial expansion of $(a + bx)^r$ is valid when $r \notin Z^+ \cup \{0\}$, is $\left(-\frac{|a|}{|b|},\frac{|a|}{|b|}\right)$.

22. Find the

i) 9th term of $\left(2+\frac{x}{3}\right)^{-5}$

ii) 10th term of $\left(1-\frac{3x}{4}\right)^{4/5}$

iii) 8th term of $\left(1-\frac{5x}{2}\right)^{-3/5}$

iv) 6th term of $\left(3+\frac{2x}{3}\right)^{3/2}$

i) 9th term of $\left(2+\frac{x}{3}\right)^{-5}$

Sol. $\left(2+\frac{x}{3}\right)^{-5} = \left[2\left(1+\frac{x}{6}\right)\right]^{-5} = 2^{-5}\left(1+\frac{x}{6}\right)^{-5}$ ...(1)

Compare $\left(1+\frac{x}{6}\right)^{-5}$ with $(1 + x)^{-n}$,

we get X = x/6, n = 5

The general term in the binomial expansion of $(1 + x)^{-n}$ is

$T_{r+1} = (-1)^{n\ (n+r-1)}C_r \cdot x^r$

Put r = 8

$T_9 = (-1)^{8\ (5+8-1)}C_8 \cdot x^8 = {}^{12}C_8\left(\frac{x}{6}\right)^8$

From (1), the 9th term of $\left(2+\frac{x}{3}\right)^{-5}$ is

$= 2^{-5}\ {}^{13}C_8\left(\frac{x}{6}\right) = \frac{495}{32}\cdot\left(\frac{x}{6}\right)^8$

ii) 10th term of $\left(1-\frac{3x}{4}\right)^{4/5}$

Sol. Compare $\left(1-\frac{3x}{4}\right)^{4/5}$ with $(1 - x)^{p/q}$, we get $x = \frac{3x}{4}, p = 4, q = 5, \frac{x}{q} = \frac{3x}{20}$.

The general term in $(1 - x)^{p/q}$ is

$T_{r+1} = \frac{(-1)^4\left[p(p-q)(p-2q)...p-(r-1)q\right]}{r!}\left(\frac{x}{q}\right)^r$

Put r = 9

$T_{10} = \frac{(-1)^9\left[4(4-5)(4-10)...(4-40)\right]}{9!}\left(\frac{3x}{20}\right)^9$

$$= -4\frac{(-10(-6)(-11)(-16)(-21)}{(-26)(-31)(-36)}\left(\frac{3x}{20}\right)^9$$

$$= \frac{-4\times1\times6\times11\times16\times21\times26\times31\times36}{9!}\left(\frac{3x}{20}\right)^9$$

iii) 8th term of $\left(1-\frac{5x}{2}\right)^{-3/5}$

Sol. Compare $\left(1-\frac{5x}{2}\right)^{-3/5}$ with $(1-x)^{-p/q}$, we get $X=\frac{5x}{2}, p=3, q=5, \frac{x}{q}=\frac{\frac{5x}{2}}{5}=\frac{x}{2}$.

The general term in $(1-x)^{-p/q}$ is

$$T_{r+1}=\frac{[p(p+q)(p+2q)...p+(r-1)q]}{r!}\left(\frac{x}{q}\right)^r$$

Put r = 7

$$T_8=\frac{(3)(3+5)(3+2\times5)....[3+(7-1)5]}{7!}\left(\frac{x}{2}\right)^7$$

$$=\frac{(3\cdot8\cdot13\cdot18\cdot23\cdot28\cdot33)}{7!}\left(\frac{x}{2}\right)^7$$

iv) 6th term of $\left(3+\frac{2x}{3}\right)^{3/2}$

Sol. $\left(3+\frac{2x}{3}\right)^{3/2}=\left[3\left(1+\frac{2x}{9}\right)\right]^{3/2}$

$$=3^{3/2}\left(1+\frac{2x}{9}\right)^{3/2}...(1)$$

Compare $\left(1+\frac{2x}{9}\right)^{3/2}$ with $(1+x)^{p/q}$, we get

$$X=\frac{2x}{9}, p=3, q=2 \Rightarrow \frac{x}{q}=\frac{(2x/9)}{2}=\frac{x}{9}$$

The general term of $(1+x)^{p/q}$ is

$$T_{r+1}=\frac{[p(p-q)(p-2q)...p-(r-1)q]}{r!}\left(\frac{x}{q}\right)^r$$

Put r = 5, we get

$$T_6=\frac{(3)(3-2)(3-2\times2)(3-3\times2)(3-4\times2)}{5!}\left(\frac{x}{9}\right)^5$$

$$=\frac{(3)(1)(-1)(-3)(-5)}{5!}\left(\frac{x}{9}\right)^5=-\frac{3}{8}\left(\frac{x}{9}\right)^5$$

From (1), the 6th term of $\left(3+\frac{2x}{3}\right)^{3/2}$ is

$$=3^{3/2}\left(-\frac{3}{8}\right)\left(\frac{x}{9}\right)^5=-\frac{9\sqrt{3}}{8}\left(\frac{x}{9}\right)^5$$

23. Write the first 3 terms in the expansion of

(i) $\left(1+\frac{x}{2}\right)^{-5}$ **, (ii)** $(3+4x)^{-2/3}$ **, (iii)** $(4-5x)^{-1/2}$

i) $\left(1+\frac{x}{2}\right)^{-5}$

Sol. We have

$$(1+X)^{-n}=1-nX+\frac{(n)(n+1)}{1\cdot 2}(X)^2+...$$

$$\therefore\left(1+\frac{x}{2}\right)^{5}=1-\frac{5x}{2}+\frac{(5)(6)}{1\cdot 2}\left(\frac{x}{2}\right)^2-...$$

$$=1-\frac{5x}{2}+\frac{15}{4}x^2-...$$

$\therefore$ The first terms in the expansion of

$\left(1+\frac{x}{2}\right)^{-5}$ are $1,\ \frac{-5x}{2},\frac{15}{4}x^2$

ii) $(3+4x)^{-2/3}$

Sol. $(3+4x)^{-2/3}=\left[3\left(1+\frac{4}{3}\right)\right]^{-2/3}$

$$=3^{-2/3}\left(1+\frac{4}{3}x\right)^{-2/3}\quad ...(1)$$

We have

$$(1+X)^{-p/q}=1-\frac{p}{1}\frac{x}{q}+\frac{(p)(p+q)}{1\cdot 2}\left(\frac{x}{q}\right)^2-...$$

$$\therefore\left(1+\frac{4x}{3}\right)^{-2/3}=1-\frac{2}{1}\cdot\frac{4x}{9}+\frac{2\cdot 5}{1\cdot 2}\left(\frac{4x}{9}\right)^2-...$$

$\therefore$ From (1), the first 3 terms of $(3+4x)^{-2/3}$ is

$$3^{-2/3}\left\{1-\frac{8x}{9}+\frac{80}{81}x^2-.....\right\}$$

i.e. $3^{-2/3},-3^{-2/3-2}(8x),3^{-2/3-4}(80x^2)$

$$\Rightarrow 3^{-2/3}-3^{-8/3}(8x),3^{-14/3}(80x^2)$$

iv) $(4-5x)^{-1/2}$

Sol. $(4-5x)^{-1/2}=\left[4\left(1-\frac{5}{4}x\right)\right]^{-1/2}$

$$=4^{-1/2}\left(1-\frac{5}{4}x\right)^{-1/2}\quad ...(1)$$

We have

$$(1-X)^{-p/q}=1+\frac{p}{1}\frac{x}{q}+\frac{(p)(p+q)}{1\cdot 2}\left(\frac{x}{q}\right)^2+...$$

Here p = 1, q = 2, $X = \frac{5}{4}x \Rightarrow \frac{x}{q} = \frac{5}{8}x$

$$\therefore \left(1-\frac{5}{4}x\right)^{-1/2} = 1+\frac{1}{1}\left(\frac{5x}{8}\right)+\frac{1\cdot 3}{1\cdot 2}\left(\frac{5x}{8}\right)^2+\ldots$$

$$= 1+\frac{5x}{8}+\frac{75}{128}x^2+\ldots$$

From (1),

$$(4-5x)^{-1/2} = 2^{-1/2}\left(1+\frac{5x}{8}+\frac{75}{128}x^2+\ldots\right)$$

$\therefore$ The first 3 terms of $(4-5x)^{-1/2}$ are :

$$\frac{1}{2}, \frac{5x}{16}, \frac{75}{256}x^2$$

24. Write the general term of

(i) $\left(3+\frac{x}{2}\right)^{-2/3}$ **(ii)** $\left(2+\frac{3x}{4}\right)^{4/5}$

(iii) $(1-4x)^{-3}$ **(iv)** $(2-3x)^{-1/3}$

Sol. i) $\left(3+\frac{x}{2}\right)^{-2/3}$

$$\left(3+\frac{x}{2}\right)^{-2/3} = \left[3\left(1+\frac{x}{6}\right)\right]^{-2/3}$$

$$= 3^{-2/3}\left(1+\frac{x}{6}\right)^{-2/3} \quad \ldots(1)$$

The general term of $(1+x)^{-p/q}$

$$T_{r+1} = (-1)^r \left\{\frac{(p)(p+q)(p+2q)....(p)+(r-1)q}{(r)!}\left(\frac{x}{q}\right)^r\right\}$$

Here p = 2, q = 3, $X = \frac{x}{6} \Rightarrow \frac{x}{q} = \frac{\left(\frac{x}{6}\right)}{3} = \frac{x}{18}$

$\therefore$ T_{r+1} of $\left(3+\frac{x}{2}\right)^{-2/3}$ is

$$T_{r+1} = (3)^{-2/3}(-1)^r \left[\frac{(2)(2+3)(2+2\cdot 3)+\ldots[2+(r-1)3]}{r!}\left(\frac{x}{18}\right)^r\right]$$

$$\frac{1}{\sqrt{27}}\left\{\frac{(-1)^r(2)(5)(8)\ldots(3r-1)}{r!}\left(\frac{x}{18}\right)^r\right\}$$

ii) $\left(2+\frac{3x}{4}\right)^{4/5}$

Sol. $\left(2+\frac{3x}{4}\right)^{4/5} = \left[2\left(1+\frac{3x}{8}\right)\right]^{4/5}$

$= 2^{4/5}\left(1+\frac{3x}{8}\right)^{4/5} \quad \ldots (1)$

T_{r+1} of $(1 + X)^{p/q}$ is

$T_{r+1} = \frac{[p(p-q)(p-2q)...(p-(r-1)q)]}{r!}\left(\frac{X}{q}\right)^r$

Here p = 4, q = 5,

$X = \frac{3x}{8}, \frac{x}{q} = \frac{\left(\frac{3x}{8}\right)}{5} = \frac{3x}{40}$

$\therefore$ T_{r+1} of $\left(1+\frac{3x}{8}\right)^{4/5}$ is

$T_{r+1} = \frac{(4)(4-5)(4-2\times5)...(4-(r-1)5)}{r!}\left(\frac{3x}{40}\right)^r$

$= \frac{4(-1)(-6).....(-5r+9)}{r!}\left(\frac{3x}{40}\right)^r$

$= (-1)^{r-1}\frac{(4)(1)(6)...(5r-9)}{r!}\left(\frac{3x}{40}\right)^r$

$\therefore$ The general term of $\left(2+\frac{3x}{4}\right)^{4/5}$ is

$2^{4/5}\left[(-1)^{r-1}\frac{4\cdot1\cdot6...(5r-9)}{r!}\right]\left(\frac{3x}{40}\right)^r$

iii) $\mathbf{(1 - 4x)^{-3}}$

Sol. $(1-4x)^{-3} = (1-X)^{-n}$, here X = 4x, n = 3.

The general term of $(1 - X)^{-n}$ is

$T_{r+1} = {}^{n+r-1}C_r \cdot X^r$

$= {}^{(3+r-1)}C_r(4x)^r$

$= {}^{(r+2)}C_r(4x)^r$

$\therefore$ General term of $(1 - 4x)^{-3}$ is

$T_{r+1} = {}^{(r+2)}C_r(4x)^r$

iv) $\mathbf{(2 - 3x)^{-1/3}}$

Sol. $(2-3x)^{-1/3} = \left[2\left(1-\frac{3}{2}x\right)\right]^{-1/3}$

$= 2^{-1/3}\left(1-\frac{3}{2}x\right)^{-1/3}$

General term of $(1 - x)^{-p/q}$

$$T_{r+1} = \frac{(p)(p+q)(p+2q)....(p)+(r-1)q}{(r)!}\left(\frac{x}{q}\right)^r$$

Here p = 1, q = 3, $X = \frac{3}{2}x \Rightarrow \frac{x}{q} = \frac{\frac{3}{2}x}{3} = \frac{x}{2}$

$\therefore$ General term of $(2-3x)^{-1/3}$ is

$$T_{r+1} = 2^{-1/3}\left[\frac{(1)(1+3)(1+2\cdot 3)...[1+(r-1)3]}{r!}\left(\frac{x}{2}\right)^r\right]$$

$$= \frac{1}{\sqrt[3]{2}}\left[\frac{(1)(4)(7)...(3r-2)}{r!}\right]\left(\frac{x}{2}\right)^2$$

25. Find the coefficient of x^{12} in $\frac{1+3x}{(1-4x)^4}$.

Sol. $\frac{1+3x}{(1-4x)^4} = (1+3x)(1-4x)^{-4}$

$$= (1+3x)\left[\sum_{r=0}^{\infty} {}^{(n+r-1)}C_r \cdot X^r\right]$$

Here X = 4x, n = 4

$$= (1+3x)\left[\sum_{r=0}^{\infty} {}^{(4+r-1)}C_r \cdot (4x)^r\right]$$

$$= (1+3x)\left[\sum_{r=0}^{\infty} {}^{(r+3)}C_r \cdot (4)^r (x)^r\right]$$

$\therefore$ The coefficient of x^{12} in $\frac{1+3x}{(1-4x)^4}$ is

$$= (1)\cdot {}^{(12+3)}C_{12} \cdot 4^{12} + 3\cdot {}^{(11+3)}C_3 \cdot 4^{11}$$

$$= {}^{15}C_3 \cdot 4^{12} + 3\cdot {}^{14}C_3 \cdot 4^{11}$$

$$= 455\times 4^{12} + (1092)4^{11} = 728\times 4^{12}$$

26. Find coefficient of x^6 in the expansion of $(1-3x)^{-2/5}$.

Sol. General term of $(1-x)^{-p/q}$ is

$$T_{r+1} = \frac{(p)(p+q)(p+2q)+...+[p+(r-1)q]}{(r)!}\left(\frac{x}{q}\right)^r$$

Here X = 3x, p = 2, q = 5, r = 6, $\frac{X}{q} = \frac{3x}{5}$

$$T_{6+1} = \frac{(2)(2+5)(2+2.5)...[2+(6-1)5]}{6!}\left(\frac{3x}{5}\right)^6$$

$$T_7 = \frac{(2)(7)(12)...(27)}{6!}\left(\frac{3x}{5}\right)^6$$

$\therefore$ Coefficient of x^6 in $(1-3x)^{-2/5}$ is

$$= \frac{(2)(7)(12)...(27)}{6!}\left(\frac{3}{5}\right)^6$$

27. Find the sum of the infinite series $1+\frac{2}{3}\cdot\frac{1}{2}+\frac{2\cdot5}{3\cdot6}\left(\frac{1}{2}\right)^2+\frac{2\cdot5\cdot8}{3\cdot6\cdot9}\left(\frac{1}{2}\right)^3+...\infty$

Sol. Let $S=1+\frac{2}{3}\cdot\frac{1}{2}+\frac{2\cdot5}{3\cdot6}\left(\frac{1}{2}\right)^2+\frac{2\cdot5\cdot8}{3\cdot6\cdot9}\left(\frac{1}{2}\right)^3+...$

$=1+\frac{2}{1}\cdot\frac{1}{6}+\frac{2\cdot5}{1\cdot2}\left(\frac{1}{6}\right)^2+\frac{2\cdot5\cdot8}{1\cdot2\cdot3}\left(\frac{1}{6}\right)^3+...$

$\because 1+\frac{p}{1!}\left(\frac{x}{q}\right)+\frac{p(p+q)}{1\cdot2}\left(\frac{x}{q}\right)^2+...=(1-x)^{-p/q}$

Here $p=2,\ q=3, \frac{x}{q}=\frac{1}{6}\Rightarrow x=\frac{3}{6}=\frac{1}{2}$

$=(1-x)^{-p/q}=\left(1-\frac{1}{2}\right)^{-2/3}=2^{2/3}=\sqrt[3]{4}$

28. Find the sum of the series

$\frac{3\cdot5}{5\cdot10}+\frac{3\cdot5\cdot7}{5\cdot10\cdot15}+\frac{3\cdot5\cdot7\cdot9}{5\cdot10\cdot15\cdot20}+...\infty$

Sol. Let $S=\frac{3\cdot5}{5\cdot10}+\frac{3\cdot5\cdot7}{5\cdot10\cdot15}+\frac{3\cdot5\cdot7\cdot9}{5\cdot10\cdot15\cdot20}+...$

$\frac{3\cdot5}{1\cdot2}\left(\frac{1}{5}\right)^2+\frac{3\cdot5\cdot7}{1\cdot2\cdot3}\left(\frac{1}{5}\right)^3+\frac{3\cdot5\cdot7\cdot9}{1\cdot2\cdot3\cdot4}\left(\frac{1}{5}\right)^4+...$

Add $1+3\cdot\frac{1}{5}$ on both sides

$1+\frac{3}{5}+S=1+\frac{3}{1}\left(\frac{1}{5}\right)+\frac{3\cdot5}{1\cdot2}\left(\frac{1}{5}\right)^2+.....$

$=1+\frac{p}{1}\left(\frac{x}{q}\right)+\frac{p(p+q)}{1\cdot2}\left(\frac{x}{q}\right)^2+.....$

Here $p=3, q=2, \frac{x}{q}=\frac{1}{5}\Rightarrow x=\frac{2}{5}$

$=(1-x)^{-p/q}$

$=\left(1-\frac{2}{5}\right)^{-3/2}=\left(\frac{5}{3}\right)^{3/2}=\frac{5\sqrt{5}}{3\sqrt{3}}$

$\Rightarrow\frac{8}{5}+S=\frac{5\sqrt{3}}{3\sqrt{3}}\Rightarrow S=\frac{5\sqrt{3}}{3\sqrt{3}}-\frac{8}{5}$

29. If x = $\frac{1}{5}+\frac{1\cdot3}{5\cdot10}+\frac{1\cdot3\cdot5}{5\cdot10\cdot15}+...\infty$, find $3x^2 + 6x$.

Sol. Given that

$$x=\frac{1}{5}+\frac{1\cdot3}{5\cdot10}+\frac{1\cdot3\cdot5}{5\cdot10\cdot15}+.....$$

$$=\frac{1}{5}+\frac{1\cdot3}{1\cdot2}\left(\frac{1}{5}\right)^2+\frac{1\cdot3\cdot5}{1\cdot2\cdot3}\left(\frac{1}{5}\right)^3+.....$$

$$\Rightarrow 1+x=1+1\cdot\frac{1}{5}+\frac{1\cdot3}{1\cdot2}\left(\frac{1}{5}\right)^2+\frac{1\cdot3\cdot5}{1\cdot2\cdot3}\left(\frac{1}{5}\right)^3+...$$

$$=1+\frac{p}{1!}\frac{1}{5}+\frac{p(p+q)}{2!}\left(\frac{1}{5}\right)^2$$

$$+\frac{p(p+q)(p+2q)}{3!}\left(\frac{1}{5}\right)^3=(1-x)^{-p/q}$$

Here $p=1, q=2, \frac{x}{q}=\frac{1}{5}\Rightarrow x=\frac{2}{5}$

$$=\left(1-\frac{2}{5}\right)^{-1/2}=\left(\frac{3}{5}\right)^{-1/2}=\sqrt{\frac{5}{3}}$$

$$\Rightarrow 1+x=\sqrt{\frac{5}{3}}\Rightarrow 3(1+x)^2=5$$

$$\Rightarrow 3x^2+6x+3=5\Rightarrow 3x^2+6x=2$$

30. Find an approximate value of $\sqrt[6]{63}$ correct to 4 decimal places.

Sol. $\sqrt[6]{63}=(63)^{1/6}=(64-1)^{1/6}$

$$=(64)^{1/6}\left(1-\frac{1}{64}\right)^{1/6}$$

$$=2\left[1-(0.5)^6\right]^{1/6}$$

$$=2\left[i-\frac{\left(\frac{1}{6}\right)(0.5)^6}{1!}+\frac{\left(\frac{1}{6}\right)\left(\frac{1}{6}-1\right)}{2!}(0.5)^{12}+...\right]$$

$$=2[1-0.0026041]=2[0.9973959]$$

$$=1.9947918=1.9948 \text{ (correct to 4 decimals)}$$

31. If |x| is so small that x^2 and higher powers of x may be neglected, then find an approximate values of $\frac{\left(1+\frac{3x}{2}\right)^{-4}(8+9x)^{1/3}}{(1+2x)^2}$.

Sol. $\frac{\left(1+\frac{3x}{2}\right)^{-4}(8+9x)^{1/3}}{(1+2x)^2}$

$=\left(1+\frac{3x}{2}\right)^{-4}\left[8\left(1+\frac{9}{8}x\right)\right]^{1/3}(1+2x)^{-2}$

$=\left(1+\frac{3x}{2}\right)^{-4}\cdot 8^{1/3}\left(1+\frac{9}{8}x\right)^{1/3}(1+2x)^{-2}$

$=2\left[1-\frac{4}{1}\left(\frac{3x}{2}\right)\right]\left[1+\frac{1}{3}\left(\frac{9x}{8}\right)\right][1+(-2)(2x)]$

$\because x^2$ and higher powers of x are neglecting

$=2(1-6x)\left(1+\frac{3x}{8}\right)(1-4x)$

$=2\left(1-6x+\frac{3x}{8}\right)(1-4x)$

($\because x^2$ and higher powers of x are neglecting)

$=2\left(1-\frac{45}{8}x\right)(1-4x)=2\left(1-4x-\frac{45}{8}x\right)$

$\because x^2$ and higher powers of x are neglecting

$=2\left(1-\frac{77}{8}x\right)$

$\therefore \frac{\left(1+\frac{3x}{2}\right)^{-4}(8+9x)^{1/3}}{(1+2x)^2}=2\left(1-\frac{77}{8}x\right)$

32. If |x| is so small that x^4 and higher powers of x may be neglected, then find the approximate value of $\sqrt[4]{x^2+81}-\sqrt[4]{x^2+16}$.

Sol. $\sqrt[4]{x^2+81}-\sqrt[4]{x^2+16}$

$=(81+x^2)^{1/4}-(16+x^2)^{1/4}$

$$= (81+x^2)^{1/4} - (16+x^2)^{1/4}$$

$$= \left[81\left(1+\frac{x^2}{81}\right)\right]^{1/4} - \left[16\left(1+\frac{x^2}{16}\right)\right]^{1/4}$$

$$= 3\left(1+\frac{x^2}{81}\right)^{1/5} - 2\left(1+\frac{x^2}{16}\right)^{1/4}$$

$$= 3\left(1+\frac{1}{4}\cdot\frac{x^2}{81}\right) - 2\left(1+\frac{1}{4}\cdot\frac{x^2}{16}\right)$$

$$= 3+\frac{3}{4}\cdot\frac{x^2}{81} - 2 - \frac{2}{4}\frac{x^2}{16} = 1+\left(\frac{1}{108}-\frac{1}{32}\right)x^2$$

$$= 1-\frac{19}{864}x^2 \text{ (after neglecting } x^4 \text{ and higher powers of x)}$$

$$\therefore \sqrt[4]{x^2+81} - \sqrt[4]{x^2+16} = 1-\frac{19}{864}x^2$$

33. Suppose that x and y are positive and x is very small when compared to y. Then find the approximate value of $\left(\frac{y}{y+x}\right)^{3/4} - \left(\frac{y}{y+x}\right)^{4/5}$.

Sol. $\left(\frac{y}{y+x}\right)^{3/4} - \left(\frac{y}{y+x}\right)^{4/5}$

$$= \left(\frac{y}{y\left(1+\frac{x}{y}\right)}\right)^{3/4} - \left(\frac{y}{y\left(1+\frac{x}{y}\right)}\right)^{4/5}$$

$$= \left(1+\frac{x}{y}\right)^{-3/4} - \left(1+\frac{x}{y}\right)^{-4/5}$$

$$= \left\{1+\left(\frac{-3}{4}\right)\left(\frac{x}{y}\right)+\frac{\left(-\frac{3}{4}\right)\left(\frac{-3}{4}-1\right)}{1\cdot 2}\left(\frac{x}{y}\right)^2+\ldots\right\}$$

$$-\left\{1+\left(\frac{-4}{5}\right)\left(\frac{x}{y}\right)+\frac{\left(\frac{-4}{5}\right)\left(\frac{-4}{5}-1\right)}{1\cdot 2}\left(\frac{x}{y}\right)^2+\ldots\right\}$$

(by neglecting $(x/y)^3$ and higher powers of x/y

$$= \left[1-\frac{3}{4}\left(\frac{x}{y}\right)-\frac{21}{32}\left(\frac{x}{y}\right)^2\right] - \left[1-\frac{4}{5}\left(\frac{x}{y}\right)+\frac{18}{25}\left(\frac{x}{y}\right)^2\right]$$

$$= \left(\frac{4}{5}-\frac{3}{4}\right)\frac{x}{y} - \left(\frac{21}{32}+\frac{18}{25}\right)\left(\frac{x}{y}\right)^2$$

$$= \frac{1}{20}\left(\frac{x}{y}\right) - \frac{1101}{800}\left(\frac{x}{y}\right)^2$$

34. Expand $5\sqrt{5}$ in increasing power of $\frac{4}{5}$.

Sol. $5\sqrt{5} = 5^{3/2} = \left(\frac{1}{5}\right)^{-3/2}$

$$= \left(1-\frac{4}{5}\right)^{-3/2}$$

$$= 1+\frac{\left(\frac{3}{2}\right)}{1!}\left(\frac{4}{5}\right)+\frac{\frac{3}{2}\cdot\frac{5}{2}}{2!}\left(\frac{4}{5}\right)^2+\ldots+\frac{\frac{3}{2}\cdot\frac{5}{2}\ldots\left(\frac{3}{2}+r-1\right)}{r!}\left(\frac{4}{5}\right)^r+\ldots\infty$$

$$= 1+\frac{3}{1!2}\frac{4}{5}+\frac{3\cdot5}{2!2^2}\left(\frac{4}{5}\right)^2+\ldots+\frac{3\cdot5\ldots(2r-1)}{r!2^r}\left(\frac{4}{5}\right)^r+\ldots$$

35. Find the sum of the infinitive terms

$$\frac{5}{6\cdot12}+\frac{5\cdot8}{6\cdot12\cdot18}+\frac{5\cdot8\cdot11}{6\cdot12\cdot18\cdot24}+\ldots\infty$$

Sol. Let $S = \frac{5}{6\cdot12}+\frac{5\cdot8}{6\cdot12\cdot18}+\frac{5\cdot8\cdot11}{6\cdot12\cdot18\cdot24}+\ldots$

$$\Rightarrow 2S = \frac{2\cdot5}{1\cdot2}\left(\frac{1}{6}\right)^2+\frac{2\cdot5\cdot8}{1\cdot2\cdot3}\left(\frac{1}{6}\right)^3+\frac{2\cdot5\cdot8\cdot11}{1\cdot2\cdot3\cdot4}\left(\frac{1}{6}\right)^4+\ldots\ldots$$

$$\Rightarrow 1+\frac{2}{1}\left(\frac{1}{6}\right)+2S = 1+\frac{2}{1}\left(\frac{1}{6}\right)+\frac{2\cdot5}{1\cdot2}\left(\frac{1}{6}\right)^2+\frac{2\cdot5\cdot8}{1\cdot2\cdot3}\left(\frac{1}{6}\right)^3+\ldots\ldots$$

$$\Rightarrow \frac{4}{3}+2S = 1+\frac{2}{1}\left(\frac{1}{6}\right)+\frac{2\cdot5}{1\cdot2}\left(\frac{1}{6}\right)^2+\frac{2\cdot5\cdot8}{1\cdot2\cdot3}\left(\frac{1}{6}\right)^3+\ldots\ldots$$

Comparing $\frac{4}{3}+2S$ with $(1-x)^{-p/q}$

$$= 1+\frac{p}{1}\left(\frac{x}{q}\right)+\frac{p(p+q)}{1\cdot2}\left(\frac{x}{q}\right)^2+\ldots$$

Here p = 2, q = 3, $\frac{x}{q} = \frac{1}{6} \Rightarrow x = \frac{q}{6} = \frac{3}{6} = \frac{1}{2}$

$$\therefore \frac{4}{3}+2S = (1-x)^{-p/q} = \left(1-\frac{1}{2}\right)^{-2/3}$$

$$= \left(\frac{1}{2}\right)^{-2/3} = (2)^{2/3} = \sqrt[3]{4}$$

$$\therefore 2S = \sqrt[3]{4}-\frac{4}{3} \Rightarrow S = \frac{\sqrt[3]{4}}{2}-\frac{2}{3} = \frac{1}{\sqrt[3]{2}}-\frac{2}{3}$$

$$\therefore \frac{5}{6\cdot12}+\frac{5\cdot8}{6\cdot12\cdot18}+\frac{5\cdot8\cdot11}{6\cdot12\cdot18\cdot24}+\ldots = \frac{1}{\sqrt[3]{2}}-\frac{2}{3}$$