BINOMIAL THEOREM - 2

EXERCISE – 5(b)

1. Find the range of x for which the binomial expansions of the following are valid.

(i) $(2 + 3x)^{-2/3}$ (ii) $(5 + x)^{3/2}$

(iii) $(7 + 3x)^{-5}$ (iv) $\left(4-\frac{x}{3}\right)^{-1/2}$

Sol. (i) $(2 + 3x)^{-2/3}$ =

$$\left[2\left(1+\frac{3}{2}x\right)\right]^{-2/3} = 2^{-2/3}\left(1+\frac{3}{2}x\right)^{-2/3}$$

$\therefore$ The binomial expansion of $(2 + 3x)^{-2/3}$ is valid when $\left|\frac{3}{2}x\right|<1$.

i.e. $|x| < \frac{2}{3}$

i.e. $x\in\left(-\frac{2}{3},\frac{2}{3}\right)$

ii) $(5+x)^{3/2} = \left[5\left(1+\frac{x}{5}\right)\right]^{3/2} = 5^{3/2}\left(1+\frac{x}{5}\right)^{3/2}$

$\therefore$ The binomial expansion of $(5 + x)^{3/2}$ is valid when $\left|\frac{x}{5}\right|<1$.

i.e. $|x| < 5$

i.e. $x \in (–5, 5)$

iii) $(7+3x)^{-5} = 7\left[\left(1+\frac{3}{7}x\right)\right]^{-5} = 7^{-5}\left(1+\frac{3}{7}x\right)^{-5}$

$(7 + 3x)^{-5}$ is valid when $\left|\frac{3x}{7}\right|<1$

$\Rightarrow |x| < \frac{7}{3} \Rightarrow x\in\left(\frac{-7}{3},\frac{7}{3}\right)$

iv) $\left(4-\frac{x}{3}\right)^{-1/2} = \left[4\left(1-\frac{x}{12}\right)\right]^{-1/2}$

$\left(4-\frac{x}{3}\right)^{-1/2}$ is valid when $\left|\frac{-x}{12}\right|<1$

$\Rightarrow |x| < 12$

$\Rightarrow x \in (–12, 12)$

2. Find the (i) 6th term of $\left(1+\frac{x}{2}\right)^{-5}$.

Sol. T_{r+1} in $(1 + x)^{-n}$

$$= (-1)^r \frac{(n)(n+1)(n+2)...(n+r-1)}{1\cdot 2\cdot 3\cdot ...r}\cdot x^r$$

Put r = 5, n = 5, x by x/2

$$T_6 = (-1)^5 \frac{(5)(5+1)(5+2)(5+3)(5+4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}\cdot\left(\frac{x}{2}\right)^5$$

$$= \frac{-5\cdot 6\cdot 7\cdot 8\cdot 9}{1\cdot 2\cdot 3\cdot 4\cdot 5}\cdot\left(\frac{1}{2}\right)^5\cdot x^5 = \frac{-63}{16}\cdot x^5$$

ii) 7th term of $\left(1-\frac{x^2}{3}\right)^{-4}$

Sol. T_{r+1} in $(1 - x)^{-n}$ =

$$= \frac{(n)(n+1)(n+2)...(n+r-1)}{1\cdot 2\cdot 3\cdot ...r}\cdot x^r$$

Put r = 6, n = 4, x by $\frac{x^2}{3}$

Then 7th term in $\left(1-\frac{x^2}{3}\right)^{-4}$ is

$$= \frac{(4)(4+1)(4+2)(4+3)(4+4)(4+5)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\left(\frac{-x^2}{3}\right)^6$$

$$= \frac{4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\cdot\frac{x^{12}}{3^6} = \frac{28}{243}\cdot x^{12}$$

iii) 10th term of $(3 - 4x)^{-2/3}$.

Sol. $(3-4x)^{-2/3} = \left[3\left(1-\frac{4}{3}x\right)\right]^{-2/3}$

$$= (3)^{-2/3}\left(1-\frac{4}{3}x\right)^{-2/3} \quad ...(1)$$

First find 10th term of $\left(1-\frac{4}{3}x\right)^{-2/3}$

The general term of $(1 - x)^{-p/q}$ is

$$T_{r+1} = \frac{(p)(p+q)(p+2q)+...+[p+(r-1)q]}{(r)!}\left(\frac{x}{q}\right)^r$$

Here p = 2, q = 3, r = 9

$$\frac{x}{q} = \left(\frac{(4/3)x}{3}\right)\left(\frac{4}{9}x\right)$$

$$T_{10} = \frac{(2)(2+3)(2+6)...[2+(9-1)3]}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9}\left(\frac{4}{9}x\right)^9$$

$$= \frac{2\cdot5\cdot8\cdot...(26)}{9!}\left(\frac{4x}{9}\right)^9$$

10^{th} term in $(3-4x)^{-2/3} = 3^{-2/3}\left[\frac{2\cdot5\cdot8\cdot...(26)}{9!}\left(\frac{4x}{9}\right)^9\right]$

iv) 5^{th} term of $\left(7+\frac{8y}{3}\right)^{7/4}$

Sol. $\left(7+\frac{8y}{3}\right)^{7/4} = \left[7\left(1+\frac{8y}{21}\right)\right]^{7/4}$

General term of $(1+x)^{p/q}$

$$T_{r+1} = \frac{(p)(p-q)(p-2q)+...+[p-(r-1)q]}{(r)!}\left(\frac{x}{q}\right)^r$$

Here p = 7, q = 4, r = 4, $\frac{x}{q} = \frac{(8y/21)}{4} = \frac{2y}{21}$

$\therefore$ T_5 of $\left(1+\frac{8y}{21}\right)^{7/4}$ is

$$= \frac{(7)(7-4)(7-2\times4)(7-3\times4)}{1\times2\times3\times4}\left(\frac{2y}{21}\right)^4$$

$$= \frac{7(3)(-1)(-5)}{1\times2\times3\times4}\cdot\frac{2^4y^4}{(21)^4} = 70\left(\frac{y}{21}\right)^4$$

$\therefore$ 5^{th} term of $\left(7+\frac{8y}{3}\right)^{7/4}$ is $7^{7/4}(70)\left(\frac{y}{21}\right)^4$

$\therefore$ T_5 in $\left(7+\frac{8y}{3}\right)^{7/4} = 7^{7/4}(70)\left(\frac{y}{21}\right)^4$

3. Write down the first 3 terms in the expansion of (i)$(3+5x)^{-7/3}$, (ii) $(1+4x)^{-4}$, (iii) $(8-5x)^{2/3}$, (iv) $(2-7x)^{-3/4}$.

Sol. i)$(3+5x)^{-7/3} = \left[3\left(1+\frac{5}{3}x\right)\right]^{-7/3}$

$$= (3)^{-7/3}\left(1+\frac{5}{3}x\right)^{-7/3}$$

Now we have

$$(1+x)^{-p/q} = 1-\frac{p}{11}\left(\frac{x}{q}\right)+\frac{p(p+q)}{1\cdot2}\left(\frac{x}{q}\right)^2+...$$

Here p = 7, q = 3, $\frac{x}{q} = \frac{(5/3)x}{3} = \frac{5}{9}x$

$\therefore (3 + 5x)^{-7/3} =$

$$(3)^{-7/3}\left[1 - \frac{7}{1!}\left(\frac{5}{9}x\right) + \frac{(7)(10)}{1\cdot 2}\left(\frac{5}{9}x\right)^2 + \ldots\right]$$

$$= 3^{-7/3}\left[1 - \frac{35}{9}x + \frac{875}{81}x^2 - \ldots\ldots\right]$$

$\therefore$ The first 3 terms of $(3 + 5x)^{-7/3}$ are

$$3^{-7/3}, \frac{-3^{7/3}\cdot 35x}{9}, 3^{-7/3}\frac{875}{81}x^2$$

ii) $(1 + 4x)^{-4}$

Sol. We have $(1 + x)^{-n} = 1 - nx + \frac{n(n+1)}{1\cdot 2}x^2 + \ldots$

Here n = 4, x = 4x

$$\therefore (1+4x)^{-4} = 1 - 4(4x) + \frac{4\cdot 5}{1\cdot 2}\cdot(4x)^2$$

$$= 1 - 16x + 160x^2 - \ldots\ldots$$

$\therefore$ The first 3 terms of $(1 + 4x)^{-4}$ are

$1, -16x, 160x^2$.

iii) $(8 - 5x)^{2/3}$

Sol. $$\left[8\left(1 - \frac{5}{8}x\right)^{2/3}\right] = (2^3)^{2/3}\left[1 - \frac{5}{8}x\right]^{2/3}$$

$$= 4\left[\left(1 - \frac{5x}{8}\right)^{2/3}\right]$$

We know that

$$(1 - X)^{p/q} = 1 - p\left(\frac{X}{q}\right) + \frac{(p)(p-q)}{1\cdot 2}\left(\frac{X}{q}\right)^2 - \ldots$$

Here $X = \frac{5x}{8}, p = 2, q = 3, \frac{X}{q} = \frac{(5x/8)}{3} = \frac{5x}{24}$

$\therefore (8 - 5x)^{2/3} =$

$$4\left[1 - 2\left(\frac{5x}{24}\right) + \frac{(2)(2-3)}{1\cdot 2}\left(\frac{5x}{24}\right)^2 - \ldots\right]$$

$$= 4\left[1 - \frac{5x}{12} - \left(\frac{5x}{24}\right)^2 + \ldots\right]$$

$\therefore$ The first 3 terms of $(8 - 5x)^{2/3}$ are :

$$4, \frac{-5x}{3}, \frac{-25}{144}x^2$$

iv) $(2 - 7x)^{-3/4}$

Sol. Write $(2-7x)^{-3/4} = \left[2\left(1-\frac{7}{2}x\right)\right]^{-3/4}$

$$= (2)^{-3/4}\left(1-\frac{7}{2}x\right)^{-3/4}$$

Now $(1-X)^{-p/q} = 1 + p\left(\frac{x}{q}\right) + \frac{(p)(p+q)}{1\cdot 2}\left(\frac{x}{q}\right)^2 + \ldots$

Here p = 3, q = 4, X = $\frac{7}{2}x$, $\frac{X}{q} = \frac{(7/2)x}{4} = \frac{7}{8}x$

$\therefore\ 2^{-3/4}\left(1-\frac{7}{2}x\right)^{-3/4}$

$$= 2^{-3/4}\left[1 + 3\left(\frac{7}{8}x\right) + \frac{3(3+4)}{1\cdot 2}\left(\frac{7}{8}x^2\right) + \ldots\right]$$

$$= 2^{-3/4}\left[1 + \frac{21}{8}x + \frac{1029}{128}x^2 + \ldots\ldots\right]$$

$\therefore$ First 3 terms of $(2 - 7x)^{-3/4}$ are :

$$2^{-3/4}, 2^{-3/4}\left(\frac{21}{8}x\right), 2^{-3/4}\left(\frac{1029}{128}\right)x^2$$

4. Find the general term $(r + 1)^{th}$ term in the expansion of

(i) $(4 + 5x)^{-3/2}$ **(ii)** $\left(1-\frac{5x}{3}\right)^{-3}$

(iii) $\left(1+\frac{4x}{5}\right)^{5/2}$ **(iv)** $\left(3-\frac{5x}{4}\right)^{-1/2}$

i) $(4 + 5x)^{-3/2}$

Sol. Write $(4 + 5x)^{-3/2} = \left[4\left(1+\frac{5}{4}x\right)\right]^{-3/2}$

$$= (2^2)^{-3/2}\left[\left(1+\frac{5}{4}x\right)^{-3/2}\right] = \frac{1}{8}\left[\left(1+\frac{5}{4}x\right)^{-3/2}\right]$$

General term of $(1 + x)^{-p/q}$ is :

$T_{r+1} = (-1)^r$

$$\frac{(p)(p+q)(p+2q)+\ldots+[p+(r-1)q]}{(r)!}\left(\frac{x}{q}\right)^r$$

Here p = 3, q = 2, $\frac{X}{p} = \frac{\left(\frac{5x}{4}\right)}{2} = \frac{5x}{8}$

$\therefore$ T_{r+1} in $(4 + 5x)^{-3/2}$ is

$$(-1)^r \frac{1}{8}\left[\frac{(3)(3+2)(3+2\times2)...[3+(r-1)2]}{r!}\right]\left(\frac{5x}{8}\right)^r$$

$$= (-1)^r \frac{3\cdot5\cdot7.....(2r+1)}{r!}\frac{(5x)^r}{(8)^{r+1}}$$

ii) $\left(1-\frac{5x}{3}\right)^{-3}$

Sol. General term of $(1 - x)^{-n}$ is

$$T_{r+1} = \frac{(n)(n+1)(n+2)...(n+r-1)}{1\cdot2\cdot3\cdot...r}\cdot X^r$$

Here $X = \frac{5x}{3}, n = 3$

$\therefore$ General term of $\left(1-\frac{5x}{3}\right)^{-3}$ is

$$T_{r+1} = \frac{(3)(4)(5).....(3+r-1)}{r!}\left(\frac{5x}{3}\right)^r$$

$$= \frac{3\cdot4\cdot5\cdot.....(r+2)}{(r)!}\left(\frac{5x}{3}\right)^r$$

$$= \frac{(r+1)(r+2)}{1\cdot2}\left(\frac{5x}{3}\right)^r = {}^{r+2}C_2\left(\frac{5x}{3}\right)^r$$

iii) $\left(1+\frac{4x}{5}\right)^{5/2}$

Sol. General term of $(1 + X)^{p/q}$ is

$$T_{r+1} = \frac{(p)(p-q)(p-2q)+...+[p-(r-1)q]}{(r)!}\left(\frac{x}{q}\right)^r$$

Here p = 5, q = 2, X = $\frac{4x}{5}$, $\frac{X}{q} = \frac{(4x/5)}{2} = \frac{2x}{5}$

T_{r+1} of $\left(1+\frac{4x}{5}\right)^{5/2}$ is :

$$= \frac{5(5-2)(5-2\times2)(5-2\times3)...[5-(r-1)2]}{r!}\left(\frac{2x}{5}\right)^r$$

$$= \frac{5(+3)(1)(-1)(-3)...[5-2r+2]}{r!}\left(\frac{2x}{5}\right)^r$$

$$= (-1)^{r-3}\frac{5\cdot3\cdot1\cdot1\cdot3\cdot5.....(2r-7)}{r!}\left(\frac{2x}{5}\right)^r$$

iv) $\left(3-\frac{5x}{4}\right)^{-1/2}$

Sol. Write $\left(3-\frac{5x}{4}\right)^{-1/2} = \left[3\left(1-\frac{5x}{12}\right)^{-1/2}\right]$

$= 3^{-1/2}\left[\left(1-\frac{5x}{12}\right)^{-1/2}\right]$

General term of $(1 - X)^{-p/q}$ is

$$T_{r+1} = \frac{(p)(p-q)(p-2q)+...+[p-(r-1)q]}{(r)!}\left(\frac{x}{q}\right)^r$$

Here p = 1, q = 2, $X = \frac{5x}{12}, \frac{x}{q} = \frac{(5x/12)}{2} = \frac{5x}{24}$

$\therefore T_{r+1}$ of $\left(3-\frac{5x}{4}\right)^{-1/2}$ is

$$= 3^{-1/2}\left[\frac{(1)(1+2)(1+2\times 2)...[1+(r-1)2]}{r!}\right]\left(\frac{5x}{24}\right)^r$$

$$= \frac{1}{\sqrt{3}}\left[\frac{1\cdot 3\cdot 5.....(2r-1)}{r!}\right]\left(\frac{5x}{24}\right)^r$$

II.

1. Find the coefficient of x^{10} in the expansion of $\frac{1+2x}{(1-2x)^2}$.

Sol. $\frac{1+2x}{(1-2x)^2} = (1+2x)(1-2x)^{-2}$

$= (1+2x)[1+2(2x)+3(2x)^2+4(2x)^3+5(2x)^4 ss+6(2x)^5+7(2x)^6+8(2x)^7+9(2x)^8+10(2x)^9$

$+11(2x)^{10}+...+(r+1)(2x)^r+...]$

$\therefore$ The coefficient of x^{10} in $\frac{1+2x}{(1-2x)^2}$ is

$= (11)(2)^{10}+10(2)(2^9)$ s

$= 2^{10}(11+10) = 2\times 1^{10}$

2. Find the coefficient of x^4 in the expansion of $(1 - 4x)^{-3/5}$.

Sol. General term in $(1 - x)^{-p/q}$ is

$$T_{r+1} = \frac{(p)(p-q)(p-2q)+...+[p-(r-1)q]}{(r)!}\left(\frac{x}{q}\right)^r$$

Here p = 3, q = 5, $\frac{X}{q} = \left(\frac{4x}{5}\right)$

Put r = 4

$$T_{4+1} = \frac{(3)(3+5)(3+2\times5)(3+3\times5)}{1\times2\times3\times4}\left(\frac{4x}{5}\right)^4$$

$\therefore$ Coefficient of x^4 in $(1-4x)^{-3/5}$ is

$$\frac{(3)(8)(13)(18)}{1\times2\times3\times4}\left(\frac{4}{5}\right)^4 = \frac{234\times256}{625} = \frac{59904}{625}$$

3. Find the sum of the infinite series

i) $1+\frac{1}{3}+\frac{1\cdot3}{3\cdot6}+\frac{1\cdot3\cdot5}{3\cdot6\cdot9}+...$

Sol. The given series can be written as

$$S = 1+\frac{1}{1}\cdot\frac{1}{3}+\frac{1\cdot3}{1\cdot2}\left(\frac{1}{3}\right)^2+\frac{1\cdot3\cdot5}{1\cdot2\cdot3}\left(\frac{1}{3}\right)^2+...$$

The series of the right is of the form

$$1+\frac{p}{1}\left(\frac{x}{q}\right)+\frac{p(p+q)}{1\cdot2}\left(\frac{x}{q}\right)^2+\frac{p(p+q)(p+2q)}{1\cdot2\cdot3}\left(\frac{x}{q}\right)^3+...$$

Here p = 1, q = 2, $\frac{x}{q}=\frac{1}{3}\Rightarrow x=\frac{2}{3}$

The sum of the given series

$$S = (1-x)^{-p/q} = \left(1-\frac{2}{3}\right)^{-1/2} = \left(\frac{1}{3}\right)^{-1/2} = \sqrt{3}$$

ii) $\frac{3}{4}+\frac{3\cdot5}{4\cdot8}+\frac{3\cdot5\cdot7}{4\cdot8\cdot12}+.....$

Sol. Let $S = \frac{3}{4}+\frac{3\cdot5}{4\cdot8}+\frac{3\cdot5\cdot7}{4\cdot8\cdot12}+.....$

$$=\frac{3}{1}\cdot\frac{1}{4}+\frac{3\cdot5}{1\cdot2}\left(\frac{1}{4}\right)^2+\frac{3\cdot5\cdot7}{1\cdot2\cdot3}\left(\frac{1}{4}\right)^3+.....$$

$$\Rightarrow 1+S = 1+\frac{3}{1}\cdot\frac{1}{4}+\frac{3\cdot5}{1\cdot2}\left(\frac{1}{4}\right)^2+.....$$

Comparing (1 + S) with

$$(1-x)^{-p/q} = 1+\frac{p}{1}\left(\frac{x}{q}\right)+\frac{p(p+q)}{1\cdot2}\left(\frac{x}{q}\right)^2+...$$

Here p = 3, q = 2, $\frac{x}{p}=\frac{1}{4}\Rightarrow x=\frac{1}{2}$

$$\therefore\ 1+S = (1-x)^{-p/q} = \left(1-\frac{1}{2}\right)^{-3/2}$$

$$= \left(\frac{1}{2}\right)^{-3/2} = 2^{3/2} = \sqrt{8}$$

$$\therefore S = 2\sqrt{2}-1$$

iii) $1-\frac{4}{5}+\frac{4\cdot7}{5\cdot10}-\frac{4\cdot7\cdot10}{5\cdot10\cdot15}+.....$

Sol. Let S = $1-\frac{4}{5}+\frac{4\cdot7}{5\cdot10}-\frac{4\cdot7\cdot10}{5\cdot10\cdot15}+.....$

$=1+\frac{4}{1}\left(-\frac{1}{5}\right)+\frac{4\cdot7}{1\cdot2}\left(-\frac{1}{5}\right)^2+\frac{4\cdot7\cdot10}{1\cdot2\cdot3}\left(-\frac{1}{5}\right)^3+...$

Comparing S with

$(1-x)^{-p/q}=1+\frac{p}{1}\left(\frac{x}{q}\right)+\frac{p(p+q)}{1\cdot2}\left(\frac{x}{q}\right)^2+...$

Here p = 4, q = 3, $\frac{x}{q}=-\frac{1}{5}\Rightarrow x=\frac{-3}{5}$

$\therefore S=(1-x)^{-p/q}=\left(1+\frac{3}{5}\right)^{-4/3}=\left(\frac{8}{5}\right)^{-4/3}$

$=\left(\frac{5}{8}\right)^{4/3}=\frac{5^{4/3}}{8^{4/3}}=\frac{\sqrt[3]{5^4}}{2^4}=\frac{\sqrt[3]{625}}{16}$

$\therefore 1-\frac{4}{5}+\frac{4\cdot7}{5\cdot10}-\frac{4\cdot7\cdot10}{5\cdot10\cdot15}+....=\frac{\sqrt[3]{625}}{16}=\frac{5^{4/3}}{16}$

iv) $\frac{3}{4\cdot8}-\frac{3\cdot5}{4\cdot8\cdot12}+\frac{3\cdot5\cdot7}{4\cdot8\cdot12\cdot16}-....$

Sol. Let S = $\frac{3}{4\cdot8}-\frac{3\cdot5}{4\cdot8\cdot12}+\frac{3\cdot5\cdot7}{4\cdot8\cdot12\cdot16}-....$

= $\frac{1\cdot3}{4\cdot8}-\frac{1\cdot3\cdot5}{4\cdot8\cdot12}+\frac{1\cdot3\cdot5\cdot7}{4\cdot8\cdot12\cdot16}-....$

Add $1-\frac{1}{4}$ on both sides,

$1-\frac{1}{4}+S=1-\frac{1}{4}+\frac{1\cdot3}{4\cdot8}-\frac{1\cdot3\cdot5}{4\cdot8\cdot12}+....$

$\Rightarrow\frac{3}{4}+S=1-\frac{1}{1}\cdot\frac{1}{4}+\frac{1\cdot3}{1\cdot2}\left(\frac{1}{4}\right)^2-\frac{1\cdot3\cdot5}{1\cdot2\cdot3}\left(\frac{1}{4}\right)^3+...$

$=1-\frac{p}{1}\frac{x}{q}+\frac{(p)(p+q)}{1\cdot2}\left(\frac{x}{q}\right)^2-\frac{(p)(p+q)(p+2q)}{1\cdot2\cdot3}\left(\frac{x}{q}\right)^3+...$

Here p = 1, q = 2, $\frac{x}{q}=\frac{1}{4}\Rightarrow x=\frac{1}{2}$

$=(1+x)^{-p/q}=\left(1+\frac{1}{2}\right)^{-1/2}=\left(\frac{3}{2}\right)^{-1/2}=\sqrt{\frac{2}{3}}$

$\therefore\ S=\sqrt{\frac{2}{3}}-\frac{3}{4}$

III.

1. If x = $\frac{1\cdot3}{3\cdot6}+\frac{1\cdot3\cdot5}{3\cdot6\cdot9}+\frac{1\cdot3\cdot5\cdot7}{3\cdot6\cdot9\cdot12}+...$ then prove that $9x^2 + 24x = 11$.

Sol. Given $x = \frac{1\cdot3}{3\cdot6}+\frac{1\cdot3\cdot5}{3\cdot6\cdot9}+\frac{1\cdot3\cdot5\cdot7}{3\cdot6\cdot9\cdot12}+...$

$$=\frac{1\cdot3}{1\cdot2}\left(\frac{1}{3}\right)^2+\frac{1\cdot3\cdot5}{1\cdot2\cdot3}\left(\frac{1}{3}\right)^2+...$$

$$=1+\frac{1}{1}\cdot\frac{1}{3}+\frac{1\cdot3}{1\cdot2}\left(\frac{1}{3}\right)^2+\frac{1\cdot3\cdot5}{1\cdot2\cdot3}\left(\frac{1}{3}\right)^2+...-\left[1+\frac{1}{3}\right]$$

Here p = 1, q = 2, $\frac{x}{q}=\frac{1}{3}\Rightarrow x=\frac{2}{3}$

$$=(1-x)^{-p/q}-\frac{4}{3}=\left(1-\frac{2}{3}\right)^{-1/2}-\frac{4}{3}$$

$$=\left(\frac{1}{3}\right)^{-1/2}-\frac{4}{3}=\sqrt{3}-\frac{4}{3}$$

$$\Rightarrow 3x+4=3\sqrt{3}$$

Squaring on both sides

$$(3x+4)^2=(3\sqrt{3})^2$$

$$\Rightarrow 9x^2+24x+16=27$$

$$\Rightarrow 9x^2+24x=11$$

2. (i) Find the coefficient of x^5 in $\frac{(1-3x)^2}{(3-x)^{3/2}}$.

Sol. $\frac{(1-3x)^2}{(3-x)^{3/2}}=\frac{(1-3x)^2}{\left[3\left(1-\frac{x}{3}\right)\right]^{3/2}}=\frac{(1-3x)^2}{3^{3/2}\left(1-\frac{x}{3}\right)^{3/2}}$

$$=\frac{1}{3^{3/2}}(1-3x)^2\left(1-\frac{x}{3}\right)^{-3/2}$$

$$=\frac{1}{\sqrt{27}}\left[(1-6x+9x^2)\left\{1+\frac{3}{2}\left(\frac{x}{3}\right)+\frac{\frac{3}{2}\cdot\frac{5}{2}}{1\cdot2}\left(\frac{x}{3}\right)^2+\frac{\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}}{1\cdot2\cdot3}\left(\frac{x}{3}\right)^3+\frac{\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdot\frac{9}{2}}{1\cdot2\cdot3\cdot4}\left(\frac{x}{3}\right)^4+.....\right\}\right]$$

$$=\frac{1}{\sqrt{27}}\left[(1-6x+9x^2)\left(1+\frac{x}{2}+\frac{5}{24}x^2+\frac{35x^3}{16\times27}+\frac{35}{8\times16\times9}x^4\frac{77}{8\times32\times27}x^5+...\right)\right]$$

$\therefore$ The coefficient of x^5 in $\frac{(1-3x)^2}{(3-x)^{3/2}}$ is

$$= \frac{1}{\sqrt{27}}\left[\frac{77}{8\times32\times27} - \frac{6(35)}{8\times16\times9} + \frac{9(35)}{16\times27}\right]$$

$$= \frac{1}{\sqrt{27}}\left[\frac{77-1260+5040}{8\times32\times27}\right] = \frac{3857}{\sqrt{27}\times8\times32\times27}$$

ii) Find the coefficient of x^8 in $\frac{(1+x)^2}{\left(1-\frac{2}{3}x\right)^3}$.

Sol. $\frac{(1+x)^2}{\left(1-\frac{2}{3}x\right)^3} = (1+x)^2\left(1-\frac{2}{3}x\right)^{-3}$

$$= (1+2x+x^2)\left[1+3\left(\frac{2x}{3}\right)+\frac{(3)(4)}{1\cdot2}\left(\frac{2x}{3}\right)^2+\frac{3\cdot4\cdot5}{1\cdot2\cdot3}\left(\frac{2x}{3}\right)^3+\frac{3\cdot4\cdot5\cdot6}{1\cdot2\cdot3\cdot4}\left(\frac{2x}{3}\right)^4+\frac{3\cdot4\cdot5\cdot6\cdot7}{1\cdot2\cdot3\cdot4\cdot5}\left(\frac{2x}{3}\right)^5\right.$$

$$\left.+\frac{3\cdot4\cdot5\cdot6\cdot7\cdot8}{1\cdot2\cdot3\cdot4\cdot5\cdot6}\left(\frac{2x}{3}\right)^6+\frac{3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7}\left(\frac{2x}{3}\right)^7+\frac{3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8}\left(\frac{2x}{3}\right)^8+...\right.$$

$\therefore$ Coefficient of x^8 in $\frac{(1+x)^2}{\left(1-\frac{2}{3}x\right)^3}$ is

$$= 45\left(\frac{2}{3}\right)^8 + 2\times36\left(\frac{2}{3}\right)^7 + 28\left(\frac{2}{3}\right)^6$$

$$= \left(\frac{2}{3}\right)^6\left[45\times\frac{4}{9} + 72\times\frac{2}{3} + 28\right]$$

$$= \left(\frac{2}{3}\right)^6(20+48+28) = \frac{96\times2^6}{3^6} = \frac{2048}{243}$$

iii) Find the coefficient of x^7 in $\frac{(2+3x)^3}{(1-3x)^4}$.

Sol. $\frac{(2+3x)^3}{(1-3x)^4} = (2+3x)^3(1-3x)^{-4}$

$$= (8+36x+54x^2+27x^3)$$

$$[1+{}^4C_1(3x)+{}^5C_2(3x)^2+{}^6C_3(3x)^3+{}^7C_4(3x)^4+{}^8C_5(3x)^5+{}^9C_6(3x)^6+...]$$

$\therefore$ Coefficient of x^7 in $\frac{(2+3x)^3}{(1-3x)^4}$ is

$$= 8\cdot\left({}^{10}C_7\cdot 3^7\right)+36\cdot\left({}^{9}C_6(3)^6\right)+54\left({}^{8}C_5(3^5)\right)+27\left({}^{7}C_4(3^4)\right)$$

$$= 8\left({}^{10}C_3 3^7\right)+36\left({}^{9}C_3 3^6\right)+54\left({}^{8}C_3 3^5\right)+27\left({}^{7}C_3 3^4\right)$$

3. Find the coefficient of x^3 in the expansion of $\frac{(1-5x)^3(1+3x^2)^{3/2}}{(3+4x)^{1/3}}$.

Sol. $\frac{(1-5x)^3(1+3x^2)^{3/2}}{(3+4x)^{1/3}} = \frac{(1-5x)^3(1+3x^2)^{3/2}}{\left[3\left(1+\frac{4}{3}\right)\right]^{1/3}}$

$$= \frac{1}{3^{1/3}}(1-5x)^3(1+3x^2)^{3/2}\left(1+\frac{4}{3}\right)^{-1/3}$$

$$= \frac{1}{3^{1/3}}[1-15x+75x^2-125x^3]$$

$$\left[1+\frac{3}{2}(3x^2)+\frac{\left(\frac{3}{2}\right)\left(\frac{3}{2}-1\right)}{1\cdot 2}(3x^2)^2+\ldots\right]$$

$$\left[1+\left(\frac{-1}{3}\right)\frac{4x}{3}+\frac{\left(\frac{-1}{3}\right)\left(\frac{-1}{3}-1\right)}{1\cdot 2}\left(\frac{4x}{3}\right)^2+\frac{\left(\frac{-1}{3}\right)\left(\frac{-1}{3}-1\right)\left(\frac{-1}{3}-2\right)}{1\cdot 2\cdot 3}\left(\frac{4x}{3}\right)^3+\ldots\right]$$

$$= \frac{1}{3^{1/3}}(1-15x+75x^2-125x^3)\left(1+\frac{9}{2}x^2+\ldots\right)$$

$$\left(1-\frac{4x}{9}+\frac{32}{81}x^2-\frac{896}{2187}x^3+\ldots\right)$$

$$= \frac{1}{3^{1/3}}\left[1-15x+75x^2-125x^3+\frac{9}{2}x^2-\frac{135}{2}x^3+\ldots\right]$$

$$\left[1-\frac{4x}{9}+\frac{32}{81}x^2-\frac{896}{2187}x^3+\ldots\right]$$

$$= \frac{1}{3^{1/3}}\left[1-15x+\frac{159}{2}x^2-\frac{385}{2}x^3+\ldots\right]$$

$$\left[1-\frac{4x}{9}+\frac{32}{81}x^2-\frac{896}{2187}x^3+\ldots\right]$$

$\therefore$ Coefficient of x^3 in $\frac{(1-5x)^3(1+3x^2)^{3/2}}{(3+4x)^{1/3}}$ is

$$= \frac{1}{3^{1/3}}\left[-\frac{385}{2}-\frac{159}{2}\times\frac{4}{9}-15\times\frac{32}{81}-\frac{896}{2187}\right]$$

$$= \frac{1}{3^{1/3}}\left[-\frac{385}{2}-\left\{\frac{77274+12960+896}{2187}\right\}\right]$$

$$= \frac{1}{3^{1/3}}\left[\frac{-841995-182260}{4374}\right] = -\frac{1024255}{\sqrt[3]{3}(4274)}$$

4. If x = $\frac{5}{(2!)\cdot 3}+\frac{5\cdot 7}{(3!)\cdot 3^2}+\frac{5\cdot 7\cdot 9}{(4!)3^3}+\ldots$, then find the value of $x^2 + 4x$.

Sol. $x = \frac{5}{(2!)\cdot 3}+\frac{5\cdot 7}{(3!)\cdot 3^2}+\frac{5\cdot 7\cdot 9}{(4!)3^3}+\ldots$

$$= \frac{3\cdot 5}{2!3^2}+\frac{3\cdot 5\cdot 7}{3!3^3}+\frac{3\cdot 5\cdot 7\cdot 9}{4!3^4}+\ldots$$

$$= \frac{3\cdot 5}{2!}\left(\frac{1}{3}\right)^2+\frac{3\cdot 5\cdot 7}{3!}\left(\frac{1}{3}\right)^3+\frac{3\cdot 5\cdot 7\cdot 9}{4!}\left(\frac{1}{3}\right)^4\ldots$$

$$= 1+\frac{3}{1}\left(\frac{1}{3}\right)+x$$

$$= 1+\frac{3}{1}\left(\frac{1}{3}\right)+\frac{3\cdot 5}{2!}\left(\frac{1}{3}\right)^2+\frac{3\cdot 5\cdot 7}{3!}\left(\frac{1}{3}\right)^3+\ldots$$

$$\Rightarrow 2+x = 1+\frac{3}{1}\left(\frac{1}{3}\right)+\frac{3\cdot 5}{2!}\left(\frac{1}{3}\right)^2+\ldots$$

Comparing $x + 2$ with $(1 - y)^{-p/q}$

$$= 1+\frac{p}{1}\left(\frac{y}{q}\right)+\frac{p(p+q)}{1\cdot 2}\left(\frac{y}{q}\right)^2+\ldots$$

Here $p = 3$, $q = 2$, $\frac{y}{q}=\frac{1}{3}\Rightarrow y=\frac{q}{3}=\frac{2}{3}$

$$\therefore x+2 = (1-y)^{-p/q} = \left(1-\frac{2}{3}\right)^{-3/2}$$

$$= \left(\frac{1}{3}\right)^{-3/2} = (3)^{3/2} = \sqrt{27}$$

Squaring on both sides

$$x^2+4x+4 = 27$$

$$\therefore x^2+4x = 23$$

5. Find the sum of the infinite series $\frac{7}{5}\left(1+\frac{1}{10^2}+\frac{1\cdot3}{1\cdot2}\frac{1}{10^4}+\frac{1\cdot3\cdot5}{1\cdot2\cdot3}\frac{1}{10^6}+...\right)$.

Sol. $1+\frac{1}{10^2}+\frac{1\cdot3}{1\cdot2}\frac{1}{10^4}+\frac{1\cdot3\cdot5}{1\cdot2\cdot3}\frac{1}{10^6}+...$

$=1+\frac{1}{1!}\left(\frac{1}{100}\right)+\frac{1\cdot3}{2!}\left(\frac{1}{100}\right)^2+\frac{1\cdot3\cdot5}{3!}\left(\frac{1}{100}\right)^3+...$

Comparing with $(1-x)^{-p/q}$

$=1+\frac{p}{1!}\left(\frac{x}{q}\right)+\frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^2$

$p = 1,\ p + q = 3,\ q = 2$

$\frac{x}{q}=\frac{1}{100}\Rightarrow x=\frac{q}{100}=\frac{2}{100}=0.02$

$\therefore 1+\frac{1}{10^2}+\frac{1\cdot3}{1\cdot2}\cdot\frac{1}{10^4}+...=(1-x)^{-p/q}$

$=(1-0.02)^{-1/2}=(0.98)^{-1/2}$

$=\left(\frac{49}{50}\right)^{-1/2}=\left(\frac{50}{49}\right)^{1/2}=\frac{5\sqrt{2}}{7}$

$\therefore \frac{7}{5}\left[1+\frac{1}{10^2}+\frac{1\cdot3}{1\cdot2}\frac{1}{10^4}+\frac{1\cdot3\cdot5}{1\cdot2\cdot3}\frac{1}{10^6}+...\right]$

$=\frac{7}{5}\frac{5\sqrt{2}}{7}=\sqrt{2}$

6. Show that

$1+\frac{x}{2}+\frac{x(x-1)}{2\cdot4}+\frac{x(x-1)(x-2)}{2\cdot4\cdot6}+...$

$=1+\frac{x}{3}+\frac{x(x+1)}{3\cdot6}+\frac{x(x+1)(x+2)}{3\cdot6\cdot9}+...$

Sol. L.H.S. $=1+\frac{x}{2}+\frac{x(x-1)}{2\cdot4}+\frac{x(x-1)(x-2)}{2\cdot4\cdot6}+...$

$=1+x\left(\frac{1}{2}\right)\frac{(x)(x-1)}{1\cdot2}\left(\frac{1}{2}\right)^2$

$+\frac{x(x-1)(x-2)}{1\cdot2\cdot3}\left(\frac{1}{2}\right)^3+...$

Comparing with

$(1+x)^n=1+{}^nC_1\cdot x+{}^nC_2x^2+.....$

$=1+\frac{n}{1!}\cdot x+\frac{n(n-1)}{1\cdot2}x^2+....$

Here $x=\frac{1}{2}, n=x=\left(1+\frac{1}{2}\right)^{x}=\left(\frac{3}{2}\right)^{x}$

R.H.S. $= 1+\frac{x}{3}+\frac{x(x+1)}{3\cdot 6}+\frac{x(x+1)(x+2)}{3\cdot 6\cdot 9}+\ldots$

$=1+\frac{x}{1}\left(\frac{x}{3}\right)+\frac{(x)(x+1)}{1\cdot 2}\left(\frac{1}{3}\right)^{2}+\frac{(x)(x+1)(x+2)}{1\cdot 2\cdot 3}\left(\frac{1}{3}\right)^{3}+\ldots$

Comparing with $(1-x)^{-n}$

$=1+n(x)+\frac{n(n+1)}{1\cdot 2}x^{2}+\ldots\ldots$

We get $x=\frac{1}{3}, n=x$

$=\left(1-\frac{1}{3}\right)^{-x}=\left(\frac{2}{3}\right)^{-x}=\left(\frac{3}{2}\right)^{x}$

$\therefore$ L.H.S. = R.H.S.

EXERCISE – 5(C)

I.

1. Find an approximate value of the following corrected to 4 decimal places.

(i) $\sqrt[5]{242}$ **(ii)** $\sqrt[7]{127}$

(iii) $\sqrt[5]{32.16}$ **(iv)** $\sqrt[7]{199}$

(v) $\sqrt[3]{1002}-\sqrt[3]{998}$

Sol. i) $\sqrt[5]{242} = (243-1)^{1/5}$

$=(243)^{1/5}\cdot\left(1-\frac{1}{243}\right)^{1/5}$

$=(3^{5})^{1/5}\left[1-\frac{1}{5}\cdot\frac{1}{243}+\frac{\frac{1}{5}\left(\frac{1}{5}-1\right)}{1\cdot 2}\left(\frac{1}{243}\right)^{2}-\ldots\right]$

$=3\left\{1-\frac{1}{5}(0.00243)-\frac{2}{25}(0.00243)^{2}-\ldots\right\}$

$\because \frac{1}{243}=\left(\frac{1}{3}\right)^{5}=(0.3)^{5}=0.00243$

$$\sqrt[5]{242} = 3 - \frac{3}{5}(0.00243) - \frac{6}{25}(0.00243)^2 - \ldots$$

$$= 3 - 0.001458 - 0.000001417176$$

$$= 2.998541$$

ii) $\sqrt[7]{127}$

Sol. $\sqrt[7]{127} = (128-1)^{1/7} = (128)^{1/7}\left[1 - \frac{1}{128}\right]^{1/7}$

$$= 2\left[1 - (0.5)^7\right]^{1/7}$$

$$= 2\left[1 - \frac{1}{7}(0.5)^7 + \frac{\frac{1}{7}\left(\frac{1}{7} - 1\right)}{1 \cdot 2}(0.5)^{14} \ldots\right]$$

$$= 2(1 - 0.0078125) = (2 - 0.0156250)$$

$$= 1.984375 = 1.9844 \text{ (corrected to 4 decimals)}$$

iii) $\sqrt[5]{32.16}$

Sol. $\sqrt[5]{32.16} = (32 + 0.16)^{1/5}$

$$= (32)^{1/5} \cdot \left(1 + \frac{0.16}{32}\right)^{1/5}$$

$$= 2\left[1 + 0.005\right]^{1/5}$$

$$= 2\left[1 + \frac{1}{5}(0.005) + \frac{\frac{1}{5}\left(-\frac{4}{5}\right)}{2!}(0.005)^2 + \ldots\right]$$

$$= 2\left[1 + 0.001 - \frac{2}{25}(0.000025) + \ldots\right]$$

$$= 2(1.001) = 2.002$$

iv) $\sqrt{199}$

Sol. $\sqrt{199} = (196+3)^{1/2}$

$$= (196)^{1/2}\left(1+\frac{3}{196}\right)^{1/2}$$

$$= 14\left[1+0.0153\right]^{1/2}$$

$$= 14\left[1+\frac{1}{2}(0.0153)+\frac{\frac{1}{2}\left(\frac{-1}{2}\right)}{2!}(0.0153)^2+....\right]$$

$$= 14\left[1+0.00765-\frac{1}{8}(0.00023409)+...\right]$$

$$= 14(1.00765) = 14.1071$$

$\therefore\ \sqrt{199} = 14.1071$

v) $\sqrt[3]{1002}-\sqrt[3]{998}$

Sol. $\sqrt[3]{1000}\sqrt[3]{1+\frac{2}{1000}}-\sqrt[3]{1000}\sqrt[3]{1-\frac{2}{1000}}$

$$= 10\left[(1+0.002)^{1/3}-(1-0.002)^{1/3}\right]$$

$$= 10\left[\left(1+\frac{1}{3}(0.002)+\frac{\frac{1}{3}\left(\frac{-2}{3}\right)}{2!}(0.002)^2+...\right)-\left(1-\frac{1}{3}(0.002)+\frac{\frac{1}{3}\left(\frac{-2}{3}\right)}{2!}(0.002)^2+...\right)\right]$$

$$= 10\left[\frac{2}{3}(0.002)\right] = 10\left[\frac{0.004}{3}\right] = \frac{0.04}{3} = 0.0133$$

$\therefore\ \sqrt[3]{1002}-\sqrt[3]{998} = 0.0133$

2. If |x| is so small that x^2 and higher powers of x may be neglected then find the approximate values of the following.

i) $\dfrac{(4+3x)^{1/2}}{(3-2x)^2}$

Sol. $\dfrac{(4+3x)^{1/2}}{(3-2x)^2} = \dfrac{\left[4\left(1+\dfrac{3}{4}x\right)\right]^{1/2}}{\left[3\left(1-\dfrac{2}{3}x\right)\right]^2}$

$= \dfrac{2}{9}\left(1+\dfrac{3}{4}x\right)^{1/2}\left(1-\dfrac{2}{3}\right)^{-2}$

$= \dfrac{2}{9}\left(1+\dfrac{1}{2}\cdot\dfrac{3}{4}x\right)\left(1-(-2)\dfrac{2}{3}x\right)$

(after neglecting x^2 and higher powers of x)

$= \dfrac{2}{9}\left(1+\dfrac{3}{8}x\right)\left(1+\dfrac{4}{3}x\right) = \dfrac{2}{9}\left(1+\dfrac{3}{8}x+\dfrac{4}{3}x\right)$

(Again by neglecting x^2 term)

$= \dfrac{2}{9}\left(1+\dfrac{41}{24}x\right) = \dfrac{2}{9}+\dfrac{41}{108}x$

$\therefore \dfrac{(4+3x)^{1/2}}{(3-2x)^2} = \dfrac{2}{9}+\dfrac{82}{108}x = \dfrac{2}{9}+\dfrac{41}{108}x$

ii) $\dfrac{\left(1-\dfrac{2x}{3}\right)^{3/2}(32+5x)^{1/5}}{(3-x)^3}$

Sol. $\dfrac{\left(1-\dfrac{2x}{3}\right)^{3/2}(32+5x)^{1/5}}{(3-x)^3}$

$= \dfrac{\left(1-\dfrac{2}{3}x\right)^{3/2}(32)^{1/5}\left(1+\dfrac{5}{32}x\right)^{1/5}}{3^3\left(1-\dfrac{x}{3}\right)^3}$

$= \dfrac{2}{27}\left(1-\dfrac{2x}{3}\right)^{3/2}\left(1+\dfrac{5}{32}x\right)^{1/5}\left(1-\dfrac{x}{3}\right)^{-3}$

$= \dfrac{2}{27}\left(1-\dfrac{3}{2}\cdot\dfrac{2x}{3}\right)\left(1+\dfrac{1}{5}\dfrac{5}{32}x\right)\left(1+3\dfrac{x}{3}\right)$

(by neglecting x^2 and higher powers of x)

$$= \frac{2}{27}(1-x)\left(1+\frac{x}{32}\right)(1+x)$$

$$= \frac{2}{27}\left(1-x^2\right)\left(1+\frac{x}{32}\right) = \frac{2}{27}\left(1+\frac{x}{32}\right)$$

iii) $\sqrt{4-x}\left(3-\frac{x}{2}\right)^{-1}$

Sol. $\sqrt{4-x}\left(3-\frac{x}{2}\right)^{-1} = (4-x)^{1/2}\left(3-\frac{x}{2}\right)^{-1}$

$$= 4^{1/2}\left(1-\frac{x}{4}\right)^{1/2} 3^{-1}\left(1-\frac{x}{6}\right)^{-1}$$

$$= \frac{2}{3}\left(1-\frac{x}{4}\right)^{1/2}\left(1-\frac{x}{6}\right)^{-1}$$

$$= \frac{2}{3}\left(1+\frac{1}{2}\left(-\frac{x}{4}\right)\right)\left(1+(-1)\left(-\frac{x}{6}\right)\right)$$

$$= \frac{2}{3}\left(1-\frac{x}{8}\right)\left(1+\frac{x}{6}\right)$$

$$= \frac{2}{3}\left(1-\frac{x}{8}+\frac{x}{6}\right) = \frac{2}{3}\left(1+\frac{x}{24}\right)$$

iv) $\dfrac{\sqrt{4+x}+\sqrt[3]{8+x}}{(1+2x)+(1-2x)^{-1/3}}$

Sol. $\dfrac{\sqrt{4+x}+\sqrt[3]{8+x}}{(1+2x)+(1-2x)^{-1/3}}$

$$= \frac{\sqrt{4}\left(1+\frac{x}{4}\right)^{1/2} + \sqrt[3]{8}\left(1+\frac{x}{8}\right)^{1/3}}{(1+2x)+\left[1+\frac{1}{3}(2x)\right]}$$

$$= \left[\frac{2\left[\left(1+\frac{x}{8}\right)+\left(1+\frac{x}{24}\right)\right]}{2+\frac{8x}{3}} \right] = \frac{2\left(2+\frac{3x+x}{24}\right)}{2\left(1+\frac{4x}{3}\right)}$$

$$= \left(2+\frac{x}{6}\right)\left(1+\frac{4x}{3}\right)^{-1} = \left(2+\frac{x}{6}\right)\left(1-\frac{4x}{3}\right)$$

$$= \left(2-\frac{8x}{3}+\frac{x}{6}\right)$$

($\because$ By neglecting x^2 and higher powers of x)

$$= 2-\frac{15x}{6} = 2-\frac{5x}{2}$$

$$\therefore \frac{\sqrt{4+x}+\sqrt[3]{8+x}}{(1+2x)+(1-2x)^{-1/3}} = 2-\frac{5x}{2} = \frac{4-5x}{2}$$

v) $\dfrac{(8+3x)^{2/3}}{(2+3x)\sqrt{4-5x}}$

Sol. $\dfrac{(8+3x)^{2/3}}{(2+3x)\sqrt{4-5x}} = \dfrac{8^{2/3}\left(1+\frac{3x}{8}\right)^{2/3}}{2\left(1+\frac{3x}{2}\right)2\left(1-\frac{5x}{4}\right)^{1/2}}$

$$= \left(1+\frac{3x}{8}\right)^{2/3}\left(1+\frac{3x}{2}\right)^{-1}\left(1-\frac{5x}{4}\right)^{-1/2}$$

$$= \left[1+\frac{2}{3}\left(\frac{3x}{8}\right)\right]\left[1-\frac{3x}{2}\right]\left[1+\frac{1}{2}\left(\frac{5x}{4}\right)\right]$$

(By neglecting x^2 and higher powers of x)

$$= \left(1+\frac{x}{4}\right)\left(1-\frac{3x}{2}\right)\left(1+\frac{5x}{8}\right)$$

$$= \left(1+\frac{x}{4}-\frac{3x}{2}\right)\left(1+\frac{5x}{8}\right)$$

$$= \left(1-\frac{5x}{4}\right)\left(1+\frac{5x}{8}\right) = 1-\frac{5x}{4}+\frac{5x}{8}$$

(By neglecting x^2 and higher power of x)

$$= 1-\frac{5x}{8}$$

$$\therefore \frac{(8+3x)^{2/3}}{(2+3x)\sqrt{4-5x}} = 1-\frac{5x}{8}$$

3. Suppose s and t are positive and t is very small when compared to s, then find an approximate value of $\left(\frac{s}{s+t}\right)^{1/3}-\left(\frac{s}{s-t}\right)^{1/3}$.

Sol. Since t is very small when compared with s, t/s is very very small.

$$\left(\frac{s}{s+t}\right)^{1/3}-\left(\frac{s}{s-t}\right)^{1/3}=\left[\frac{1}{1+\frac{t}{s}}\right]^{1/3}-\left[\frac{1}{1-\frac{t}{s}}\right]^{1/3}$$

$$=\left(1+\frac{t}{s}\right)^{-1/3}-\left(1-\frac{t}{s}\right)^{-1/3}$$

$$=\left(1+\left(-\frac{1}{3}\right)\left(\frac{t}{s}\right)+\frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)}{1\cdot 2}\left(\frac{t}{s}\right)^2+\frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)\left(-\frac{1}{3}-2\right)}{3!}\left(\frac{t}{s}\right)^3+\ldots\right)$$

$$=\left(1-\left(-\frac{1}{3}\right)\left(\frac{t}{s}\right)+\frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)}{1\cdot 2}\left(\frac{t}{s}\right)^2-\frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)\left(-\frac{1}{3}-2\right)}{3!}\left(\frac{t}{s}\right)^3+\ldots\right)$$

$$=2\left[-\frac{1}{3}\left(\frac{t}{s}\right)-\frac{1\cdot 4\cdot 7}{27\times 6}\frac{t^3}{s^3}\right]=\frac{-2}{3}\frac{t}{s}-\frac{28}{81}\frac{t^3}{s^3}$$

4. Suppose p, q are positive and p is very small when compared to q. Then find an approximate value of $\left(\frac{q}{q+p}\right)^{1/2}+\left(\frac{q}{q-p}\right)^{1/2}$.

Sol. $\left(\frac{q}{q+p}\right)^{1/2}+\left(\frac{q}{q-p}\right)^{1/2}$

$$=\left[\frac{q}{q\left(1+\frac{p}{q}\right)}\right]^{1/2}+\left[\frac{q}{q\left(1-\frac{p}{q}\right)}\right]^{1/2}$$

$$=\left(1+\frac{p}{q}\right)^{-1/2}+\left(1-\frac{p}{q}\right)^{-1/2}$$

$$=\left[1+\left(-\frac{1}{2}\right)\left(\frac{p}{q}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}\left(\frac{p}{q}\right)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!}\left(\frac{p}{q}\right)^3+\ldots\right]$$

$$+\left[1+\left(-\frac{1}{2}\right)\left(-\frac{p}{q}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}\left(-\frac{p}{q}\right)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!}\left(-\frac{p}{q}\right)^3+\ldots\right]$$

$$=\left\{1-\frac{p}{2q}+\frac{3}{8}\left(\frac{p}{q}\right)^2+\ldots\right\}+\left\{1+\frac{p}{2q}+\frac{3}{8}\left(\frac{p}{q}\right)^2-\ldots\right\}$$

$$=2+2\left(\frac{3}{8}\right)\left(\frac{p}{q}\right)^2=2+\left(\frac{3}{4}\right)\left(\frac{p}{q}\right)^2$$

$$\therefore\left(\frac{q}{q+p}\right)^{1/2}+\left(\frac{q}{q-p}\right)^{1/2}=2+\frac{3}{4}\left(\frac{p}{q}\right)^2$$

5. By neglecting x^4 and higher powers of x, find an approximate value of $\sqrt[3]{x^2+64}-\sqrt[3]{x^2+27}$.

Sol. $\sqrt[3]{x^2+64}-\sqrt[3]{x^2+27}$

$$=(64+x^2)^{1/3}-(27+x^2)^{1/3}$$

$$=(64)^{1/3}\left(1+\frac{x^2}{64}\right)^{1/3}-(27)^{1/3}\left(1+\frac{x^2}{27}\right)^{1/3}$$

$$=4\left(1+\frac{x^2}{192}\right)-3\left(1+\frac{x^2}{81}\right)$$

(By neglecting x^4 and higher powers of x)

$$=4+\frac{x^2}{48}-3-\frac{x^2}{27}=1+\frac{(27-48)}{48\times27}x^2$$

$$=1+\left(\frac{-21}{48\times27}\right)x^2=1-\frac{7x^2}{432}=1-\frac{7}{432}x^2$$

$$\therefore\sqrt[3]{x^2+64}-\sqrt[3]{x^2+27}=1-\frac{7}{432}x^2$$

6. Expand $3\sqrt{3}$ in increasing powers of 2/3.

Sol. $3\sqrt{3} = 3^{3/2} = \left(\frac{1}{3}\right)^{-3/2} = \left(1-\frac{2}{3}\right)^{-3/2}$

$$= 1 + \frac{\frac{3}{2}}{1}\cdot\left(\frac{2}{3}\right) + \frac{\frac{3}{2}\left(\frac{3}{2}+1\right)}{1\cdot 2}\left(\frac{2}{3}\right)^2 + \ldots\ldots + \frac{\frac{3}{2}\left(\frac{3}{2}+1\right)\ldots\ldots\left(\frac{3}{2}+r-1\right)}{(1\cdot 2\cdot 3\ldots r)2^r}\left(\frac{2}{3}\right)^r + \ldots\ldots$$

$$= 1 + \frac{3}{1\cdot 2}\left(\frac{2}{3}\right) + \frac{3\cdot 5}{(1\cdot 2)2^2}\left(\frac{2}{3}\right)^2 + \ldots\ldots + \frac{3\cdot 5\ldots(2r+1)}{(1\cdot 2\cdot\ldots r)2^r}\left(\frac{2}{3}\right)^r + \ldots\ldots$$

$$= 1 + 3\left(\frac{1}{3}\right) + \frac{3\cdot 5}{2!}\left(\frac{1}{3}\right)^2 + \ldots\ldots + \frac{3\cdot 5\cdot 7\ldots(2r+1)}{r!}\left(\frac{1}{3}\right)^r + \ldots$$