SIMPLE HARMONIC MOTION

3. SIMPLE PENDULUM

POINTS TO REMEMBER

1. Simple Pendulum :

a. Time period of a simple pendulum is given by $T = 2\pi\sqrt{\frac{\ell}{g}}$.

b. Restoring Force on the bob of the pendulum is $F = mg\sin\theta$.

c. For small amplitudes time period is independent of
a) Mass of the bob

b) Size (or) shape of the bob

c) Density of the bob

d) Amplitude of oscillation.

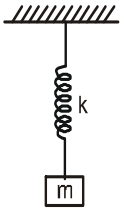
d. A pendulum whose time period is 2 sec is called seconds pendulum.

$$\ell = \frac{gT^2}{4\pi^2} = \frac{g}{\pi^2}.$$

e. $\ell - T$ Graph is parabola and $\ell - T^2$ graph is a straight line passing through origin.

2. Springs:

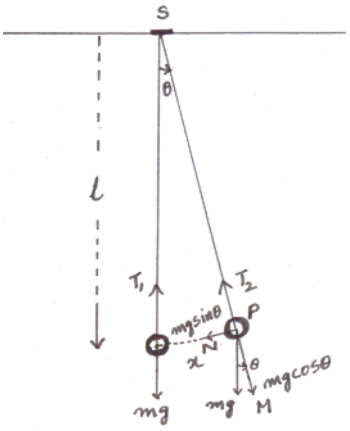
The spring constant or stiffness constant of a spring may be defined as the force required to produce an extension of on unit in the spring K = F / x.



Time period of the loaded spring is given by $T = 2\pi\sqrt{\frac{m}{k}}$

LONG ANSWER QUESTION

1. Show that the motion of a simple pendulum is simple harmonic and hence derive an equation for its time period. What is second's pendulum? (March2011, June2010, May2009, March-2000, March 06)



Ans : Consider a simple pendulum of length *l* with a metal bob of mass m which is suspended from a point S as shown in figure. Let the bob is given a small angular displacement 'θ' and left free to oscillate about the equilibrium position O.

Let the bob be at any point P during the oscillation.

At point 'P' the forces acting on the bob are

(i) The weight of the bob mg vertically down wards and This can be resolved into two components namely a) $mg\cos\theta$ which is giving the tension T_2 in the string b) $mg\sin\theta$ which provides the restring force.

$$\therefore The\ acceleration\ of\ the\ bob = \frac{\text{Re}storing\ \ force}{mass} \qquad \Rightarrow\ a = \frac{mg\sin\theta}{m}$$

Or $a = -g\sin\theta$

When θ is very small, $\sin\theta \approx \theta$

Then $a = -g\theta$

$$a = -g\frac{x}{l}\left(\because \theta = \frac{Arc\ length}{Radius} = \frac{x}{l}\right)$$

Negative sign indicates that a and θ are in opposite directions.

$\therefore a \propto -x$ i.e, acceleration is directly proportional to the displacement x and is directed towards the equilibrium position O. Hence the pendulum executes simple harmonic motion.

$\therefore$ the time period of oscillation of the simple pendulum is given by

$$T = 2\pi\sqrt{\frac{displacement}{acceleration}} = 2\pi\sqrt{\frac{x}{a}}$$

From equation (5) , $\frac{x}{a} = \frac{l}{g}$ (taking magnitudes only)

$$\therefore T = 2\pi\sqrt{\frac{l}{g}}$$

For smaller amplitudes the time period is independent of the amplitude of oscillation..

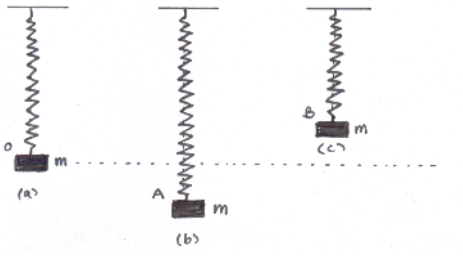
Second's pendulum: - A simple pendulum whose time period is two seconds is called a second's pendulum.

SHORT ANSWER QUESTIONS:

1. Show that the oscillations of a loaded spring are simple harmonic. Hence deduce a relation between the time periods and force constant of the spring. ?

Ans: Consider a spring of spring constant 'K' which is suspended vertically from a rigid support and loaded with a mass 'm'. The spring is given a small displacement down from equilibrium position 'O' so that it executes vertical oscillations as shown.

At any instant of time't' the restoring force 'F' in the spring is given by $F = -ky$ where 'y' be the displacement of the mass



$a = \frac{F}{m} = \frac{-k}{m}.y \Rightarrow a\alpha - y$ Since the acceleration is proportional and opposite to the displacement, the loaded spring executes SHM.

Comparing the above equation with $a = -\omega^2 y$

$$\omega = \sqrt{\frac{k}{m}}$$

Using, $T = \frac{2\pi}{\omega}$

$$T = 2\pi\sqrt{\frac{m}{k}}$$

VERY SHORT ANSWER QUESTIONS

1) The bob of a Pendulum is made of a hollow brass Sphere. What happens to the time period of the pendulum, if the bob is filled with Water completely? Why?

A. Remains same. Because centre of mass is same for hollow sphere or sphere with complete water and hence the length of the pendulum does not change ($T. \propto \sqrt{l}$) .

2) A pendulum clock gives correct time at the equator. Will it gain or lose time if it is taken to the poles? Why?

A) Time period $T = 2\pi\sqrt{\frac{l}{g}}$

It depends on g value, if one moves from equator to the poles, value of g increases. Hence the time period decreases and hence gains time.

3) Two identical springs of force constant k are joined one at the end of the other (in series). Find the effective force constant of the combination.

Ans: $\frac{1}{k^1} = \frac{1}{k} + \frac{1}{k} \Rightarrow \frac{1}{k^1} = \frac{k+k}{k^2} = \frac{2k}{k^2} \Rightarrow k^1 = \frac{k}{2}$

4. What happens to the time period of a simple pendulum kept inside a lift, When the lift (i) moves up with acceleration a? (ii) Moves down with acceleration 'a'?

A) Time period $T = 2\pi\sqrt{\frac{l}{g}}$

(i) Time period $T = 2\pi\sqrt{\frac{l}{g+a}}$ (When lift moves up with acceleration a)

T decreases.

(ii) $T = 2\pi\sqrt{\frac{l}{g-a}}$ (When lift moves down with acceleration a)

T increases.

5) A loaded bus is more comfortable in motion than an empty bus. Why?

A) If the loaded bus has mass greater than an empty and hence frequency of SHM deceases

$n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

Where n is frequency m is mass of loaded the bus.

SOLVED PROBLEMS

1. what is the length of the pendulum of a clock which beats seconds, given that the acceleration due to gravity at the place is equal to 9.8 ms^{-2}

Sol. T = 2 s ; $g = 9.8 ms^{-2}$

$T = 2\pi\sqrt{\frac{l}{g}} \Rightarrow l = \frac{gT^2}{4\pi^2}$

$\Rightarrow l = \frac{9.8\times 2^2}{4\times 9.86} = 1m$

2. The length of a second's pendulum at a place is 1.02 m. Find the time period of another pendulum of length 0.51 m at the same place. (March2011, March2011)

3. Sol. T = $2\pi\sqrt{\frac{l}{g}}$ i.e., $\frac{l}{T^2} = \frac{g}{4\pi^2}$ = constant, at a given place.

$\frac{l_1}{T_1^2} = \frac{l_2}{T_2^2}$

T_1 = 2 s ; $l_1 = 1.02$ m ; $l_2 = 0.51$m ; T_2 = ?

$$\frac{1.02}{2^2}=\frac{0.51}{T_2^2} \text{ or } T_2^2=\frac{0.51}{1.02}\times 4=2 \quad \text{or} \quad T_2=\sqrt{2}=1.414\,s$$

3. What is the change in the time period of a simple pendulum if its length changes by 1 %?

Sol. $T=2\pi\sqrt{\frac{l}{g}} \Rightarrow T\alpha\sqrt{l}$ *or* $\frac{\Delta T}{T}=\frac{1}{2}\frac{\Delta l}{l}=\frac{1}{2}\times 1=0.5\%$

4. When the length of a simple pendulum is increased by 21% what is the change in its time period?

Sol. $l_1=l\ ;\ l_2=\frac{121}{100}l$

$$T=2\pi\sqrt{\frac{l}{g}} \qquad \Rightarrow \frac{T_2}{T_1}=\sqrt{\frac{l_2}{l_1}} \Rightarrow \frac{T_2}{T_1}=\sqrt{\frac{121l}{100l}}=\frac{11}{10}$$

Increase in time period = 10%.

5. A second's pendulum is taken into a mine of depth 640m and oscillated. What is its time period?

Sol. $T=2\pi\sqrt{\frac{l}{g}} \Rightarrow \frac{T_1}{T_2}=\sqrt{\frac{g_2}{g_1}}$

$$g_d=g\left(1-\frac{d}{R}\right)$$

$$\frac{T_1}{T_2}=\sqrt{\frac{g\left(1-\frac{d}{R}\right)}{g}} \text{ or } \frac{T_1}{T_2}=\sqrt{1-\frac{d}{R}}$$

$T_1 = 2$ s; d = 640 m = 0.640 km; R = 6400 km; T_2 = ?

$$\frac{2}{T_2}=\sqrt{1-\frac{0.640}{6400}}=\sqrt{1-0.0001} \Rightarrow T_2=2.0001s$$

6. A clock regulated by a second's pendulum keeps correct time. During summer the length of the pendulum increases to 1.01 m. How much will the clock gain of lose in one day?

Sol. $T=2\pi\sqrt{\frac{l}{g}} \qquad \Rightarrow \frac{dT}{T}=\frac{1}{2}\frac{dl}{l}$

For a seconds pendulum, T = 2 s and $l=\frac{g}{\pi^2}=0.9927\,m$

$dl = 1.01 - 0.9927 = 0.0173 m.$

$$\frac{dT}{2} = \frac{1}{2}\left(\frac{0.0173}{0.9927}\right)$$

Change in time period for 1 oscillation i.e. for 2 seconds, $dT = \frac{0.0173}{0.9927} s.$

The gain or loss of time per day $= 43200 \times \frac{0.0173}{0.9927} = 752.9 s$

7. A spring when compressed by 0.2 m develops a restoring force of 50 N. If a body of mass 10 kg is placed on it, calculate (i) the force constant (K) of the spring (ii) the depression of the spring (iii) time period of oscillation when the weight is slightly displaced from equilibrium position.?

Sol. (i) The force constant $k = \frac{F}{x} = \frac{50}{0.2} = 250\, Nm^{-1}$

(ii) Depression of the spring $= \frac{mg}{k} = \frac{10 \times 9.8}{250} = 0.392 m$

(iii) Time period $T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{10}{250}} = 1.2568 s$

8. A body is suspended at the end of a spring hanging vertically and it oscillates with a time period of 1.414 s. The spring is then cut into two equal parts and the same body attached to one of the pieces is made to oscillate. What will be the time period of oscillation?

Sol. $T = 2\pi\sqrt{\frac{m}{k}}$

When the spring is cut into two pieces the spring constant of each piece is $k' = 2k$

$$T' = 2\pi\sqrt{\frac{m}{k'}}$$

$$\therefore \frac{T}{T'} = \sqrt{\frac{k'}{k}}$$

But T = 1.414 s, $T' = ?$

$$\frac{1.414}{T'} = \sqrt{\frac{2k}{k}} = \sqrt{2} \qquad \Rightarrow \quad T' = 1 s$$

9. A light vertical spring is stretched by 2 cm when a weight of 10 g is attached to its free end. The weight is further pulled down by 1.0 cm and released. Compute the frequency and amplitude of motion.

Sol. i) $k = \frac{F}{x} = \frac{10\times10^{-3}\times9.8}{2\times10^{-2}} = 4.9 Nm^{-1}$

But, $T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{10\times10^{-3}}{4.9}}$

The frequency, $v = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{4.9}{10^{-2}}} = 3.523\,Hz$

ii) The amplitude of motion A = 1.0 cm.

10. A second's pendulum oscillates with amplitude of 0.4 m. If the mass of the pendulum is 0.2 kg find (i) K.E. at the mean positions, (ii) PE at the extreme positions, (iii) PE at the displacement 0.2 m and (iv) KE at the displacement 0.2 m. ?

Sol. T = 2 s ; $\omega = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$ rad s^{-1} ; A = 0.4 m ; m = 0.2 kg.

i) KE at the mean position $= KE_{max} = \frac{1}{2}m\omega^2A^2 = \frac{1}{2}\times0.2\times\pi^2\times(0.4)^2 = 0.1578\,J$

ii) PE at the extreme position $= PE_{max} = \frac{1}{2}m^2\omega^2A^2 = \frac{1}{2}\times0.2\times\pi^2\times(0.4)^2 = 0.1578\,J$

iii) PE at the displacement 0.2 m $= \frac{1}{2}\times m^2\omega^2y^2 = \frac{1}{2}\times0.2\times\pi^2\times(0.2)^2 = 0.03944\,J$

iv) KE at the displacement 0.2 m $= \frac{1}{2}\times m^2\omega^2\left(A^2-y^2\right) = \frac{1}{2}\times0.2\times\pi^2\left((0.4)^2-(0.2)^2\right)$

$=0.11832$ J

EXERCISE 1

1. **The mass and radius of a planet are double that of the earth. If the time period of a simple pendulum on the earth is T, find the time period on the planet**

Sol : $g = \frac{GM}{R^2}$

And $g' = \frac{G(2M)}{(2R)^2} = \frac{1}{2}\frac{GM}{R^2} = \frac{1}{2}g$

$$\therefore \frac{T}{T^1} = \sqrt{\frac{g^1}{g}} = \sqrt{\frac{g/2}{g}} \Rightarrow T' = T\sqrt{2}$$

2 **Calculate the change in the length of a simple pendulum of length 1m, when its period of oscillation changes from 2 s to 1.5 s. ?**

Sol: $l_1 = 1m,\ T_1 = 2\sec,\ T_2 = 1.5\sec$

$$T\alpha\sqrt{l} \quad \Rightarrow \qquad \frac{T_1}{T_2} = \sqrt{\frac{l_1}{l_2}}$$

$$\Rightarrow \frac{2}{1.5} = \sqrt{\frac{1}{l_2}} \Rightarrow l_2 = \frac{9}{16}m$$

$\therefore$ Change in length $= l_1 - l_2 = 1 - \frac{9}{16} = \frac{7}{16}$

3. **A freely falling body takes 2 seconds to reach the ground on a planet, when it is dropped from a height of 8m. If the period of a simple pendulum is π seconds on the planet, calculate the length of the pendulum.**

Sol : t = 2 sec, s = 8m

For a freely falling body, $s = \frac{1}{2}at^2$

$8 = \frac{1}{2}g\left(4^2\right) \Rightarrow$ g = $4m/\sec^2$

Now , $T = 2\pi\sqrt{\frac{l}{g}}$ or $\pi = 2\pi\sqrt{\frac{l}{4}} \Rightarrow \quad l = 1m = 100cm$

4. **A simple pendulum is of length 50 cm. Find its time period and frequency of oscillation.(g=9.8 m/s^2)?**

Sol: $l = 50cm = 0.5m$

$$T = 2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{0.5}{9.8}} = 2\pi\sqrt{\frac{1}{19.6}} = 1.4 \text{ sec}$$

Frequency $\nu = \frac{1}{T} = \frac{1}{1.4} = 0.71$ sec

5. **The period of a simple pendulum is found to increase by 50% when the length of the pendulum is increased by 0.6 m. Calculate the initial length and the initial period of oscillation at a place where g** $= 9.8 m/s^2$ **. ?**

Sol: Let $l_1 = l$ and $l_2 = l + 0.6$

$T_1 = T$ And $T_2 = \dfrac{150T}{100}$

$T \alpha \sqrt{l} \Rightarrow \dfrac{T_1}{T_2} = \sqrt{\dfrac{l_1}{l_2}} \Rightarrow \dfrac{100}{150} = \sqrt{\dfrac{l}{l+0.6}} \Rightarrow \dfrac{4}{9} = \dfrac{l}{l+0.6}$

Solving we get, $l = 0.48m$

But $T = 2\pi\sqrt{\dfrac{l}{g}} = 2\pi\sqrt{\dfrac{0.48}{9.8}} = 1.3897$ sec

6. **The bob of a simple pendulum of mass 100 g is oscillating with a time period of 1.42s. If the bob is replaced by another bob of mass 150 g but of same radius, what will be the new time period of oscillation?**

Sol: The time period is independent of mass of the bob.

7. The mass and diameter of a planet are twice those of the earth. What will be the period of oscillation of a pendulum on this planet, if it is a second's pendulum on the earth?

Sol: Let g and g' be the acceleration due to gravities on the earth and the planet.

$g = \dfrac{GM}{R^2}$

$M_P = 2M$, $R_P = 2R$

$\therefore g' = \dfrac{G(2M)}{(2R)^2} = \dfrac{GM}{2R^2} = \dfrac{g}{2}$

or $T \alpha \dfrac{1}{\sqrt{g}} \Rightarrow \dfrac{T_1}{T_2} = \sqrt{\dfrac{g_2}{g_1}} \Rightarrow T_2 = 2\sqrt{2}$ sec

8. **A pendulum that beats seconds on the surface of the earth was taken to a depth of** $\left(\dfrac{1}{4}\right)$ **th the radius of the earth. What will be its time period of oscillation?**

Sol : $g' = g\left(1 - \dfrac{d}{R}\right) = g\left(1 - \dfrac{R}{4 \times R}\right) = g\left(1 - \dfrac{1}{4}\right) = \dfrac{3}{4}$ g

But $T\alpha\frac{1}{\sqrt{g}} \Rightarrow \frac{T_1}{T_2} = \sqrt{\frac{g_2}{g_1}} \Rightarrow T_2 = \frac{4}{\sqrt{3}}\text{sec} = 2.309\,\text{sec}$

9. The pendulum of a certain clock has time period 2.04s. How fast or slow does the clock run during 24 hour?

Sol: The time period is less by is 2. 04 - 2 = 0.04 sec

The number of oscillations in one day is $\frac{1}{2}\times 86,400 = 43,200$

Hence the time period is less by 43, 200 × 0.04= 1728 sec = 28.8 min

10. **A clock regulated by a second's pendulum keeps correct time. During summer the length of the pendulum increases to 1.02m. How much will the clock gain or lose in one day?**

Sol : $T\alpha\sqrt{l}$ or $\frac{\Delta T}{T} = \frac{1}{2}\frac{\Delta l}{l}$

$l = 0.9927m$

$\Delta l = 1.02 - 0.9927 = 0.0273$

$T = 2\,\text{sec}$

$\frac{\Delta T}{2} = \frac{1}{2}\times\frac{0.0273}{0.9927}$

The number of oscillation in one day are 43, 200

$\therefore$Loss in time in day = $43200\times\frac{0.0273}{0.9928}$ = 1188 sec

11. A simple pendulum of length 1 m is oscillated at a place where $g = 9.8m/s^{2.}$ Find the maximum velocity of the bob of the simple pendulum if the amplitude of oscillation is 3cm.?

Sol: $l = 1m,\ g = 9.8m/s^2,\ A = 3cm = 0.03m$

$T = 2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{1}{9.8}}$

$\omega = \frac{2\pi}{T} = 3.13\ rad/\text{sec}$

Maximum particle velocity = $A\omega = (0.03)(3.13)$ = 9.391 cm /sec

12. **The force on a body executing SHM is 2 N when the displacement is 2 cm. If the amplitude of oscillations is 5 cm, what is the total energy associated with the SHM?**

Sol: F = 2 N, when x = 2 cm = 0.02 m

$\Rightarrow F = m\omega^2 x \Rightarrow 2 = m\omega^2 (0.02)$

$\Rightarrow m\omega^2 = \frac{2}{0.02} = 100$

A = 5 cm = 0.05 m

Total energy = $\frac{1}{2} m\omega^2 A^2 = \frac{1}{2}(100)(0.05)^2 = 50(0.0025) = 0.125J = 12.5\times10^{-2} J$

13. **The time period of a body suspended from a spring is T. What will be the new time period, if the spring is cut into two equal parts and (i) the mass is suspended from one part? (ii) The mass is suspended simultaneously from both the parts?**

Sol: (i) If the spring is cut into two equal parts, the new spring constant $k' = 2k$

$\therefore$New Time period $T' = 2\pi\sqrt{\frac{m}{k'}} = 2\pi\sqrt{\frac{m}{2k}} = \frac{T}{\sqrt{2}}$

(ii) When two springs are attached parallel then

Spring constant $k'' = k_1 + k_2$ or $k'' = 2k + 2k = 4$k

$\therefore$ Time period of $T'' = 2\pi\sqrt{\frac{m}{4k}} = \frac{T}{2}$

14. A spring balance has a scale that reads 0 to 20 kg. The length of the scale is 10 cm. A body suspended from this balance, when displaced and released, oscillates with period of $\frac{\pi s}{10}$. What is the mass of the body?

Sol: m = 20 kg ; x = 10 cm = 0.1 m

$\Rightarrow k = \frac{F}{x} = \frac{200}{0.1} = 2000$

$T = 2\pi\sqrt{\frac{m}{k}} \Rightarrow T = 2\pi\sqrt{\frac{20}{2000}}$

$\therefore T = \frac{2\pi}{10}$

Also, $T \propto \sqrt{m}$

$$\frac{T_2}{T_1} = \sqrt{\frac{m_2}{m_1}} \Rightarrow m_2 = 5\,kg$$

$\therefore$ The mass attached to the spring is 5 kg.

ASSESS YOURSELF

1. What is the restoring force responsible for the oscillations of simple pendulum?

A. The restoring force responsible for the oscillations of simple pendulum is the component of the gravitational force mg sin θ due to the earth on it.

2. Why should the angular displacement be small for a simple pendulum?

A. As the angular displacement θ increases sin θ largely differs from 'θ'. Therefore, for the oscillations to be along a straight line and for the condition of SHM that a $\propto$ -x to be valid θ must be very small.

3. How does the time period of a simple pendulum depend on the length of the pendulum?

A. The time period of simple pendulum is directly proportional to the square root of its length.

4. How does the time period of a simple pendulum depend on the acceleration due to gravity at the given place?

A. The time period of simple pendulum is inversely proportional to the square root of the acceleration due to gravity.

5. what is the force responsible for the oscillations of a loaded spring?

A. The elastic restoring force is responsible for the oscillations of a loaded spring.

6. what happens to the value of the force constant of a spring, if the mass attached to it is doubled?

A. The force constant of the spring is independent of the mass attached to it. Therefore it does not change with the mass attached.

7. Which of the two, a small spring in a ball point pen, and a shock absorbed spring in a motor cycle, has a greater force constant?

A. The force constant of a shock absorber spring is greater.