BINOMIAL THEOREM --1

Binomial :- An expression which contains two terms is called a binomial

Pascal Triangle:-

Index **coefficient**

1 1 1

2 1 2 1

3 1 3 3 1

4 1 4 6 4 1

5 **1 5 10 10 5 1**

In each row 1st and last elements '1' remaining terms are obtained by adding the 2 terms

Theorem :-

If n is a positive integer x, a are real number then

$$(x+a)^n = nc_0 x^n + nc_1 x^{n-1}a + nc_2 x^{n-2}a^2 + nc_3 x^{n-3}a^3 + \ldots + nc_n a^n$$

Note 1: This theorem is called binomial theorem for positive integer index.

Note 2: This expansion contains (n +1) terms

Note 3: In the expansion the sum of powers of x and 'a' in each term is equal to n.

Note 4: In this expansion the power term x is decreased by 1 and the power of a is increased by 1 except in the first term

Notes 5: In this expansion $(r+1)^{th}$ term is called general term denoted by T_{r+1} where $T_{r+1} = nc_r . x^{n-r}\ a^r$

Note 6: In this expansion the coefficients $nc_0, nc_1, nc_2 ..... nc_n$ are called binomial coefficients and these are simply denoted by $C_0, C_1, C_2 ..... C_n$

Note 7: $(x+a)^n = \sum_{r=0}^{n} nc_r , x^{n-r}\ a^r$

Note 8 : $(x-a)^n = \sum_{r=0}^{n} (-1)^n\ nc_r . x^{n-r}\ a^r$

Note 9 : $(1+x)^n = \sum_{r=0}^{n} nc_r\ x^r$

Note 10 : $(1-x)^n = \sum_{r=0}^{n} (-1)^n\ nc_r\ x^r$

Note 11 :

Expansion	**Nature of n**	**No. of terms**
$(x+a)^n + (x-a)^n$	even	$\frac{n}{2}+1$
	Odd	$\frac{n+1}{2}$
$(x+a)^n - (x-a)^n$	even	$\frac{n}{2}$
	Odd	$\frac{n+1}{2}$

Note 12: $(x+a)^n = (a+x)^n$ even though the above expansion are equal T_{r+1} in $(x+a)^n \neq T_{r+1}$ in $(a+x)^n$

Note 13 : $(a+x)^n = a^n \left\{1+\frac{x}{a}\right\}^n = a^n (1+t)^n$ where $t = \frac{x}{a}$

Note 14: The number of terms in the expansion $(x+a)^n+(x-a)^n+(x+ai)^n+(x-ai)^n$ is $\left[\frac{n+4}{4}\right]$ where [.] is greatest integer function

Note 15: $(a+b+c)^n=[a+(b+c)]^n=\sum_{r=0}^{n} nc_r\, a^{n-r}\,(b+c)^r$

Note 16: Number of terms of $(a+b+c)^n$ is $(n+2)k_2$

Note 17 : Number of terms of $(a_1+a_2+a_3+....+a_r)^n$ is $(n+r-1)c_{r-1}$

The coefficient of x^m in $\left(ax^p+\frac{b}{x^q}\right)^n$ is $nc_r a^{n-r}\,b^{n-r}$ where $r=\frac{np-m}{p+q}$

The constant term in $\left(ax^p+\frac{b}{x^q}\right)^n$ is $nc_r\,a^{n-r}\,b^r$ where $r=\frac{np}{p+q}$

Middle term(s) of $(x+a)^n$

Case (i) **If n is even**

Here number of terms = n + 1

$\therefore n+1$ is odd

Hence only one middle term exist

$\therefore$ middle term $=\left(\frac{n+1+1}{2}\right)^{th}$ term $=\left(\frac{n}{2}+1\right)^{th}$ term

Case (ii) **If n is odd**

Number term n + 1 is even

$\therefore$ two middle terms $\left(\frac{n+1}{2}\right)^{th}, \left(\frac{n+3}{2}\right)^{th}$ terms

Numerically greatest terms :-

Theorem 1 : If $\frac{(n+1)|x|}{|x|+1} = p$ a positive integer then p^{th} and $(p+1)^{th}$ term are the numerically greatest terms n the expansion of $(1+x)^n$

ii) If $\frac{(n+1)|x|}{|x|+1} = p + F$ where p is a positive integer and o < F < 1 then $(p+1)^{th}$ term is the numerically greatest term in the expansion of $(1+x)^n$

Note 1 : The numerically greatest term of $(a+x)^n$ can be found by writing $(a+x)^n = a^n\left(1+\frac{x}{a}\right)^n$ since a^n is constant it will not effect the relative numerical b value of the terms

Note 2 : In the expansion of $(1+x)^n$ if we take x = 1 then $\frac{(n+1)|x|}{1+|x|} = \frac{n+1}{2}$ and the terms are just the binomial coefficient $nc_0, nc_1, nc_2 .....nc_n$ thus if n is odd then $\frac{n+1}{2} = p$ a positive integer and hence $T_p = \frac{nc_{n-1}}{2}$ and $T_{p+1} = \frac{nc_{n+1}}{2}$ are the numerically greatest terms

If n is even then $\frac{n+1}{2} = p + \frac{1}{2}$ is not a positive integer and hence $T_{p+1} = nc_{n/2}$ is the numerically greatest coefficient

BINOMIAL COEFFICIENTS

Theorem : If c_r denotes nc_r then

i) $c_0 + c_1 + c_2 + ... + c_n = 2^n$

ii) $c_0 - c_1 + c_2 - c_3 + ... + (-1)^n c_n = 0$

iii) $c_0 + c_2 + c_4 + ... = c_1 + c_3 + c_5 + ... = 2^{n-1}$

Note i) : $\sum_{r=0}^{n} nc_r = 2^n$

ii) $\sum_{r=0}^{n} (-1)^r nc_r = 0$

Theorem 2 : If c_r denotes nc_r then
$ac_0 + (a+d).c_1 + (a+2d).c_2 + ... + (a+nd).c_n = (2a+nd).2^{n-1}$

Theorem 3: If c_r denotes nc_r then $c_1 + 2.c_2 x + 3.c_3 x^2 + ... + nc_n .x^{n-1} = n(1+x)^{n-1}$

Corollary: Prove that $c_1 + 2.c_2 + 3.c_3 + ...n.c_n = n.2^{n-1}$

We know that

$c_1 + 2.c_2 + 3.c_3 x^2 + ...n\,c_n x^{n-1} = n.(1+x)^{n-1}$

Put x = 1 on both sides

$c_1 + 2.c_2 + 3.c_3 + ... + n.c_n = n.2^{n-1}$

Corollary 2 : Prove that $c_{1-1}.c_2 + 3.c_3 + ... + n(-1)^{n-1} c_n = 0$

We know that

$(1+x)^n = c_0 + c_1 x + c_2 x^2 + x_3 x^3 + ... + c_n x^n$

$n(1+x)^{n-1} = c_1 + 2.c_2 x + 3.c_3 x^2 + ... + n.c_n x^{n-1}$

Put x = -1 on both sides

$c_1 - 2\,c_2 + 3\,c_3 - 4c_4 + ... + (-1)^{n-1} n.c_n = 0$

Note i) $\sum_{r=0}^{n} r.nc_r = n.2^{n-1}$

ii) $\sum_{r=0}^{n} (-1)^r r.nc_r = 0$

Corollary 3 : Prove that $2.1c_2 + 3.2\,c_3x + 4.3\,c_4x^2 + ... + n(n-1)c_nx^{n-2} = n(n-1)(1+x)^{n-2}$

We know that

$(1+x)^n = c_0 + c_1x + c_2x^2 + c_3x^3 + ... + c_nx^n$

Different writ x

$n(1+x)^{n-1} = c_1 + 2\,c_2x + 3\,c_3x^2 + ... + n\ c_nx^{n-1}$

Different again wrt x

$n(n-1)(1+x)^{n-2} = 2.1c_2 + 3.2c_3x + 4.3c_4\,x^2 + ... + n(n-1)\,c_nx^{n-2}$

Note : $\sum_{r=2}^{n} r(r-1).nc_r = n(n-1).2^{n-2}$

Theorem 4: If c_r denotes nc_r then $c_0 + \frac{c_1}{2}x + \frac{c_2}{3}x^2 + ... + \frac{c_n}{n+1}x^n = \frac{(1+x)^{n+1}-1}{(n+1)x}$

Theorem 5: If c_r denotes nc_r then

$c_oc_r + c_1\,c_{r+1} + c_2c_{r+2} + ... + c_{n-r}c_r = 2nc_{n+r}\,\frac{(2n)!}{(n-r)!(n+r)!}$

Theorem 6: If c_r denotes nc_r then $c_0^2 + c_1^2 + c_2^2 + ... + c_n^2 = \frac{(2n)!}{(n!)^2}$

Theorem 7: If c_r denotes nc_r then prove that $c_0c_1 + c_1c_2 + c_2c_3 + ... + c_{n-1}\,c_n = 2nc_{n+1}$

BINOMIAL THEOREM FOR RATIONAL INDEX

1. If n is a rational number and $|x| < 1$ then

$1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ... = (1+x)^n$

2. $(1+x)^{-1} = 1 - x + x^2 - x^3 + ... + x(-1)^r\,x^r + ...$

3. $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + ... + (-1)^r\,(r+1)x^r + ...$

4. $(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + ... + (-1)^r\,\frac{(r+1)(r+2)}{2}x^r + ...$

5. $(1+x)^{-\frac{p}{q}} = 1 - p\left(\frac{x}{q}\right) + \frac{p(p+q)}{1.2}\left(\frac{x}{q}\right)^2 + \frac{p(p+q)(p+2q)}{1.2.3}\left(\frac{x}{q}\right)^3 + ...\infty$

6. $(1+x)^{-n} = 1 + nx + \frac{n(n+1)}{1.2}x^2 + \frac{n(n+1)(n+2)}{1.2.3}x^3 + ...$

$\frac{n(n+1)(n+2)....(n+r-1)}{1.2.3.....r}x^r + ...\infty$

7. $(1-x)^{-1} = 1 + x + x^2 + x^3 + ... + x^r + ...\infty$

8. $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + ... + (r+1)x^r + ...\infty$

9. $(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + ... + \frac{(r+1)(r+2)}{2}x^r + ...\infty$

10. $(1-x)^{-p/q} = 1 + p\left(\frac{x}{q}\right) + \frac{p(p+q)}{1.2}\left(\frac{x}{q}\right)^2 + ...\infty$

11. If n is a positive integer then

$(1+x)^{-n} = 1 - nc_1x + (n+1)c_2x^2 - (n+2)c_3x^3 + ...\infty$

$(1+x)^{-n} = 1 + nc_1x + (n+1)c_2x^2 + (n+2)c_3x^3 + ...\infty$

General term $(1+x)^{-n} = (-1)^r\,(n+r-1)c_r\,x^r$

General term $(1-x)^{-n} = (n+r-1)c_r\,x^r$

EXERCISE – 5(a)

I.

1. Expand the following using binomial theorem.

(i) $(4x + 5y)^7$ **(ii)** $\left(\frac{2}{3}x+\frac{7}{4}y\right)^5$

(iii) $\left(\frac{2p}{5}-\frac{3p}{7}\right)^6$ **(iv) $(3 + x – x^2)^4$**

i) $(4x + 5y)^7$

Sol. $(4x+5y)^7 =$

$${}^7C_0(4x)^7(5y)^0 + {}^7C_1(4x)^6(5y)^1 + {}^7C_2(4x)^5(5y)^2 + {}^7C_3(4x)^4(5y)^3 + {}^7C_4(4x)^3(5y)^4 + {}^7C_5(4x)^2(5y)^5 + {}^7C_6(4x)^1(5y)^6 + {}^7C_7(4x)^0 + (5y)^7$$

$$= \sum_{r=0}^{7} {}^7C_r(4x)^{7-r}(5y)^r$$

ii) $\left(\frac{2}{3}x+\frac{7}{4}y\right)^5$

Sol. $\left(\frac{2}{3}x+\frac{7}{4}y\right)^5 =$

$${}^5C_0\left(\frac{2}{3}x\right)^5 + {}^5C_1\left(\frac{2}{3}x\right)^4\left(\frac{7}{4}y\right) + {}^5C_2\left(\frac{2}{3}x\right)^3\left(\frac{7}{4}y\right)^2 + {}^5C_3\left(\frac{2}{3}x\right)^2\left(\frac{7}{4}y\right)^3 + {}^5C_4\left(\frac{2}{3}x\right)^1\left(\frac{7}{4}y\right)^4 + {}^5C_5\left(\frac{7}{4}y\right)^5$$

$$= \sum_{r=0}^{5} {}^5C_r\left(\frac{2}{3}x\right)^{5-r}\left(\frac{7}{4}y\right)^r$$

iii) $\left(\frac{2p}{5}-\frac{3p}{7}\right)^6$

Sol. $\left(\frac{2p}{5}-\frac{3p}{7}\right)^6 = {}^6C_0\left(\frac{2p}{5}\right)^6 - {}^6C_1\left(\frac{2p}{5}\right)^5\left(\frac{3q}{7}\right)^1 + {}^6C_2\left(\frac{2p}{5}\right)^4\left(\frac{3q}{7}\right)^2 - {}^6C_3\left(\frac{2p}{5}\right)^3\left(\frac{3q}{7}\right)^3 + {}^6C_4\left(\frac{2p}{5}\right)^2\left(\frac{3q}{7}\right)^4 - {}^6C_5\left(\frac{2p}{5}\right)^1\left(\frac{3q}{7}\right)^5 + {}^6C_6\left(\frac{3q}{7}\right)^5$

$$= \sum_{r=0}^{6} (-1)^r\, {}^6C_r\left(\frac{2p}{5}\right)^{6-r}\left(\frac{3q}{7}\right)^r$$

iv) $(3 + x - x^2)^4$

Sol. $(3+x-x^2)^4 = [(3+x)-x^2]^4$

$= {}^4C_0(3+x)^4 - {}^4C_1(3+x)^3x^2 + {}^4C_2(3+x)^2(x^2)^2 - {}^4C_3(3+x)^1(x^2)^3 + {}^4C_4(x^2)^4$

$= (3+x)^4 - 4(3+x)^3x^2 + 6(3+x)^2x^4 - 4(3+x)x^6 + x^8$

$= [{}^4C_0(3)^4 + {}^4C_1(3)^3x + {}^4C_2(3)^2x^2 + {}^4C_3(3)x^3 + {}^4C_4x^4] - 4[{}^3C_0(3)^3 +$

${}^3C_1(3)^2x + {}^3C_2(3)x^2 + {}^3C_3x^3]x^2 + 6[{}^2C_6(3)^2 + {}^2C_1(3)x + {}^2C_2x^2]x^4 - 4(3+x)x^6 + x^8$

$= 81+108x+54x^2+12x^3+x^4$

$-4x^2(27+27x+9x^2+x^3) + 6x^4(9+6x+x^2) - 4x^6(3+x) + x^8$

$= 81+108x+(54-108)x^2+(12-108)x^3+(1-36+54)x^4+(-4+36)x^5+(6-12)x^6+(-4)x^7+x^8$

$= 81+108x-54x^2-96x^3+19x^4+32x^5-6x^6-4x^7+x^8$

2. Write down and simplify

i) 6^{th} term in $\left(\frac{2x}{3}+\frac{3y}{2}\right)^9$

ii) 7^{th} term in $(3x - 4y)^{10}$

iii) 10^{th} term in $\left(\frac{3p}{4}-5q\right)^{14}$

iv) rth term in $\left(\frac{3a}{5}+\frac{5b}{7}\right)^8 (1 \le r \le 9)$

i) 6^{th} term in $\left(\frac{2x}{3}+\frac{3y}{2}\right)^9$

Sol. 6^{th} term in $\left(\frac{2x}{3}+\frac{3y}{2}\right)^9$

The general term in $\left(\frac{2x}{3}+\frac{3y}{2}\right)^9$ is

$$T_{r+1} = {}^9C_r\left(\frac{2x}{3}\right)^{9-r}\left(\frac{3y}{2}\right)^r$$

Put r = 5

$$T_6 = {}^9C_5\left(\frac{2x}{3}\right)^4\left(\frac{3y}{2}\right)^5 = {}^9C_5\left(\frac{2}{3}\right)^4\left(\frac{3}{x}\right)^5 x^4y^5$$

$$= \frac{9\times8\times7\times6}{1\times2\times3\times4}\frac{(2^4)}{3^4}\frac{3^5}{2^5}x^4y^5 = 189x^4y^5$$

ii) 7^{th} term in $(3x - 4y)^{10}$

Sol. General term in $(3x - 4y)^{10}$ is

$$T_{r+1} = (-1)^r\ {}^{10}C_r(3x)^{10-r}(4y)^r$$

Put r = 6

$T_7 = (-1)^6\ {}^{10}C_6 (3x)^4 (4y)^6 = {}^{10}C_4 (3)^4 (4)^6 x^4 y^6$

$= \frac{10\times9\times8\times7}{1\times2\times3\times4} 3^4 (4)^6 x^4 y^6 = 280(12)^5 x^4 y^6$

iii) 10th term in $\left(\frac{3p}{4} - 5q\right)^{14}$

Sol. General term in $\left(\frac{3p}{4} - 5q\right)^{14}$ is

$T_{r+1} = {}^{14}C_r \left(\frac{3p}{4}\right)^{14-r} (-5q)^r$

$= (-1)^r\ {}^{14}C_r \left(\frac{3p}{4}\right)^{14-r} (5q)^r$

Put r = 9

$T_{10} = (-1)^9\ {}^{14}C_9 \left(\frac{3p}{4}\right)^5 (5q)^9$

$= -{}^{14}C_5 \left(\frac{3}{4}\right)^5 (5)^9 p^5 q^9$

$= -\frac{14\times13\times12\times11\times10}{1\times2\times3\times4\times5} \frac{3^5}{4^5} 5^9 p^5 q^9$

$= \frac{-(2002)3^5 \cdot 5^9}{4^5} p^5 q^9$

iv) rth term in $\left(\frac{3a}{5} + \frac{5b}{7}\right)^8 (1 \le r \le 9)$

Sol. The general term in $\left(\frac{3a}{5} + \frac{5b}{7}\right)^8$ is

$T_{r+1} = {}^8C_r \left(\frac{3a}{5}\right)^{8-r} \left(\frac{5b}{7}\right)^r$

Replace r by r – 1, we get

$T_r = {}^8C_{(r-1)} \left(\frac{3a}{5}\right)^{8-r+1} \left(\frac{5b}{7}\right)^{r-1}$

$= {}^8C_{(r-1)} \left(\frac{3a}{5}\right)^{9-r} \left(\frac{5b}{7}\right)^{r-1}; 1 \le r \le 9$

3. **Find the number of terms in the expansion of**

(i) $\left(\frac{3a}{4}+\frac{b}{2}\right)^9$ **(ii)** $(3p+4q)^{14}$

(iii) $(2x+3y+z)^7$

i) $\left(\frac{3a}{4}+\frac{b}{2}\right)^9$

Sol. Number of terms in $(x + a)^n$ is $(n + 1)$, where n is a positive integer.

Hence number of terms in $\left(\frac{3a}{4}+\frac{b}{2}\right)^9$ are :$9 + 1 = 10$

ii) $(3p+4q)^{14}$

Sol. Number of terms in $(3p+4q)^{14}$ are :$14 + 1 = 15$

iii) $(2x+3y+z)^7$

Sol. Number of terms in $(a + b + c)^n$ are $\frac{(n+1)(n+2)}{2}$, where n is a positive integer.

Hence number of terms in $(2x+3y+z)^7$ are : $\frac{(7+1)(7+2)}{2}=\frac{8\times9}{2}=36$

II.

1. Find the coefficient of

i) x^{-6} in $\left(3x-\frac{4}{x}\right)^{10}$

ii) x^{11} in $\left(2x^2+\frac{3}{x^3}\right)^{13}$

iii) x^2 in $\left(7x^3-\frac{2}{x^2}\right)^9$

iv) x^{-7} in $\left(\frac{2x^2}{3}-\frac{5}{4x^5}\right)^7$

i) x^{-6} in $\left(3x-\frac{4}{x}\right)^{10}$

Sol. The general term in $\left(3x-\frac{4}{x}\right)^{10}$ is

$$T_{r+1}=(-1)^r\,{}^{10}C_r(3x)^{10-r}\left(\frac{4}{x}\right)^r$$

$$=(-1)^r\,{}^{10}C_r\,3^{10-r}(4)^r x^{10-r-r}$$

$$=(-1)^r\,{}^{10}C_r\,3^{10-r}(4)^r x^{10-2r} \quad ...(1)$$

For coefficient of x^{-6}, put $10 - 2r = -6$

$\Rightarrow 2r = 10+6 = 16 \Rightarrow r = 8$

Put r = 8 in (1)

$T_{8+1} = (-1)^8\ {}^{10}C_8 3^{10-8}(4)^8 x^{10-16} = {}^{10}C_8 3^2 4^8 x^{-6}$

$\therefore$ Coefficient of x^{-6} in $\left(3x - \frac{4}{x}\right)^{10}$ is :

${}^{10}C_8 3^2 4^8 = {}^{10}C_2 3^2 4^8$

$= \frac{10\times 9}{1\times 2}\times 9\times 4^8 = 405\times 4^8$

ii) x^{11} in $\left(2x^2 + \frac{3}{x^3}\right)^{13}$

Sol. The general term in $\left(2x^2 + \frac{3}{x^3}\right)^{13}$ is :

$T_{r+1} = {}^{13}C_r (2x^2)^{13-r}\left(\frac{3}{x^3}\right)^r$

$= {}^{13}C_r (2)^{13-r} 3^r x^{26-2r} x^{-3r}$

$= {}^{13}C_r (2)^{13-r}(3)^r x^{26-5r}$ ...(1)

For coefficient of x^{11}, put $26 - 5r = 11$

$\Rightarrow 5r = 15 \Rightarrow r = 3$

Put r = 3 in (1)

$T_{3+1} = {}^{13}C_3 (2)^{10}(3)^3 x^{26-15}$

$T_4 = \frac{13\times 12\times 11}{1\times 2\times 3}\cdot 2^{10}\cdot 3^3 \cdot x^{11}$

$\therefore$ Coefficient of x^{11} in $\left(2x^2 + \frac{3}{x^3}\right)^{13}$ is :

$(286)(2^{10})(3^3)$

iii) x^2 in $\left(7x^3 - \frac{2}{x^2}\right)^9$

Sol. The general term in $\left(7x^3 - \frac{2}{x^2}\right)^9$ is

$T_{r+1} = (-1)^r\ {}^9C_r (7x^3)^{9-r}\left(\frac{2}{x^2}\right)^r$

$= (-1)^r\ {}^9C_r (7)^{9-r}(2)^r x^{27-3r} x^{-2r}$

$= (-1)^r\ {}^9C_r (7)^{9-r}(2)^r x^{27-5r}$ ...(1)

For coefficient of x^2, put $27 - 5r = 2$

$\Rightarrow 5r = 25 \Rightarrow r = 5$

Put r = 5 in (1)

$T_{5+1} = (-1)^5\ {}^9C_5 (7)^4 (2)^5 x^{27-25}$

$= -{}^9C_4 7^4 \cdot 2^5 \cdot x^2 = -\left(\frac{9\times8\times7\times6}{1\times2\times3\times4}\right)7^4 \cdot 2^5 x^2$

$\therefore$ Coefficient of x^2 in $\left(7x^3 - \frac{2}{x^2}\right)^9$ is :$-126 \times 7^4 \times 2^5$.

iv) x^{-7} in $\left(\frac{2x^2}{3} - \frac{5}{4x^5}\right)^7$

Sol. The general term in $\left(\frac{2x^2}{3} - \frac{5}{4x^5}\right)^7$ is

$T_{r+1} = (-1)^r \cdot {}^7C_r \left(\frac{2x^2}{3}\right)^{7-r} \left(\frac{5}{4x^5}\right)^r$

$= (-1)^r \cdot {}^7C_r \left(\frac{2}{3}\right)^{7-r} \left(\frac{5}{4}\right)^r x^{14-2r} x^{-5r}$

$\therefore T_{r+1} = (-1)^r\ {}^7C_r \left(\frac{2}{3}\right)^{7-r} \left(\frac{5}{4}\right)^r x^{14-7r}$ ...(1)

For coefficient of x^{-7}, put $14 - 7r = -7$

$\Rightarrow 7r = 21 \Rightarrow r = 3$

Put r = 3 in equation (1)

$T_{3+1} = (-1)^3\ {}^7C_3 \left(\frac{2}{3}\right)^4 \left(\frac{5}{4}\right)^3 x^{14-21}$

$= \frac{-7\times6\times5}{1\times2\times3}\left(\frac{2}{3}\right)^4 \left(\frac{5}{4}\right)^3 x^{-7}$

$\therefore$ Coefficient of x^{-7} in $\left(\frac{2x^2}{3} - \frac{5}{4x^5}\right)^7$ is :

$= -35 \times \frac{1}{3^4} \cdot \frac{5^3}{2^2} = \frac{-4375}{324}$

2. Find the term independent of x in the expansion of

(i) $\left(\frac{x^{1/2}}{3} - \frac{4}{x^2}\right)^{10}$ (ii) $\left(\frac{3}{\sqrt[3]{x}} + 5\sqrt{x}\right)^{25}$

(iii) $\left(4x^3+\frac{7}{x^2}\right)^{14}$ (iv) $\left(\frac{2x^2}{5}+\frac{15}{4x}\right)^9$

i) $\left(\frac{x^{1/2}}{3}-\frac{4}{x^2}\right)^{10}$

Sol. The general term in $\left(\frac{x^{1/2}}{3}-\frac{4}{x^2}\right)^{10}$ is

$$T_{r+1}=(-1)^r\,{}^{10}C_r\left(\frac{x^{1/2}}{3}\right)^{10-r}\left(\frac{4}{x^2}\right)^r$$

$$=(-1)^r\,{}^{10}C_r\left(\frac{1}{3}\right)^{10-r}(4)^r\cdot x^{5-\frac{r}{2}}\cdot x^{-2r}$$

$$=(-1)^r\,{}^{10}C_r\left(\frac{1}{3}\right)^{10-r}(4)^r\cdot x^{5-\frac{r}{2}-2r}$$

$$=(-1)^r\,{}^{10}C_r\left(\frac{1}{3}\right)^{10-r}(4)^r\cdot x^{\frac{10-5r}{2}}\quad ...(1)$$

For the term independent of x,

Put $\frac{10-5r}{2}=0\Rightarrow 5r=10\Rightarrow r=2$

Put r = 2 in eq.(1)

$$T_{2+1}=(-1)^2\,{}^{10}C_2\left(\frac{1}{3}\right)^8 4^2\cdot x^0$$

$$T_3=\frac{80}{729}$$

ii) $\left(\frac{3}{\sqrt[3]{x}}+5\sqrt{x}\right)^{25}$

Sol. The general term in $\left(\frac{3}{\sqrt[3]{x}}+5\sqrt{x}\right)^{25}$ is

$$T_{r+1}={}^{25}C_r\left(\frac{3}{\sqrt[3]{x}}\right)^{25-r}(5\sqrt{x})^r$$

$$={}^{25}C_r(3)^{25-r}(5)^r\cdot x^{-1/3(25-r)}x^{r/2}$$

$$={}^{25}C_r(3)^{25-r}(5)^r\cdot x^{-\frac{25}{3}+\frac{r}{3}+\frac{r}{2}}$$

$= {}^{25}C_r (3)^{25-r} (5)^r \cdot x^{-\frac{50+2r+3r}{6}}$ ...(1)

For term independent of x, put

$$\frac{-50+5r}{6} = 0 \Rightarrow 5r = 50 \Rightarrow r = 10$$

Put r = 10 in equation (1),

$T_{10+1} = {}^{25}C_{10} (3)^{15} (5)^{10} x^0$

i.e. $T_{11} = {}^{25}C_{10} (3)^{15} (5)^{10}$

iii) $\left(4x^3 + \frac{7}{x^2}\right)^{14}$

Sol. The general term in $\left(4x^3 + \frac{7}{x^2}\right)^{14}$ is

$$T_{r+1} = {}^{14}C_r (4x^3)^{14-r} \left(\frac{7}{x^2}\right)^r$$

$$= {}^{14}C_r (4)^{14-r} (7)^r x^{42-3r} x^{-2r}$$

$$= {}^{14}C_r (4)^{14-r} (7)^r x^{42-5r} \quad ...(1)$$

For term independent of x,

Put $4x - 5r = 0 \Rightarrow r = 42/5$ which is not an integer.

Hence term independent of x in the given expansion is zero.

iv) $\left(\frac{2x^2}{5} + \frac{15}{4x}\right)^9$

Sol. The general term in $\left(\frac{2x^2}{5} + \frac{15}{4x}\right)^9$ is

$$T_{r+1} = {}^9C_r \left(\frac{2x^2}{5}\right)^{9-r} \left(\frac{15}{4x}\right)^r$$

$$= {}^9C_r \left(\frac{2}{5}\right)^{9-r} \left(\frac{15}{4}\right)^r x^{18-2r} \cdot x^{-r}$$

$$= {}^9C_r \left(\frac{2}{5}\right)^{9-r} \left(\frac{15}{4}\right)^r x^{18-3r} ...(1)$$

For term independent of x,

put $18 - 3r = 0 \Rightarrow r = 6$

Put r = 6 in equation (1)

$$T_{6+1} = {}^9C_6\left(\frac{2}{5}\right)^3\left(\frac{15}{4}\right)^6 x^0 = {}^9C_6 \cdot \frac{2^3}{5^3} \cdot \frac{3^6 \times 5^6}{4^6}$$

$$= \frac{9\times8\times7}{1\times2\times3} \cdot \frac{3^6\times5^6}{4^6} = \frac{3^7\times5^3\times7}{2^7}$$

3. Find the middle term(s) in the expansion of

(i) $\left(\frac{3x}{7} - 2y\right)^{10}$ **(ii)** $\left(4a + \frac{3}{2}b\right)^{11}$

(iii) $(4x^2 + 5x^3)^{17}$ **(iv)** $\left(\frac{3}{a^3} + 5a^4\right)^{20}$

Sol. The middle term in $(x + a)^n$ when n is even is $T_{\left(\frac{n+1}{2}\right)}$, when n is odd, we have two middle terms, i.e. $T_{\left(\frac{n+1}{2}\right)}$ and $T_{\left(\frac{n+3}{2}\right)}$.

i) $\left(\frac{3x}{7} - 2y\right)^{10}$

Sol. n = 10 is even, we have only one middle term.

i.e. $\frac{10}{2} + 1 = 6^{th}$ term

$\therefore$ T_6 in $\left(\frac{3x}{7} - 2y\right)^{10}$ is :

$$= {}^{10}C_5\left(\frac{3x}{7}\right)^5 (-2y)^5 = -({}^{10}C_5)\frac{3^5}{7^5} \cdot 2^5 (xy)^5$$

$$= -{}^{10}C_5\left(\frac{6}{7}\right)^5 x^5y^5$$

ii) $\left(4a + \frac{3}{2}b\right)^{11}$

Sol. Here n = 11 is an odd integer, we have two middle terms, i.e. $\frac{n+1}{2}$ and $\frac{n+3}{2}$ terms

= 7^{th} and 7^{th} terms are middle terms.

T_6 in $\left(4a + \frac{3}{2}b\right)^{11}$ is :

$= {}^{11}C_5(4a)^6\left(\frac{3}{2}b\right)^5 = {}^{11}C_5(4)^6\frac{3^5}{2^5}a^6b^5$

$= \frac{11\times10\times9\times8\times7}{1\times2\times3\times4\times5}2^7\cdot3^5\cdot a^6b^5$

$= 77\times2^8\times3^6\times a^6b^5$

T_7 in $\left(4a+\frac{3}{2}b\right)^{11}$ is :

$= {}^{11}C_6(4a)^5\left(\frac{3}{2}b\right)^6 = {}^{11}C_5(4)^5\frac{3^6}{2^6}a^5b^6$

$= \frac{11\times10\times9\times8\times7}{1\times2\times3\times4\times5}2^4\cdot3^6\cdot a^5b^6$

$= 77\times2^5\times3^7\times a^5b^6$

iii) $(4x^2+5x^3)^{17}$

Sol. $(4x^2+5x^3)^{17} = [x^2(4+5x)]^{17}$

$= x^{34}(4+5x)^{17}$ ...(1)

Consider $(4 + 5x)^{17}$

$\because$ n = 17 is odd positive integer, we have two middle terms. They are 9^{th} and 10^{th} terms are middle terms.

T_9 in $(4 + 5x)^{17}$ is

$= {}^{17}C_8(4)^{17-8}(5x)^8 = {}^{17}C_8(4)^9 5^8x^8$

$\therefore$ T_9 in $(4x^2 + 5x^3)^{17}$ is

$= x^{34}\left[{}^{17}C_8\cdot4^9\cdot5^8\cdot x^8\right] = {}^{17}C_8 4^9\cdot5^8\cdot x^{42}$

T_{10} in $(4 + 5x)^{17}$ is

$= {}^{17}C_9 4^{17-9}\cdot(5x)^9 = {}^{17}C_9\cdot4^8\cdot5^9\cdot x^9$

$\therefore$ T_{10} in $(4x^2+5x^3)^{17}$ is

$= x^{34}\cdot{}^{17}C_9 4^8\cdot5^9\cdot x^9 = {}^{17}C_9\cdot4^8\cdot5^9\cdot x^{43}$

iv) $\left(\frac{3}{a^3}+5a^4\right)^{20}$

Sol. Here n = 20 is even positive integer, we have only one middle term, i.e. $\left(\frac{20+1}{2}\right)$ =

11^{th} term

T_{11} in $\left(\frac{3}{a^3}+5a^4\right)^{20}$ is

$= {}^{20}C_{10}\left(\frac{3}{a^3}\right)^{20-10}(5a^4)^{10}$

$= {}^{20}C_{10}\frac{3^{10}}{a^{30}}\cdot 5^{10}\cdot a^{40} = {}^{20}C_{10}(15)^{10}\cdot a^{10}$

4. Fin the numerically greatest term (s) in the expansion of

i) $(4 + 3x)^{15}$ when x = $\frac{7}{2}$

ii) $(3x + 5y)^{12}$ when x = $\frac{1}{2}$ and y = $\frac{4}{3}$

iii) $(4a – 6b)^{13}$ when a = 3, b = 5

iv) $(3 + 7x)^n$ when x = $\frac{4}{5}$, n = 15

i) $(4 + 3x)^{15}$ when x = $\frac{7}{2}$

Sol. Write $(4 + 3x)^{15} = \left[4\left(1+\frac{3}{4}x\right)\right]^{15}$

$= 4^{15}\left(1+\frac{3}{4}x\right)^{15}$ …(1)

First we find the numerically greatest term in the expansion of $\left(1+\frac{3}{4}x\right)^{15}$

Write X = $\frac{3}{4}x$ and calculate $\frac{(n+1)|x|}{1+|x|}$

Here $|X| = \left(\frac{3}{4}X\right) = \frac{3}{4}\times\frac{7}{2} = \frac{21}{8}$

Now $\frac{(n+1)|x|}{1+|x|} = \frac{15+1}{1+\frac{21}{8}}\cdot\frac{21}{8}$

$= \frac{16\times 21}{29} = \frac{336}{29} = 11\frac{17}{29}$

Its integral part $m = \left[11\frac{17}{29}\right] = 11$

T_{m+1} is the numerically greatest term in the expansion $\left(1+\frac{3}{4}x\right)^{15}$ and

$T_{m+1} = T_{12} = {}^{15}C_{11}\left(\frac{3}{4}x\right)^4 = {}^{15}C_{11}\left(\frac{3}{4}\cdot\frac{7}{2}\right)^{11}$

$\therefore$ Numerically greatest term in $(4 + 3x)^{15}$

$$= 4^{15}\left[{}^{15}C_{11}\left(\frac{21}{8}\right)^{11}\right] = {}^{15}C_4\,\frac{(21)^{11}}{2^3}$$

ii) **$(3x + 5y)^{12}$ when $x = \frac{1}{2}$ and $y = \frac{4}{3}$**

Sol. Write $(3x + 5y)^{12} = \left[3x\left(1+\frac{5y}{3x}\right)\right]^{12}$

$$= 3^{12}x^{12}\left(1+\frac{5}{3}\frac{y}{x}\right)^{12}$$

On comparing $\left(1+\frac{5}{3}\frac{y}{x}\right)^{12}$ with $(1 + x)^n$, we get

n = 17, $x = \frac{5}{3}\cdot\frac{y}{x} = \frac{5}{3}\frac{(4/3)}{(1/2)} = \frac{5}{3}\cdot\frac{8}{3} = \frac{40}{9}$

Now $\frac{(n+1)|x|}{1+|x|} = \frac{(12+1)\left(\frac{40}{9}\right)}{1+\frac{40}{9}}$

$$= \frac{13\times 40}{49} = \frac{520}{49} = 10\frac{30}{49}$$

which is not an integer.

$\therefore\ m = \left[10\frac{30}{49}\right] = 10$

N.G. term in $\left(1+\frac{5y}{3x}\right)^{12}$ is

$T_{m+1} = T_{11} = {}^{12}C_{10}\left(\frac{5}{3}\frac{y}{x}\right)^{10} = {}^{12}C_{10}\left(\frac{5}{3}\times\frac{(4/3)}{(1/2)}\right)^{10}$

$= {}^{12}C_{10}\left(\frac{5}{3}\times\frac{8}{3}\right)^{10} = {}^{12}C_{10}\left(\frac{40}{9}\right)^{10}$

$\therefore$ N.G. term in $(3x + 5y)^{12}$ is

$= 3^{12}\left(\frac{1}{2}\right)^{12}{}^{12}C_{10}\left(\frac{40}{9}\right)^{10}$

$= {}^{12}C_{10}\,\frac{3^{12}}{2^{12}}\,\frac{(2^2)^{10}\times(10)^{10}}{(3^2)^{10}} = {}^{12}C_{10}\left(\frac{3}{2}\right)^2\left(\frac{20}{3}\right)^{10}$

iii) $(4a - 6b)^{13}$ when a = 3, b = 5

Sol. Write $(4a - 6b)^{13} = \left[4a\left(1-\frac{6b}{4a}\right)\right]^{13}$

$$= (4a)^{13}\left(1-\frac{3}{2}\frac{b}{a}\right)^{13}$$

On comparing $\left(1-\frac{3}{2}\frac{b}{a}\right)^{13}$ with $(1 + x)^n$

We get n = 13, $x = \frac{-3}{2}\left(\frac{b}{a}\right)$

$$x = \frac{-3}{2}\times\frac{5}{3} = \frac{-5}{2}$$

Now $\frac{(n+1)|x|}{1+|x|} = \frac{(13+1)\left|\frac{-5}{2}\right|}{1+\left|\frac{-5}{2}\right|} = \frac{14\times\frac{5}{2}}{1+\frac{5}{2}}$

$= \frac{70}{7} = 10$ which is an integer.

Hence we have two numerically greatest terms namely T_{10} and T_{11}.

T_{10} in $\left(1-\frac{3}{2}\frac{b}{a}\right)^{13} = {}^{13}C_9\left|-\frac{3}{2}\cdot\frac{b}{a}\right|^9$

$$= {}^{13}C_9\left(\frac{3}{2}\cdot\frac{5}{3}\right)^9 = {}^{13}C_9\left(\frac{5}{2}\right)^9$$

T_{10} in $(4a - 6b)^{13}$ is

$$= (4a)^{13}\cdot{}^{13}C_9\left(\frac{5}{2}\right)^9 = (4\times3)^{13}\cdot{}^{13}C_9\left(\frac{5}{2}\right)^9$$

$$= {}^{13}C_9(12)^4(12)^9\left(\frac{5}{2}\right)^9 = {}^{13}C_9(12)^4(30)^9$$

T_{11} in $\left(1-\frac{3}{2}\frac{b}{a}\right)^{13}$ is $= {}^{13}C_{10}\left(\frac{-3}{2}\cdot\frac{b}{a}\right)^{10}$

$$= {}^{13}C_{10}\left(\frac{3}{2}\times\frac{5}{3}\right)^{10} = {}^{13}C_{10}\left(\frac{5}{2}\right)^{10}$$

$\therefore$ N.G. term in $(4a - 6b)^{13}$ is

$$= (4a)^{13} \cdot {}^{13}C_{10} \left(\frac{5}{2}\right)^{10} = (4\times 3)^{13} \cdot {}^{13}C_{10} \left(\frac{5}{2}\right)^{10}$$

$$= (12)^{13} \cdot {}^{13}C_{10} \frac{5^{10}}{2^{10}} = {}^{13}C_{10} (12)^3 \cdot (12)^{10} \cdot \frac{5^{10}}{2^{10}}$$

$$= {}^{13}C_{10} (12)^3 (30)^{10}$$

iv) $(3 + 7x)^n$ when $x = \frac{4}{5}$, n = 15

Sol. Write $(3 + 7x)^n = \left[3\left(1+\frac{7}{3}x\right)\right]^n$

$$= 3^n \left(1+\frac{7}{3}x\right)^n$$

Now we first find N.G. term in $\left(1+\frac{7}{3}x\right)^n$

On comparing with $(1 + x)^n$, we get

$X = \frac{7}{3}x = \frac{7}{3}\times\frac{4}{5} = \frac{28}{15} \Rightarrow n = 15$

Now $\frac{(n+1)|x|}{1+|x|} = \frac{(15+1)\left(\frac{28}{15}\right)}{1+\frac{28}{15}}$

$$= \frac{16\times 28}{43} = \frac{448}{43} = 10\frac{18}{43}$$

Its integral part (m) = 10

$\therefore T_{m+1} = T_{11}$ is the N.G. term

T_{11} in $\left(1+\frac{7}{3}x\right)^{15} = {}^{15}C_{10}\left(\frac{7}{3}x\right)^{10}$

$$= {}^{15}C_{10}\left(\frac{7}{3}\times\frac{4}{5}\right)^{10} = {}^{15}C_{10}\left(\frac{28}{15}\right)^{10}$$

$\therefore$ N.G. term in $(3 + 7x)^n$ is

$$= 3^{15} \cdot {}^{15}C_{10}\left(\frac{28}{15}\right)^{10} = {}^{15}C_{10} \cdot 3^5 \left(\frac{28}{5}\right)^{10}$$

5. Prove the following

i) $2 \cdot C_0 + 5 \cdot C_1 + 8 \cdot C_2 + ... + (3n+2) \cdot C_n$

$$= (3n+4) \cdot 2^{n-1}$$

Sol. Let $S = 2 \cdot C_0 + 5 \cdot C_1 + 8 \cdot C_2 + ...$

$$...+(3n-1)\cdot C_{n-1}+(3n+2)C_n$$

$\because C_n = C_0, C_{n-1} = C_1 ...$

$S = (3n+2)C_0 + (3n-1)C_1 + (3n-4)C_2 + ...$

$$....+5C_{n-1}+2\cdot C_n$$

Add ———————————————————

$2S = (3n+4)C_0 + (3n+4)C_1 + (3n+4)C_2 + ... + (3n+4)C_n$

$= (3n+4)(C_0 + C_1 + C_2 + ... + C_n) = (3n+4)2^n$

$\therefore S = (3n+4)\cdot 2^{n-1}$

ii) $C_0 - 4\cdot C_1 + 7\cdot C_2 - 10\cdot C_3 + ... = 0$

Sol. 1, 4, 7, 10 … are in A.P.

$T_{n+1} = a + nd = 1 + n(3) = 3n + 1$

$\therefore\ C_0 - 4\cdot C_1 + 7\cdot C_2 - 10\cdot C_3 + ...(n+1)\text{terms}$

$= C_0 - 4\cdot C_1 + 7\cdot C_2 - 10\cdot C_3 + ... + (-1)^n(3n+1)C_n$

$= \sum_{r=0}^{n}(-1)^r(3r+1)C_r$

$= \sum_{r=0}^{n}\left\{(-1)^r(3r)C_r + (-1)^r C_r\right\}$

$= 3\cdot\sum_{r=0}^{n}(-1)^r r\cdot C_r + \sum_{r=0}^{n}(-1)^r\cdot C_r$

$= 3(0) + 0 = 0$

$\therefore C_0 - 4\cdot C_1 + 7\cdot C_2 - 10\cdot C_3 + ... = 0$

iii) $\frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \frac{C_7}{8} + ... = \frac{2^n - 1}{n+1}$

Sol.

$\frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \frac{C_7}{8} + ...$

$= \frac{{}^nC_1}{2} + \frac{{}^nC_3}{4} + \frac{{}^nC_5}{6} + \frac{{}^nC_7}{8} + ...$

$= \frac{n}{2} + \frac{n(n-1)(n-2)}{4\times 3!} + \frac{n(n-1)(n-2)(n-3)(n-4)}{6\times 5!} + ...$

$= \frac{1}{n+1}\left[\frac{(n+1)n}{2!} + \frac{(n+1)n(n-1)(n-2)}{4!} + ... \frac{(n+1)n(n-1)(n-2)(n-3)(n-4)}{6!} + ...\right]$

$= \frac{1}{n+1}\left[{}^{(n+1)}C_2 + {}^{(n+1)}C_4 + {}^{(n+1)}C_6 + ...\right]$

$$= \frac{1}{n+1}\left[{}^{(n+1)}C_0 + {}^{(n+1)}C_2 + {}^{(n+1)}C_4 + \ldots - {}^{(n+1)}C_0\right] = \frac{1}{n+1}\left[2^n - 1\right] = \frac{2^n - 1}{n+1}$$

$$\therefore \ \frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \frac{C_7}{8} + \ldots = \frac{2^n - 1}{n+1}$$

iv) $C_0 + \frac{3}{2}C_1 + \frac{9}{3}C_2 + \frac{27}{4}C_3 + \ldots + \frac{3^n}{n+1}C_n = \frac{4^{n+1} - 1}{3(n+1)}$

Sol. Let S =

$$C_0 + \frac{3}{2}C_1 + \frac{3^2}{3}C_2 + \frac{3^3}{4}C_3 + \ldots + C_n \frac{3^n}{n+1} \ldots (1)$$

$$\Rightarrow 3\ S =$$

$$C_0 \cdot 3 + \frac{3^2}{2}C_1 + \frac{3^3}{3}C_2 + \frac{3^4}{4}C_3 + \ldots + C_n \frac{3^{n+1}}{n+1} \ldots (2)$$

$$\Rightarrow (n+1)3 \cdot S$$

$$= (n+1)C_0 \cdot 3 + (n+1)C_1 \cdot \frac{3^2}{2} + (n+1)C_2 \cdot \frac{3^3}{3} + (n+1)C_3 \cdot \frac{3^4}{3} + \ldots + (n+1)C_n \cdot \frac{3^{n+1}}{n+1}$$

$$\Rightarrow (n+1)3 \cdot S$$

$$= {}^{(n+1)}C_1 \cdot 3 + {}^{(n+1)}C_2 \cdot 3^2 + {}^{(n+1)}C_3 \cdot 3^3 + \ldots\ldots + {}^{(n+1)}C_{n+1} \cdot 3^{n+1}$$

$$= (1+3)^{n+1} - {}^{(n+1)}C_0 = 4^{n+1} - 1$$

$$\therefore S = \frac{4^{n+1} - 1}{3(n+1)}$$

v) $C_0 + 2 \cdot C_1 + 4 \cdot C_2 + 8 \cdot C_3 + \ldots + 2^n \cdot C_n = 3^n$

Sol. L.H.S. $= C_0 + 2 \cdot C_1 + 4 \cdot C_2 + 8 \cdot C_3 + \ldots + 2^n \cdot C_n$

$= C_0 + C_1(2) + C_2(2^2) + C_3(2^3) + \ldots + C_n(2^n)$

$= (1 + 2)^n = 3^n$

Note : $(1+x)^n = C_0 + C_1 \cdot x + C_2 x^2 + \ldots + C_n x^n$

6. Using binomial theorem, prove that $50^n - 49n - 1$ is divisible by 49^2 for all positive integers n.

Sol. $50^n - 49n - 1 = (49 + 1)^n - 49n - 1$

$$= [{}^nC_0(49)^n + {}^nC_1(49)^{n-1} + {}^nC_2(49)^{n-2} + \ldots + {}^nC_{n-2}(49)^2 + {}^nC_{n-1}(49) + {}^nC_n(1)] - 49n - 1$$

$$= (49)^n + {}^nC_1(49)^{n-1} + {}^nC_2(49)^{n-2} + \ldots + {}^nC_{n-2}(49)^2 + (n)(49) + 1 - 49n - 1$$

$$= 49^2[(49)^{n-2} + {}^nC_1(49)^{n-3} + {}^nC_2(49)^{n-4} + \ldots + \ldots\ldots + \ldots\ldots + {}^nC_{n-2}]$$

$= 49^2$ [a positive integer]

Hence $50^n - 49n - 1$ is divisible by 49^2 for all positive integers of n.

7. Using binomial theorem, prove that
$5^{4n} + 52n - 1$ is divisible by 676 for all positive integers n.

Sol. $5^{4n} + 52n - 1 = (5^2)^{2n} + 52n - 1$

$= (25)^{2n} + 52n - 1$

$= (26-1)^{2n} + 52n - 1$

$= [\,{}^{2n}C_0(26)^{2n} - {}^{2n}C_1(26)^{2n-1} + {}^{2n}C_2(26)^{2n-2} - ..... + {}^{2n}C_{2n-2}(26)^2 - {}^{2n}C_{2n-1}(26) + {}^{2n}C_{2n}(1)] + 52n - 1$

$= {}^{2n}C_0(26)^{2n} - {}^{2n}C_1(26)^{2n-1} + {}^{2n}C_2(26)^{2n-2} - ..... + {}^{2n}C_{2n-2} - 2n(26) + 1 + 52n - 1$

$= (26)^2[\,{}^{2n}C_0(26)^{2n-2} - {}^{2n}C_1(26)^{2n-3} + {}^{2n}C_2(26)^{2n-4} + ... + {}^{2n}C_{2n-2}]$

is divisible by $(26)^2 = 676$

$\therefore 5^{4n} + 52n - 1$ is divisible by 676, for all positive integers n.

8. If $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + ... + a_{2n}x^{2n}$, then prove that

i) $a_0 + a_1 + a_2 + ... + a_{2n} = 3^n$

ii) $a_0 + a_2 + a_4 + ... + a_{2n} = \dfrac{3^n + 1}{2}$

iii) $a_1 + a_3 + a_5 + ... + a_{2n-1} = \dfrac{3^n - 1}{2}$

iv) $a_0 + a_3 + a_6 + a_9 + ... = 3^{n-1}$

Sol. $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + ... + a_{2n}x^{2n}$

Put x = 1,

$\therefore\ a_0 + a_1 + a_2 + \ldots + a_{2n} = (1+1+1)^n = 3^n$ …(1)

Put x = –1,

$a_0 - a_1 + a_2 - \ldots + a_{2n} = (1-1+1)^n = 1$ …(2)

i) $a_0 + a_1 + a_2 + ... + a_{2n} = 3^n$

ii) $(1) + (2) \Rightarrow 2(a_0 + a_2 + a_4 + ... + a_{2n}) = 3^n + 1$

$\therefore\ a_0 + a_2 + a_4 + ... + a_{2n} = \dfrac{3^n + 1}{2}$

iii) $(1) - (2) \Rightarrow\ 2(a_1 + a_3 + a_5 + ... + a_{2n-1}) = 3^n - 1$

$\therefore\ a_1 + a_3 + a_5 + ... + a_{2n-1} = \dfrac{3^n - 1}{2}$

iv) Put x = 1

$a_0 + a_1 + a_2 + ... + a_{2n} = 3^n$ …(a)

Hint : $1 + \omega + \omega^2 = 0$; $\omega^3 = 1$

Put $x = \omega$

$a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3 + ... + a_{2n}\omega^{2n} = 0$ …(b)

Put $x = \omega^2$

$a_0 + a_1\omega^2 + a_2\omega^4 + a_3\omega^6 + ... + a_{2n}\omega^{4n} = 0$ …(c)

Adding (a), (b), (c)

$3a_0 + a_1(1+\omega+\omega^2) + a_2(1+\omega^2+\omega^4)$

$+a_3(1+\omega^3+\omega^6) + ... + a_{2n}(1+\omega^{2n}+\omega^{4n}) = 3^n$

$\Rightarrow 3a_0 + a_1(0) + a_2(0) + 3a_3 + ... + ... = 3^n$

$\therefore a_0 + a_3 + a_6 + a_9 + ... = \frac{3^n}{3} = 3^{n-1}$

9. If the coefficients of $(2r + 4)^{th}$ term and $(3r + 4)^{th}$ term in the expansion of $(1 + x)^{21}$ are equal, find r.

Sol. T_{2r+4} in $(1 + x)^{21}$ is $= {}^{21}C_{2r+3}(x)^{2r+3}$ …(1)

T_{3r+4} in $(1 + x)^{21}$ is $= {}^{21}C_{3r+3}(x)^{3r+3}$ ...(2)

$\Rightarrow$ Coefficients are equal

$\Rightarrow {}^{21}C_{2r+3} = {}^{21}C_{3r+3}$

$\Rightarrow 21 = (2r + 3) + (3r + 3)$ (or) $2r + 3 = 3r + 3$

$\Rightarrow 5r = 15 \Rightarrow r = 3$ (or) $r = 0$

Hence r = 0, 3.

III.

1. If 36, 84, 126 are three successive binomial coefficients in the expansion of $(1 + x)^n$, find n.

Sol. Let ${}^nC_{r-1}$, ${}^nC_r$, ${}^nC_{r+1}$ are three successive binomial coefficients in the expansion of $(1 + x)^n$, find n.

Then ${}^nC_{r-1} = 36$, ${}^nC_r = 84$ and ${}^nC_{r+1} = 126$

Now $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{84}{36} \Rightarrow \frac{n-r+1}{r} = \frac{7}{3}$

$3n - 3r + 3 = 7r \Rightarrow 3n = 10r - 3$

$\Rightarrow \frac{3n+3}{10} = r$ ...(1)

$\Rightarrow \frac{{}^nC_{r+1}}{{}^nC_r} = \frac{126}{84} \Rightarrow \frac{n-r}{r+1} = \frac{3}{2}$

$\Rightarrow 2n - 2r = 3r + 3$

$\Rightarrow 2n = 5r + 3$ ...(2)

$\Rightarrow 2n = 5\left(\frac{3n+3}{10}\right) + 3$ from (1)

$\Rightarrow 2n = \frac{3n+3+6}{2} \Rightarrow 4n = 3n + 9 \Rightarrow n = 9$

2.If the 2^{nd}, 3^{rd} and 4^{th} terms in the expansion of $(a + x)^n$ are respectively 240, 720, 1080, find a, x, n.

Sol. $T_2 = 240 \Rightarrow {}^nC_1\, a^{n-1}\, x = 240$ …(1)

$T_3 = 720 \Rightarrow {}^nC_2\, a^{n-2}\, x^2 = 720$ ...(2)

$T_4 = 1080 \Rightarrow {}^nC_3\, a^{n-3}\, x^3 = 1080$ ...(3)

$$\frac{(2)}{(1)} \Rightarrow \frac{{}^{n}C_2 a^{n-2}x^2}{{}^{n}C_1 a^{n-1}x} = \frac{720}{240}$$

$$\Rightarrow \frac{n-1}{2}\frac{x}{a} = 3 \Rightarrow (n-1)x = 6a \quad ...(4)$$

$$\frac{(3)}{(2)} \Rightarrow \frac{{}^{n}C_3 a^{n-3}x^3}{{}^{n}C_2 a^{n-2}x^2} = \frac{1080}{720}$$

$$\Rightarrow \frac{n-2}{3}\frac{x}{a} = \frac{3}{2}$$

$$\Rightarrow 2(n-2)x = 9a \quad ...(5)$$

$$\frac{(4)}{(5)} \Rightarrow \frac{(n-1)x}{2(n-2)x} = \frac{6a}{9a} \Rightarrow \frac{n-1}{2n-4} = \frac{2}{3}$$

$$\Rightarrow 3n-3 = 4n-8 \Rightarrow n = 5$$

From (4), $(5-1)x = 6a \Rightarrow 4x = 6a$

$$\Rightarrow x = \frac{3}{2}a$$

Substitute $x = \frac{3}{2}a$, $n = 5$ in (1)

$${}^{5}C_1 \cdot a^4 \cdot \frac{3}{2}a = 240 \Rightarrow 5 \times \frac{3}{2}a^5 = 240$$

$$a^5 = \frac{480}{15} = 32 = 2^5$$

$$\therefore a = 2, x = \frac{3}{2}a = \frac{3}{2}(2) = 3$$

$$\therefore a = 2, x = 3, n = 5$$

3. If the coefficients of r^{th}, $(r+1)^{th}$ and $(r+2)^{th}$ terms in the expansion of $(1 + x)^{th}$ are in A.P. then show that $n^2 - (4r + 1)n + 4r^2 - 2 = 0$.

Sol. Coefficient of $T_r = {}^{n}C_{r-1}$

Coefficient of $T_{r+1} = {}^{n}C_r$

Coefficient of $T_{r+2} = {}^{n}C_{r+1}$

Given ${}^{n}C_{r-1}$, ${}^{n}C_r$, ${}^{n}C_{r+1}$ are in A.P.

$\Rightarrow 2\,{}^{n}C_r = {}^{n}C_{r-1} + {}^{n}C_{r+1}$

$$\Rightarrow 2\frac{n!}{(n-r)!r!}=\frac{n!}{(n-r+1)!(r-1)!}+\frac{n!}{(n-r-1)!(r+1)!}$$

$$\Rightarrow \frac{2}{(n-r)r}=\frac{1}{(n-r+1)(n-r)}+\frac{1}{(r+1)r}$$

$$\Rightarrow \frac{1}{n-r}\left[\frac{2}{r}-\frac{1}{n-r+1}\right]=\frac{1}{(r+1)r}$$

$$\Rightarrow \frac{1}{n-r}\left[\frac{2n-2r+2-r}{r(n-r+1)}\right]=\frac{1}{r(r+1)}$$

$$\Rightarrow (2n-3r+2)(r+1)=(n-r)(n-r+1)$$

$$\Rightarrow 2nr+2n-3r^2-3r+2r+2$$

$$=n^2-2nr+r^2+n-r$$

$$\Rightarrow n^2-4nr+4r^2-n-2=0$$

$$\therefore n^2-(4r+1)n+4r^2-2=0$$

4. Find the sum of the coefficients of x^{32} and x^{-18} in the expansion of $\left(2x^3-\frac{3}{x^2}\right)^{14}$.

Sol. The general term in $\left(2x^3-\frac{3}{x^2}\right)^{14}$ is :

$$T_{r+1}={}^{14}C_r(2x^3)^{14-r}\left(-\frac{3}{x^2}\right)^r$$

$$=(-1)^r\,{}^{14}C_r(2)^{14-r}\cdot(3)^r\cdot x^{42-r}\cdot x^{-2r}$$

$$=(-1)^r\cdot{}^{14}C_r\,2^{14-r}(3)^r x^{42-5r} \qquad ...(1)$$

From coefficients of x^{32},

Put $42-5r=32\Rightarrow 5r=10\Rightarrow r=2$

Put r = 2 in equation (1)

$$T_3=(-1)^2\,{}^{14}C_2(2)^{12}(3)^2\cdot x^{42-10}$$

$$={}^{14}C_2(2)^{12}(3)^2\cdot x^{32}$$

Coefficient of x^{32} is ${}^{14}C_2(2)^{12}(3)^2$ …(2)

For coefficient of x^{-18}

Put $42-5r=-18\Rightarrow 5r=60\Rightarrow r=12$

Put r = 12 in equation (1)

$$T_{13}=(-1)^{12}\,{}^{14}C_{12}(2)^2(3)^{12}\cdot x^{42-60}$$

$$={}^{14}C_{12}(2)^2(3)^{12}\cdot x^{-18}$$

$\therefore$ Coefficient of x^{-18} is ${}^{14}C_{12}(2)^2\,3^{12}$

Hence sum of the coefficients of x^{32} and x^{-18} is ${}^{14}C_2(2)^{12}(3)^2+{}^{14}C_{12}(2)^2(3)^{12}$.

5. If P and Q are the sum of odd terms and the sum of even terms respectively in the expansion of $(x + a)^n$ then prove that
(i) $P^2 - Q^2 = (x^2 - a^2)^n$
(ii) $4PQ = (x + a)^{2n} - (x - a)^{2n}$

Sol. $(x+a)^n = {}^nC_0x^n + {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 +$

$${}^nC_3x^{n-3}a^3 + ... + {}^nC_{n-1}xa^{n-1} + {}^nC_na^n$$

$$= ({}^nC_0x^n + {}^nC_2x^{n-2}a^2 + {}^nC_4x^{n-4}a^4 + ...) + ({}^nC_1x^{n-1}a + {}^nC_3x^{n-3}a^3 + {}^nC_5x^{n-5}a^5 + ...)$$

$$= P + Q$$

$$(x-a)^n = {}^nC_0x^n - {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 - {}^nC_3x^{n-3}a^3 + ... + {}^nC_n(-1)^n a^n$$

$$= ({}^nC_0x^n + {}^nC_2x^{n-2}a^2 + {}^nC_4x^{n-4}a^4 + ...) - ({}^nC_2x^{n-1}a + {}^nC_3x^{n-3}a^3 + {}^nC_5x^{n-5}a^5 + ...)$$

$$= P - Q$$

i) $P^2 - Q^2 = (P+Q)(P-Q)$
$= (x + a)^n (x - a)^n$
$= [(x + a)(x - a)]^n = (x^2 - a^2)^n$

ii) $4PQ = (P + Q)^2 - (P - Q)^2$
$= [(x + a)^n]^2 - [(x - a)^n]^2$
$= (x + a)^{2n} - (x - a)^{2n}$

6. If the coefficients of 4 consecutive terms in the expansion of $(1 + x)^n$ are a_1, a_2, a_3, a_4 respectively, then show that

$$\frac{a_1}{a_1+a_2} + \frac{a_3}{a_3+a_4} = \frac{2a_2}{a_2+a_3}$$

Sol. Given a_1, a_2, a_3, a_4 are the coefficients of 4 consecutive terms in $(1 + x)^n$ respectively.
Let $a_1 = {}^nC_{r-1}$, $a_2 = {}^nC_r$, $a_3 = {}^nC_{r+1}$, $a_4 = {}^nC_{r+2}$

L.H.S. : $\dfrac{a_1}{a_1+a_2} + \dfrac{a_3}{a_3+a_4}$

$$= \frac{a_1}{a_1\left(1+\frac{a_2}{a_1}\right)} + \frac{a_3}{a_3\left(1+\frac{a_4}{a_3}\right)}$$

$$= \frac{1}{1+\frac{{}^nC_r}{{}^nC_{r-1}}} + \frac{1}{1+\frac{{}^nC_{r+2}}{{}^nC_{r+1}}}$$

$$= \frac{1}{1+\frac{n-r+1}{r}} + \frac{1}{1+\frac{n-r-1}{r+2}}$$

$$= \frac{r}{n+1} + \frac{r+2}{r+2+n-r-1} = \frac{r+r+2}{n+1} = \frac{2(r+1)}{n+1}$$

R.H.S : $\frac{2a_2}{a_2+a_3}=\frac{2a_2}{a_2\left(1+\frac{a_3}{a_2}\right)}$

$\frac{2}{1+\frac{{}^{n}C_{r+1}}{{}^{n}C_{r}}}=\frac{2}{1+\frac{n-r}{r+1}}=\frac{2(r+1)}{n+1}=\text{L.H.S}$

$\therefore \frac{a_1}{a_1+a_2}+\frac{a_3}{a_3+a_4}=\frac{2a_2}{a_2+a_3}$

7. Prove that

$({}^{2n}C_0)^2-({}^{2n}C_1)^2+({}^{2n}C_2)^2-({}^{2n}C_3)^2+...+({}^{2n}C_{2n})^2=(-1)^n\ {}^{2n}C_n$

Sol. $(x+1)^{2n}={}^{2n}C_0x^{2n}+{}^{2n}C_1x^{2n-1}+\ {}^{2n}C_2x^{2n-2}+...+{}^{2n}C_{2n}...(1)$

$(x-1)^{2n}={}^{2n}C_0-{}^{2n}C_1x+{}^{2n}C_2x^2+...+{}^{2n}C_{2n}x^{2n}\qquad ...(2)$

Multiplying eq. (1) and (2), we get

$({}^{2n}C_0x^{2n}+{}^{2n}C_1x^{2n-1}+{}^{2n}C_2x^{2n-2}+...+{}^{2n}C_{2n})$

$({}^{2n}C_0-{}^{2n}C_1x+{}^{2n}C_2x^2+...+{}^{2n}C_{2n}x^{2n})$

$=(x+1)^{2n}(1-x)^{2n}=[(1+x)(1-x)]^{2n}$

$=(1-x^2)^{2n}$

$=\sum_{r=0}^{2n}{}^{2n}C_r(-x^2)^r$

Equating the coefficients of x^{2n}

$({}^{2n}C_0)^2-({}^{2n}C_1)^2+({}^{2n}C_2)^2-({}^{2n}C_3)^2+....+({}^{2n}C_{2n})^2=(-1)^n\ {}^{2n}C_n$

8. Prove that

$(C_0+C_1)(C_1+C_2)(C_2+C_3)...(C_{n-1}+C_n)=\frac{(n+1)^n}{n!}\cdot C_0\cdot C_1\cdot C_2\cdot...C_n$

Sol. $(C_0+C_1)(C_1+C_2)(C_2+C_3)...(C_{n-1}+C_n)=$

$$= C_0\left(1+\frac{C_1}{C_0}\right)\cdot C_1\left(1+\frac{C_2}{C_1}\right)...C_{n-1}\left(1+\frac{C_n}{C_{n-1}}\right)$$

$$=\left(1+\frac{{}^nC_1}{{}^nC_0}\right)\left(1+\frac{{}^nC_2}{{}^nC_1}\right).....\left(1+\frac{{}^nC_n}{{}^nC_{n-1}}\right)C_0C_1C_2...C_{n-1}$$

$$=\left(1+\frac{n}{1}\right)\left(1+\frac{n-1}{2}\right)...\left(1+\frac{1}{n}\right)$$

$$C_n\cdot C_1\cdot C_2\cdot ...C_{n-1}[C_0 = C_n]$$

$$=\left(\frac{1+n}{1}\right)\left(\frac{1+n}{2}\right).....\left(\frac{1+n}{n}\right)C_1\cdot C_2\cdot ...C_{n-1}\cdot C_n$$

$$=\frac{(1+n)^n}{n!}C_1C_2...C_n$$

$$\therefore (C_0+C_1)(C_1+C_2)(C_2+C_3)...(C_{n-1}+C_n)=\frac{(n+1)^n}{n!}\cdot C_0\cdot C_1\cdot C_2\cdot ...C_n$$

9. Find the term independent of x in $(1+3x)^n\left(1+\frac{1}{3x}\right)^n$**.**

Sol. $(1+3x)^n\left(1+\frac{1}{3x}\right)^n = (1+3x)^n\left(\frac{3x+1}{3x}\right)^n$

$$=\left(\frac{1}{3x}\right)^n(1+3x)^{2n}=\frac{1}{3^n\cdot x^n}\sum_{r=0}^{2n}({}^{2n}C_r)(3x)^r$$

The term independent of x in

$(1+3x)^n\left(1+\frac{1}{3x}\right)^n$ is $\frac{1}{3^n}({}^{2n}C_n)3^n = {}^{2n}C_n$

10. If $(1+3x-2x^2)^{10}=a_0+a_1x+a_2x^2+...\ +a_{20}x^{20}$ **then prove that**

i) $a_0+a_1+a_2+...+a_{20}=2^{10}$

ii) $a_0-a_1+a_2-a_3+...+a_{20}=4^{10}$

Sol. $(1+3x-2x^2)^{10}=a_0+a_1x+a_2x^2+...+a_{20}x^{20}$

i) Put x = 1

$(1+3-2)^{10}=a_0+a_1+a_2+...+a_{20}$

$\therefore a_0+a_1+a_2+...+a_{20}=2^{10}$

ii) Put x = –1

$(1-3-2)^{10}=a_0-a_1+a_2+...+a_{20}$

$\therefore a_0-a_1+a_2-a_3+...+a_{20}=(-4)^{10}=4^{10}$

11. If R, n are positive integers, n is odd, 0 < F < 1 and if $(5\sqrt{5}+11)^n = R+F$, then prove that
i) R is an even integer and
ii) (R + F)F = 4^n.

Sol. i) Since R, n are positive integers, 0 < F < 1 and $(5\sqrt{5}+11)^n = R+F$

Let $(5\sqrt{5}-11)^n = f$

Now, $11 < 5\sqrt{5} < 12 \Rightarrow 0 < 5\sqrt{5}-11 < 1$

$\Rightarrow 0 < (5\sqrt{5}-11)^n < 1 \Rightarrow 0 < f < 1$

$\Rightarrow 0 > -f > -1$

$\therefore -1 < -f < 0$

$R + F - f = (5\sqrt{5}+11)^n - (5\sqrt{5}-11)^n$

$$= \left[\begin{array}{l} {}^nC_0(5\sqrt{5})^n + {}^nC_1(5\sqrt{5})^{n-1}(11) + \\ \qquad {}^nC_2(5\sqrt{5})^{n-2}(11)^2 + \ldots + {}^nC_n(11)^n \end{array}\right]$$

$$- \left[\begin{array}{l} {}^nC_0(5\sqrt{5})^n - {}^nC_1(5\sqrt{5})^{n-1}(11) + \\ \qquad {}^nC_2(5\sqrt{5})^{n-2}(11)^2 + \ldots + {}^nC_n(-11)^n \end{array}\right]$$

$$= 2\left[{}^nC_1(5\sqrt{5})^{n-1}(11) + {}^nC_3(5\sqrt{5})^{n-3}(11)^2 + \ldots\right]$$

= 2k where k is an integer.

$\therefore$ R + F – f is an even integer.

$\Rightarrow$ F – f is an integer since R is an integer.

But 0 < F < 1 and $-1 < -f < 0 \Rightarrow -1 < F - f < 1$

$\therefore$ F – f = 0 $\Rightarrow$ F = f

$\therefore$ R is an even integer.

ii) (R + F)F = (R + F)f, $\because$ F = f

$= (5\sqrt{5}+11)^n(5\sqrt{5}-11)^n$

$= \left[(5\sqrt{5}+11)(5\sqrt{5}-11)\right]^n = (125-121)^n = 4^n$

$\therefore$ $(R + F)F = 4^n$.

12. If I, n are positive integers, 0 < f < 1 and if $(7+4\sqrt{3})^n = I+f$, then show that (i) I is an odd integer and (ii) (I + f)(I – f) = 1.

Sol. Given I, n are positive integers and

$(7+4\sqrt{3})^n = I+f$, 0 < f < 1

Let $7-4\sqrt{3} = F$

Now $6 < 4\sqrt{3} < 7 \Rightarrow -6 > -4\sqrt{3} > -7$

$\Rightarrow 1 > 7-4\sqrt{3} > 0 \Rightarrow 0 < (7-4\sqrt{3})^n < 1$

$\therefore$ 0 < F < 1

$1 + f + F = (7+4\sqrt{3})^n(7-4\sqrt{3})^n$

$$= \begin{bmatrix} {}^{n}C_0 7^n + {}^{n}C_1 7^{n-1}(4\sqrt{3}) + {}^{n}C_2 7^{n-2}(4\sqrt{3})^2 \\ \qquad\qquad\qquad\qquad +\ldots + {}^{n}C_n (4\sqrt{3})^n \end{bmatrix}$$

$$= \begin{bmatrix} {}^{n}C_0 7^n - {}^{n}C_1 7^{n-1}(4\sqrt{3}) + {}^{n}C_2 7^{n-2}(4\sqrt{3})^2 \\ \qquad\qquad\qquad\qquad +\ldots + {}^{n}C_n (-4\sqrt{3})^n \end{bmatrix}$$

$= 2\left[{}^{n}C_0 7^n + {}^{n}C_2 7^{n-2}(4\sqrt{3})^2 \ldots\ldots \right]$

$= 2k$ where k is an integer.

$\therefore$ 1 + f + F is an even integer.

$\Rightarrow$ f + F is an integer since I is an integer.

But $0 < f < 1$ and $0 < F < 1 \Rightarrow f + F < 2$

$\therefore$ f + F = 1 …(1)

$\Rightarrow$ I + 1 is an even integer.

$\therefore$ I is an odd integer.

(I + f)(I – f) = (I + f)F, by (1)

$$= (7+4\sqrt{3})^n (7-4\sqrt{3})^n$$

$$= \left[(7+4\sqrt{3})(7-4\sqrt{3}) \right]^n$$

$$= (49-48)^n = 1$$

13. If n is a positive integer, prove that $\sum_{r=1}^{n} r^3 \left(\frac{{}^{n}C_r}{{}^{n}C_{r-1}} \right)^2 = \frac{(n)(n+1)^2(n+2)}{12}$**.**

Sol. $\sum_{r=1}^{n} r^3 \left(\frac{{}^{n}C_r}{{}^{n}C_{r-1}} \right)^2 = \sum_{r=1}^{n} r^3 \left(\frac{n-r+1}{r} \right)^2$

$$= \sum_{r=1}^{n} r(n-r+1)^2 = \sum_{r=1}^{n} r[(n+1)^2 - 2(n+1)r + r^2]$$

$$= (n+1)^2 \Sigma r - 2(n+1)\Sigma r^2 + \Sigma r^3$$

$$= (n+1)^2 \frac{(n)(n+1)}{2} - 2(n+1)\frac{(n)(n+1)(2n+1)}{6} + \frac{n^2(n+1)^2}{4}$$

$$= \frac{(n+1)^2}{2}\left[n(n+1) - \frac{2n(2n+1)}{3} + \frac{n^2}{2} \right]$$

$$= \frac{(n+1)^2}{2}\left[\frac{6n^2+6n-8n^2-4n+3n^2}{6} \right]$$

$$= \frac{(n+1)^2}{2}\left[\frac{n^2+2n}{6} \right] = \frac{n(n+1)^2(n+2)}{12}$$