SIMPLE HARMONIC MOTION

1 .S.H.M.EQUATION

POINTS TO REMEMBER

1. Any motion repeated at regular intervals of time is called periodic motion.
   Ex. Earth revolution around it self, simple pendulum.

2. If there is retracement of the path, it is called harmonic motion.
   Ex: Simple pendulum.

3. Every harmonic is periodic. But every periodic is not harmonic.
4. In SHM acceleration is directly proportional to the displacement of the particle from the fixed point, and the acceleration is always directed towards the fixed point in the path of the body.

   $a\alpha - y \qquad \Rightarrow a = -\omega^2 y$ (Or) $\frac{d^2y}{dt^2} + \omega^2 y = 0$

   This is the differential equation of a body executing SHM.

5. Displacement of a particle executing SHM is given by

   $y = A\sin(\omega t \pm \phi) = A\sin(2\pi nt \pm \phi) = A\sin\frac{2\pi}{\lambda}(vt \pm x)$

   $y_{max} = A =$ Amplitude

6. Velocity, $(v) = A\omega\cos(\omega t \pm \phi)$

   $V_{max} = A\omega$ And $V = \omega\sqrt{A^2 - y^2}$

7. Acceleration (a) = $-A\omega^2\sin(\omega t \pm \phi)$

   $a = -\omega^2 y$ (or) $a\alpha - y$

   $a_{max} = A\omega^2$

8. $\omega = \sqrt{\frac{a}{y}} = \sqrt{\frac{\text{Acceleration}}{\text{Displacement}}}$

   Time period = $2\pi\sqrt{\frac{\text{Displacement}}{\text{acceleration}}}$

LONG ANSWER QUESTIONS

1. Define Simple Harmonic Motion?

Show that the projection of uniform circular motion on any diameter is simple harmonic.

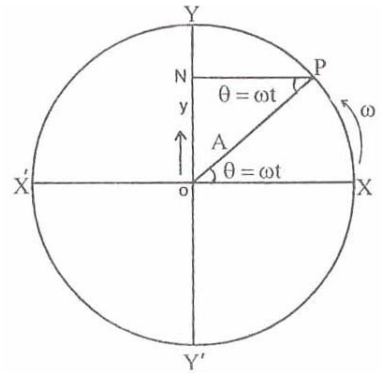
Obtain the equation of simple harmonic motion of a particle whose amplitude is 0.04m and whose frequency is 50 Hz. The initial phase is $\pi/3$.(March2010, March2009)

Ans. **Simple Harmonic Motion:** A body is said to be in simple harmonic motion such that, at any point its acceleration is directed always towards the mean position and it is directly proportional to its displacement from the mean position.

Consider a particle moving along a circle of radius A with uniform angular velocity ω. Let O be the centre of the circle and XX^1 and YY^1 are two mutually perpendicular diameters of the circle as shown in figure. Let the particle is at P at any instant of time t Let PN be the normal drawn on to the diameter YY^1 from P. As P moves on the circumference of the circle, N moves on the diameter YY^1. Let the angular displacement of the particle P is $\angle XOP = \theta$

The displacement of N with respect to the fixed point O in the path is given by

ON = Y = $A\sin\theta$ (or) $Y = A\sin\omega t$ $\left(\because \theta = \omega t\right)$ ..... (1)



The acceleration of N $\left(a_N\right)$ is equal to the component of the centripetal acceleration of P parallel to the diameter YY^1.

The centripetal acceleration of P along PO is given by

$a_p = A\omega^2$.....(2)

The centripetal acceleration parallel to NO.

$a_N = a_P \sin\theta$.....(3)

From equations (2) and (3) , we get,

$a_N = A\omega^2 \sin\omega t$

Using equation (1) , $a_N = -\omega^2 y$

The negative sign indicates that a_N and Y are in opposite directions. Since ' ω ' is a constant then we can write $a_N \alpha - y$

i.e, the acceleration of N is directly proportional to the displacement in magnitude but in opposite direction, always towards the fixed point 'O' in the path.

Hence the projection of uniform circular motion on t the diameter is simple harmonic.

(b) A = 0.04 m ; v = 50Hz ; $\phi = \pi/3$

$y = A\sin(\omega t + \phi) = A\sin(2\pi\nu t + \phi)$

$y = 0.04\sin(100\pi t + \pi/3)$ m.

2. What is a Reference Circle? What is the importance of Reference Circle in SHM? Derive expression for the velocity and accelerations of a particle executing SHM.

Ans. **Reference Circle** :A uniform circular motion with its projection on a diameter of the circle serves as a mathematical model for S.H.M. Hence the circle is known as reference circle.

Importance:

Derivation for velocity and acceleration:

Velocity : The equation for the displacement of a particle executing S.H.M is given by $y = A\sin\omega t$

Velocity $V = \frac{dy}{dt}$ Or $V = A\frac{d}{dt}(\sin\omega t)$

$V = A\omega\cos\omega t$

Also, $y = A\sin\omega t \Rightarrow \sin\omega t = y/A$

$\therefore V = A\omega\sqrt{1-\frac{y^2}{A^2}} \Rightarrow V = \omega\sqrt{A^2 - y^2}$

Acceleration : The rate of change of velocity gives acceleration of the particle executing SHM which is given by

$a = \frac{dv}{dt} = \frac{d}{dt}(A\omega\cos\omega t)$

$a = -A\omega^2\sin\omega t$

Since $y = A\sin\omega t$

$a = -\omega^2 y$ (or) $a \propto -y$

Thus the acceleration is directly proportional to the displacement in magnitude and is opposite in direction at every point in the path.

SHORT ANSWER QUESTIONS :

1. Define SHM. Explain with an example.

Ans. **Simple Harmonic Motion:** A body is said to be in simple harmonic motion such that, at any point its acceleration is directed always towards the mean position and it is directly proportional to its displacement from the mean position.

Linear SHM: Oscillations of a simple pendulum with small amplitude, vibrations of strings in musical instruments

$a \propto -x$

Angular SHM: Oscillations of a balance wheel in wrist watch, oscillations as in tensional pendulum

Angular acceleration $\propto$ - (angular displacement)

2. What are the conditions for a particle to execute SHM? How do the velocity and acceleration vary with position during SHM?

Ans. The conditions for a body to execute S.H.M are

(i) The motion should be periodic about a fixed point.

(ii) Acceleration and displacements are always opposite in directions .

(iii) The acceleration should always be directed towards fixed point.

The displacement of a particle in SHM at any instant of time t is given by the equation

$y = A\sin\omega t$ Where A is the amplitude and ω is angular frequency

$Velocity = \frac{dy}{dt} = A\omega\cos\omega t$ or $V = \omega\sqrt{A^2 - y^2}$

Acceleration (a) = $\frac{dv}{dt} = -a\omega^2\sin\omega t$

$\therefore a = -\omega^2 y \left(\because y = A\sin\omega t\right)$

3. Derive equations for the (i) Velocity (ii) Acceleration (iii) Time period of a particle in SHM.

Ans. The displacement of the particle executing SHM at any instant of time is given by $y = A\sin\omega t$ Where A is the radius of reference circle and ω is the constant angular velocity.

<u>(i) Velocity</u> : Differentiating the equation for displacement w.r.t. time,

$\frac{dy}{dt} = \frac{d}{dt}\left(A\sin\omega t\right) \Rightarrow V = A\frac{d}{dt}\left(\sin\omega t\right)$

$\Rightarrow V = A\omega\cos\omega t$

$\Rightarrow V = A\omega\sqrt{1-\sin^2\omega t} = A\omega\sqrt{1-\frac{y^2}{A^2}} \quad \left(\because \sin\omega t = \frac{y}{A}\right)$

$\therefore V = \omega\sqrt{A^2 - y^2}$

(ii) Acceleration: Differentiating equation for velocity w.r.t. time

$\frac{dv}{dt} = \frac{d}{dt}\left[A\omega\cos\omega t\right] \quad \Rightarrow a = A\omega\frac{d}{dt}\cos\omega t$

$a = -A\omega^2\sin\omega t$

$\therefore a = -\omega^2 y \quad [\because y = A\sin\omega t]$

$\therefore$ Acceleration $a = -\omega^2 y$

(iii) Time period: The magnitude of acceleration of the particle executing SHM is given by $a = \omega^2 y \Rightarrow \omega^2 = \frac{a}{y}$

But , Time period $T = \frac{2\pi}{\omega} \Rightarrow T = 2\pi\sqrt{\frac{y}{a}}$

$\therefore T = 2\pi\sqrt{\frac{Displacement}{Acceleration}}$

7. What are "Phase" and "Epoch"?

Discuss the phase relations between Displacement, Velocity and Acceleration in SHM.

Ans. **Phase:** The phase of a particle executing SHM at any instant is defined as its state or condition as regards its position and direction of motion at that instant measured in radians.

Epoch : Initial phase of oscillation of a particle executing SHM is called Epoch. In SHM from the mean position t = 0 initial phase angle is equal to zero.

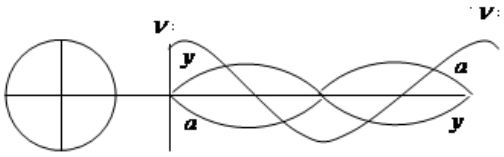
Initial displacement $y = A\sin\omega t = A\sin\phi$ where ϕ is called Epoch.

Velocity : We know that $y = A\sin\omega t$

$$V = \frac{dy}{dt} = A\omega\cos\omega t$$

Acceleration : $a = \frac{d^2y}{dt^2} \Rightarrow a = -A\omega^2\sin\omega t$ where A is Amplitude

Graphical Representation of SHM:



Phase difference between

1. s and v $\rightarrow \pi/2$ V leads y
2. v and a $\rightarrow \pi/2$ a leads v
3. s and a $\rightarrow \pi$ a leads y.

8. What is 'Reference Circle'?

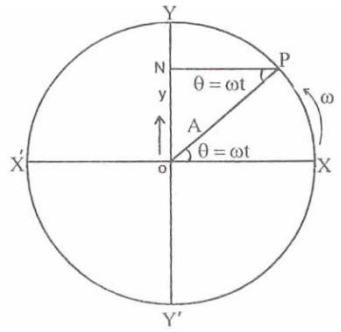
Derive an expression for the time period of SHM with the help of the reference Circle.

Ans : **Reference circle** : - A uniform circular motion, with its projection on diameter of a circle serves as a mathematical model for SHM. This circle is known as a reference circle.

Time period : The time taken to complete one oscillation about the mean position by the particle is defined as the time period of SHM. The period of oscillation of the foot of the perpendicular 'N' is equal to the period of revolution of 'P' on the reference circle .since

angle described in one complete revolution is 2π . The time period of SHM is given by

$T = \frac{2\pi}{\omega}$.....(1)



Where 'ω' is the angular velocity on the reference circle, and is a constant. Hence the time period is a constant and the oscillations occur at regular intervals of time

We know that $a = -\omega^2 y$.....(2)

From (1) & (2) $T = 2\pi\sqrt{\frac{y}{a}}$

VERY SHORT ANSWER QUESTIONS

1). what are the physical quantities having maximum value at the mean position in S.H.M?

A) Velocity and kinetic energy are maximum at the mean position in SHM.

2). what are the physical quantities having maximum value at the extreme position in S.H.M?

Ans Displacement, acceleration, potential energy (P.E) are minimum at extreme position in SHM.

3). What happens to the energy of simple harmonic oscillator if its I) amplitude is doubled ? II) Time period is doubled? Give relevant equations.

A) Energy of simple harmonic oscillator $T.E = \frac{1}{2} m\omega^2 A^2$

i) If amplitude is doubled energy becomes four times

$$E\alpha A^2 \Rightarrow \frac{E_2}{E_1} = \left(\frac{A_2}{A_1}\right)^2 \Rightarrow E_2 = 4E_1$$

ii) If time period is doubled =

$$E\alpha \frac{1}{T^2} \Rightarrow \frac{E_2}{E_1} = \left(\frac{T_1}{T_2}\right)^2 = \frac{1}{4} \Rightarrow E_2 = \frac{E_1}{4}$$

4) What is Force Constant of a spring? On What factors does it depend?

A) Restoring force per unit extension in the Spring is known as force constant of a spring.

$$Force\ cons\tan t(K) = \frac{\text{Restoring force(F)}}{\text{Extension(x)}}$$

K value depends on elasticity and Size of the spring.

5) The amplitude of Oscillations of a loaded spring is doubled. What is the effect on

(i) The force constant of the spring (ii) The frequency of oscillations?

A) Force constant and frequency do not change.

6) The acceleration of given particle in S.H.M is 0.4 ms^{-2} at a displacement of 0.1 m. What is the period of oscillation?

A) $a = 0.4ms^{-2}$; $y = 0.1m$

$$T = 2\pi\sqrt{y} = 2\pi\sqrt{\frac{0.1}{0.4}} \Rightarrow T = \pi\ Sec.$$

7) A particle executes S.H.M such that the maximum velocity during the oscillation is numerically equal to half the maximum acceleration. What is the time period?

A) $V_{\max} = \frac{1}{2}a_{\max}$

$$A\omega = \frac{1}{2}\omega^2 A \Rightarrow \omega = 2 \Rightarrow T = \pi\ second$$

8) A mass of 2kg attached to a spring of force constant 260Nm^{-1} makes100 oscillations. What is the time taken?

A) $m = 2kg$; $k = 260Nm^{-1}$

$$\text{Re}quired\ time = 100 \times Time\ Period = = 100 \times 2\pi\sqrt{\frac{m}{K}} = 100 \times 2\pi\sqrt{\frac{2}{260}} = 55.08 \text{ sec}$$

9) A simple harmonic oscillator has a time period of 2sec. what will be the change in the phase after 0.25 sec after leaving the mean position?

A) Phase difference = $\frac{2\pi}{T} \times$ Time difference Where T is the time period.

$$\phi = \frac{2\pi}{T}t = \frac{2\pi}{2} \times 0.25 = 0.25\pi\ Rad.$$

10) How does the acceleration of a particle in S.H.M Vary? Where will it be maximum?

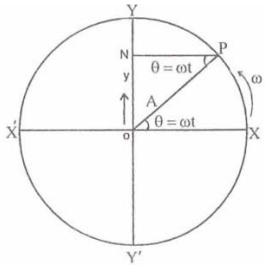
A): $\frac{d^2y}{dt^2} = a = -A\omega^2 \sin(\omega t + \phi)$ Acceleration is maximum at extreme position.

11) How does the velocity of a particle in S.H.M vary? Where will it be maximum?

A) $\frac{dy}{dt} = V = A\omega.\cos(\omega t + \phi)$ Velocity is maximum at mean position.

12) A Simple harmonic motion is represented by the equation $x = 4\sin(2\pi t + \frac{\pi}{4})$ m. What is the initial displacement.

A) *If* $t = 0$, then $x = 4\sin\frac{\pi}{4} = 2\sqrt{2}$ *m.*



13) What is the direction of acceleration and velocity of simple harmonic oscillator at i) mean position? ii) The extreme position?

A) At the mean position acceleration is zero and velocity is maximum .Its direction is towards mean position

At extreme position velocity is zero and acceleration is maximum and directed towards mean position.

14) Give two examples of periodic motion which are not oscillatory.

A) 1. Planet moving round the sun.

2. Electron revolving around nucleus.

Exercise 1

1. A particle executes SHM represented by the equation, $y = 0.02\sin\left(3.14t + \frac{\pi}{2}\right)$ m.

Find (i) Amplitude and (ii) time period (iii) frequency (iv) epoch (v) maximum velocity and (vi) maximum acceleration.

Sol. $y = 0.02\sin\left(3.14t + \frac{\pi}{2}\right)$

Comparing this with the equation, $y = A\sin(\omega t + \phi)$

(i) Amplitude A = 0.02 m.

(ii) Time period $T = \frac{2\pi}{\omega}$ or $T = \frac{2\pi}{3.14} = 2s$.

(iii) Frequency $v = \frac{1}{T} = \frac{1}{2} = 0.5Hz$.

(iv) Epoch $\phi = \pi/2 = \frac{3.14}{2} = 1.57$ rad.

(v) Maximum velocity $v_{max} = A\omega = 0.02 \times 3.14 = 0.0628 ms^{-1}$

(vi) Maximum acceleration $a_{max} = -A\omega^2 = -0.02 \times (3.14)^2 = 0.1972\, ms^{-2}$

2. The simple harmonic oscillations of a particle are according to the equation $x = 5\cos(2\pi t + \pi/4)$ m. Find the (i) displacement, (ii) velocity and (iii) acceleration at t = 0.

Sol. Comparing with, the general equation $x = A\cos(\omega t + \phi)$

Amplitude A = 5 m and $\omega = 2\pi$

(i) Displacement $x = 5\cos(2\pi t + \pi/4)$

At $t = 0, x = 5\cos\frac{\pi}{4} = 5 \times \frac{1}{\sqrt{2}} = 3.54 m.$

(ii) Velocity $v = \omega\sqrt{A^2 - x^2}$

At t = 0, $v = \omega\sqrt{A^2 - x^2} = 2\pi\sqrt{5^2 - \left(\frac{5}{\sqrt{2}}\right)^2} = 2\pi\sqrt{\frac{25}{2}} = 22.2\, ms^{-1}$

(iii) Acceleration $a = -\omega^2 x$

At t = 0, $a = -\omega^2 x = -(2\pi)^2\, 3.54 = -4\pi^2 (3.54)^2$

$\therefore a = -139.42\, ms^{-2}$

3. A body executes SHM, such that its velocity at the mean position is 1 ms^{-1} and acceleration at the extreme position is 1.57 ms^{-2}. Calculate the amplitude and the time period of oscillation.

Sol. $v_{max} = A\omega = 1 ms^{-1}$ and $a_{max} = A\omega^2 = 1.57\, ms^{-2}$

Dividing (2) by (1) we get $\frac{a_{max}}{v_{max}} = \frac{A\omega^2}{A\omega} \Rightarrow \omega = 1.57$

Time period, $T = \frac{2\pi}{\omega} = \frac{2(3.14)}{1.57} = 4 s$

Also, $v_{max} = A\omega = 1 ms^{-1}$

i.e. $A(1.57) = 1$ or $A = \frac{1}{1.57}$

or Amplitude A = 0.637 m.

4. A particle is performing SHM in a straight line. When distance of the particle from the equilibrium has the values x_1 and x_2, the corresponding values of the velocities are u_1 and u_2 respectively. Show that its time period is $T = 2\pi\left[\frac{x_2^2 - x_1^2}{u_1^2 - u_2^2}\right]^{1/2}$.

Sol. $v = \omega\sqrt{A^2 - x^2}$.

When x = x_1, v = u_1

$u_1 = \omega\sqrt{A^2 - x_1^2}$ ---------- (1)

When x = x_2. v = u_2

$u_2 = \omega\sqrt{A^2 - x_2^2}$ ---------- (2)

Squaring (1) and (2) and then subtracting,

$u_1^2 - u_2^2 = \omega^2\left(x_2^2 - x_1^2\right)$

$\omega = \left(\frac{u_1^2 - x_2^2}{x_2^2 - x_1^2}\right)^{1/2}$

Time period, $T = \frac{2\pi}{\omega} = 2\pi\left(\frac{x_2^2 - x_1^2}{u_1^2 - u_2^2}\right)^{1/2}$

5. A particle executes SHM of period 22 seconds and amplitude 10 cm. Find the distance it travels in 3.5 second staring from the point of zero displacement.

Sol. A = 10 cm = 0.01 m ; T = 22 s and $\phi = 0$

The angular velocity, $\omega = \frac{2\pi}{T} = \frac{2\times 22}{7\times 22} = \frac{2}{7}$ rad/s.

$x = A\sin\left(\omega t + \phi\right) = 0.10\sin\left(\frac{2}{7}t\right)$

But $t = 3.5s = \frac{7}{2}s$

$\therefore x = 0.10\sin\left(\frac{2}{7}\times\frac{7}{2}\right) = 0.10\sin\left(1\right) = 0.10\sin\left(57^o 17'\right) = 0.10\times 0.8414 = 0.08414\,m.$

$\left(\because 1 radian = 57^o 17'\right)$

6. A particle performs SHM of time period T. Find the time taken by the particle to go directly (i) from its mean position to half the amplitude and (ii) from half the amplitude position to its positive extreme position.

Sol. If the particle starts from the mean position, $x = A\sin\omega t$

i)x = A/2.

$$\frac{A}{2} = A\sin\omega t \Rightarrow \sin\omega t = \frac{1}{2} \Rightarrow \omega t = \frac{\pi}{6} \Rightarrow \frac{2\pi}{T}t = \frac{\pi}{6} \Rightarrow t = \frac{T}{12}$$

ii) The time taken to go from half the amplitude position to its positive extreme position.

= (time taken to go from mean to positive extreme position) - (time taken to go from mean to half amplitude position) $= \frac{T}{4} - \frac{T}{12} = \frac{T}{6}$

7. Two particles execute SHM of the same amplitude and frequency on parallel lines side by side. They cross each another when moving in opposite directions each time their displacement is half their amplitude. What is the phase difference between them?

Sol. $x = A\sin\omega t$

But , $x = \frac{A}{2}$

$\frac{A}{2} = A\sin\omega t$ Or $\sin\omega t = \frac{1}{2}$

Phase $= \omega t = 30^o or 150^o$, $\left[\because \sin\left(180^o - \theta\right) = \sin\theta; \sin\left(180^o - 30^0\right) = \sin 30^0\right]$

One of the particles has phase 30^0 and the other has phase 150^o.

Phase difference $= 120^o = \frac{2\pi}{3}$ radian.

8. When a particle is performing SHM of time period T_1 the force acting on it is F_1 for a certain displacement. When the same particle is performing SHM of time period T_2 the force acting is F_2 for the same displacement. What will be the time period of the particle when a combined force of F_1 and F_2 produces the same displacement in SHM?

Sol. The force acting on a particle executing SHM is given by

$$F = -m\omega^2 x = -m\left(\frac{2\pi}{T}\right)^2 x$$

When the force is F_1, $T = T_1$

$$F = -m\left(\frac{2\pi}{T_1}\right)^2 x$$

When the force is F_2,$T = T_2$

$$\therefore F_2 = -m\left(\frac{2\pi}{T_2}\right)^2 x$$

When the force $F_1 + F_2$, T = T

$$\therefore F_1 + F_2 = -m\left(\frac{2\pi}{T}\right)^2 x$$

Substituting the values of F_1 and F_2 from the above,

$$-m\left(\frac{2\pi}{T_1}\right)^2 x - m\left(\frac{2\pi}{T_2}\right)^2 x = -m\left(\frac{2\pi}{T}\right)^2 x \Rightarrow \frac{1}{T^2} = \frac{1}{T_1^2} + \frac{1}{T_2^2}$$

The time period of the particle, $T = \dfrac{T_1 T_2}{\sqrt{T_1^2 + T_2^2}}$

9. The displacement of a particle executing SHM is given by $x = 0.10\sin\left(\dfrac{2\pi t}{T} + \phi\right)$ m. The time period of oscillation **is 3 s. At t = 0, the particle is displaced by 0.05 m. Find i) the initial phase, ii) the phase angle corresponding to 0.0866 m and iii) the phase difference between any two positions of the particle 2 s apart.**

Sol. $x = 0.10\sin\left(\dfrac{2\pi t}{T} + \phi\right) = 0.10\sin\left(\dfrac{2\pi}{3}t + \phi\right) \quad (\because T = 3s)$

i) At t = 0, x = 0.05 m

$$0.05 = 0.10\sin\left(\frac{2\pi}{3} \times 0 + \phi\right) = 0.10\sin\phi$$

$$\sin\phi = \frac{1}{2} \Rightarrow \phi = \frac{\pi}{6} \text{ radian}$$

ii) x = 0.0866 m

$$0.0866 = 0.10\sin\left(\frac{2\pi}{3}t + \phi\right)$$

$$\sin\left(\frac{2\pi}{3}t + \phi\right) = \frac{0.0866}{0.10} = 0.866.$$

Phase angle $\left(\dfrac{2\pi}{3}t + \phi\right) = \dfrac{\pi}{3}$ radian

iii) Phase angle at t = t_1 is $\phi_1 = \left(\dfrac{2\pi}{3} \times t_1 + \phi\right)$ and at t = t_2 is $\phi_2 = \left(\dfrac{2\pi}{3} \times t_2 + \phi\right)$

Phase difference $= \phi_2 - \phi_1 = \left(\dfrac{2\pi}{3}t_2 + \phi\right) - \left(\dfrac{2\pi}{3}t_1 + \phi\right) = \dfrac{2\pi}{3}(t_2 - t_1)$

$$= \frac{2\pi}{3} \times 2 = \frac{4\pi}{3} \text{ radian.} \qquad (\because t_2 - t_1 = 2)$$

10. A particle performs linear SHM of period $\frac{2\pi}{\omega}$ about a centre 'O' and is observed to have a velocity $b\omega\sqrt{3}$ when at a distance b from 'O'. If the particle is moving towards the positive extremity at that instant, show that it will travel a further distance b in a time $\frac{\pi}{3\omega}$ before coming momentarily to rest.

Sol. $T = \frac{2\pi}{\omega}$; $v = b\omega\sqrt{3}$ and x = b

$v = \omega\sqrt{A^2 - x^2} \Rightarrow b\omega\sqrt{3} = \omega\sqrt{A^2 - b^2}$

Squaring, $3b^2\omega^2 = \omega^2\left(A^2 - b^2\right) \Rightarrow A = \pm 2b$

The time taken (t) to travel from the mean position to a distance b can be found from

$x = A\sin\omega t$

Here, x = b, A = 2 b

$\therefore b = 2b\sin\omega t \quad \Rightarrow t = \frac{\pi}{6\omega}$

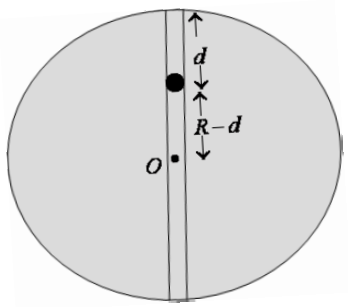
Further time taken to reach the extreme position $= \frac{T}{4} - \frac{\pi}{6\omega} = \frac{2\pi}{4\omega} - \frac{\pi}{6\omega} = \frac{\pi}{3\omega}$

It will momentarily come to rest when it reaches the positive extreme position.

Further distance travelled $= A - b = 2b - b = b$

14. If earth were a homogeneous sphere of radius 'R' and a tunnel is drilled straight along the diameter, show that a body dropped into the tunnel will execute SHM. Calculate the time period of oscillation of the body given that R = 6400 km and the acceleration due to gravity on the surface of the earth g = 9.8 ms^{-2}.

Sol. If 'ρ' is the density of the earth, $g = \frac{4}{3}\pi\rho GR$



The value of 'g' at any depth d $= g_d = \frac{4}{3}\pi\rho G\left(R - d\right)$

Dividing, $\frac{(R-D)}{R} = \frac{g_d}{g}$

Time period $T = 2\pi\sqrt{\frac{displacement}{acceleration}} = 2\pi\sqrt{\frac{(R-D)}{g_d}}$

$T = 2\pi\sqrt{\frac{R}{g}} = 2\pi\sqrt{\frac{6400\times10^3}{9.8}} = 84.57 \text{ min}.$

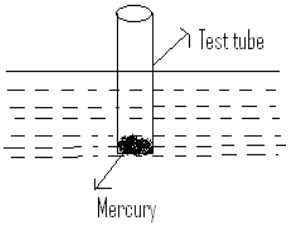
15. A test tube of mass m and of external diameter 2 r is floated in water by placing mercury of mass M at the bottom of the tube. The tube is depressed by a small amount and then released. Find its time period

Sol: A test tube of mass m contains mercury of mass M and floats in water as shown in the figure. Its external radius is R and area of cross section $A = \pi R^2$. When the tube is depressed by x, the additional upward thrust on the tube is equal to the weight of the liquid displaced due to the depression of the tube. This upward thrust acts as the restoring force on the tube.

The restoring force = -(weight of the displaced liquid due to depression)

= - (mass of the displaced liquid x density of the liquid)xg

= - (A $x\rho$ g)



Acceleration = $\frac{Force}{Mass} = -\frac{Ax\rho g}{(M+m)}$

$$a = \left(\frac{A\rho g}{(m+M)}\right)x$$

A ρ g/(m+M) is a constant. The acceleration is directly proportional to its displacement and opposite direction .Hence, it performs SHM.

Time period, $T = \frac{2\pi}{\sqrt{\frac{a}{x}}} = \frac{2\pi}{\sqrt{\frac{A\rho g}{m+M}}} = 2\pi\sqrt{\frac{m+M}{A\rho g}}$

Exercise 2

1. **The amplitude and time period of a particle of mass 0.1 kg executing simple harmonic motion are 1 m and 6.28s respectively. Find its (i) angular frequency (ii) acceleration and (iii) velocity at a displacement of 0.5m.**

Sol: A = 1m; T = 6.28 sec ; m = 0.1 kg; x = 0.5 m

(i) $\omega = \frac{2\pi}{T} = \frac{2\pi}{6.28} = 1 \ rad/\sec$

(ii) $a = -\omega^2 x = -1\times(0.5) = -0.5 m/s^2$

(iii) $v = \omega\sqrt{A^2 - x^2} = 1\sqrt{1-0.25} = \frac{\sqrt{3}}{2} m/\sec$

2. **The amplitude of oscillation of a particle is 0.05 m. If its time period is 1.57s (i) what is the velocity at the mean position? (ii) What is its acceleration at the extreme position?**

Sol. A = 0.05m ; T = 1.57 sec

(i) Formula: $v = \omega\sqrt{A^2 - x^2}$

Here x = 0

$\therefore v = \omega A = \frac{2\pi}{T}.A$

$= \frac{2\pi}{1.57} \times (0.05) = .2$ m/ s

(ii) $a = \omega^2 A = \left(\frac{2\pi}{T}\right)^2 A = \frac{4\pi^2}{T^2}.A = \frac{4\pi^2}{(1.57)^2} \times (0.05) = 0.8\ m/s^2$

3. **The displacement of an oscillating particle varies with time't' (in seconds) according as x = 2cos $(0.5\pi t)$ m. Find the (i) amplitude, (ii) time period, (iii) maximum velocity and (iv) maximum acceleration.**

Sol : x = 2 $\cos(0.5\pi t)$ m

Comparing with x = A $\cos(\omega t)$ m

(i) A = 2m

(ii) $T = \frac{2\pi}{\omega} = 4\text{sec}$

(iii) $V_{max} = A\omega = 2 \times (0.5\pi) = \pi\ m/\text{s}$

(iv) $a_{max} = \omega^2 A = (0.5\pi)^2 \times (2) = \frac{\pi^2}{2}\ m/\text{s}^2$

4. **The equation for the displacement of a particle executing SHM is** $y = 5\sin(2\pi t)\,cm$**. Find (i) the velocity at 3 cm from the mean position, (ii) acceleration after 0.5 s after leaving the mean position.**

Sol: y = 5 $\sin(2\pi t)$ cm

Comparing with, y = A $\sin(\omega t)$ cm

$\omega = 2\pi$; $A = 5\,\text{cm}$

(i) x = 3 cm

$v = \omega\sqrt{A^2 - x^2} = 2\pi\sqrt{(5)^2 - (3)^2} = 8\pi\ cm/\text{sec}$

(iii) After 0.5 sec the particle is $\left(\because T=\frac{2\pi}{\omega}=\frac{2\pi}{2\pi}=1\text{s}\right)$ at mean position

$x=0\Rightarrow$ Acceleration $a=-\omega^2 x=-\omega^2(0)$ = 0 (zero)

5. **The velocity of a particle under going SHM at the mean position is $2\,ms^{-1}$. Find the velocity of the particle at the point where the displacement from the mean position is equal to half the amplitude.**

Sol: $A\omega=2$ (velocity at mean position)

$V=\omega\sqrt{A^2-x^2}$

$\Rightarrow V=\omega\sqrt{A^2-\left(\frac{A}{2}\right)^2}$ =1.732 m/s

6. **The frequency of a particle performing SHM is 12 Hz. Its amplitude is 4 cm. Its initial displacement is 2 cm towards positive extreme position. Write its equation for displacement.**

Sol: $x=A\sin(\omega t+\phi)$

At t = 0 , $x=2cm$ = 0.02 m ; A = 0.04 m

$\frac{x}{A}=\sin(2\pi\nu t+\phi)$

$\frac{0.02}{0.04}=\sin(\phi)\Rightarrow\sin\phi=\frac{1}{2}\Rightarrow\phi=\frac{\pi}{6}$

$\therefore x=0.04\sin\left(2\pi\nu t+\frac{\pi}{6}\right)\Rightarrow x=0.04\sin\left(2\pi(12)t+\frac{\pi}{6}\right)$

$\therefore x=0.04\sin\left(24\pi t+\frac{\pi}{6}\right)m$

7. **A particle performing SHM with a frequency of 5 Hz and amplitude 2 cm is initially in the positive extreme position. What is the equation for its displacement?**

Sol: The equation of motion of a body from extreme position is $x=A\cos\omega t$

$\Rightarrow x=0.02\cos(2\pi n)t\Rightarrow x=0.02\cos(2\pi 5)t\Rightarrow x=0.02\cos(10\pi t)m$

8. **A particle is performing SHM. Its path length is 8 cm and time period 0.2 s. If the particle is initially in the mean position, find the time at which it will be at (i) 1 cm, (ii) 2 cm, (iii) 3 cm and (iv) 4 cm.**

Sol. The path length is 2 A = 8 cm $\Rightarrow A=4cm=0.04m$; And T = 0.2 sec

x = A $\sin \omega t$

(i) at x =1 cm = 0.01 m

$$0.01 = 0.04 \sin\left(\frac{2\pi}{T}\right).t \Rightarrow \frac{1}{4} = \sin\left(\frac{2\pi}{0.2}\right).t$$

$$\Rightarrow \sin(10\pi)t = \frac{1}{4} \Rightarrow t = \frac{1}{10\pi}\sin^{-1}\left(\frac{1}{4}\right) = \frac{14.47}{10\pi} = \text{ 0.008038 sec}$$

(ii) at x = 2 cm = 0.02 m

$$0.02 = 0.04 \sin\left(\frac{2\pi}{0.2}\right)t \Rightarrow t = \frac{1}{10\pi}\sin^{-1}\left(\frac{1}{2}\right) = \frac{1}{10\pi}\left(\frac{\pi}{6}\right) = \frac{1}{60} = 0.01667 \text{sec}$$

(iii) at x = 3 cm = 0.03m

$$0.03 = 0.04 \sin(10\pi)t$$

$$\Rightarrow t = \frac{1}{10\pi}\sin^{-1}\left(\frac{3}{4}\right) = 0.02699 \text{ sec}$$

(iv) at x = 4 cm = 0.04 m

$$0.04 = 0.04 \sin(10\pi t)$$

$$\Rightarrow \sin(10\pi t) = 1 \Rightarrow t = \frac{1}{20} = 0.05 \text{ sec}$$

9. **In the above problem, find the phase angles at 1 cm, 2 cm, 3 cm and 4 cm.**

Sol: The initial phase at time t = 0 is zero at mean position

Now, $x = A\sin(\omega t + \phi)$

Here when x = 1 cm = 0.01 m

$$0.01\text{m} = 0.04 \sin(\omega(0) + \phi) \Rightarrow \sin\phi = \frac{1}{4}$$

$$\Rightarrow \phi = \sin^{-1}\left(\frac{1}{4}\right) = 14.48^0$$

Similarly, for x = 2cm = 0.02 m, $\phi = \sin^{-1}\left(\frac{1}{2}\right) = 30^0$

For x = 3 cm = = 0.03 m, $\phi = \sin^{-1}\left(\frac{3}{4}\right) = 48.58^0$

and for x = 4 cm = 0.04 m , $\phi = \sin^{-1}(1) = 90^0$

10. **The time period of a particle performing linear SHM is 12s. What is the time taken by it to make a displacement equal to half its amplitude?**

Sol: T = 12 sec ; $x = \frac{A}{2}$

$$\Rightarrow \frac{A}{2} = A\sin\left(\frac{2\pi}{T}.t\right) \Rightarrow \frac{1}{2} = \sin\left(\frac{2\pi}{12}.t\right) \qquad \Rightarrow t = 1\text{sec}$$

11. **An object executing SHM has an acceleration of $0.2 m/s^2$ at a distance of 0.05 m from the mean position. Calculate the period of SHM.**

Sol: $a = 0.2\ m/\text{s}^2$; x = 0.05 m

$$a = \omega^2 x \Rightarrow 0.2 = \left(\frac{2\pi}{T}\right)^2 (0.05) \Rightarrow T = 3.142\text{sec}$$

12 **The velocities of a particle performing linear SHM are 0.13 m/s and 0.12 m/s when it is at 0.12 m and 0.13 m from mean position respectively. Find the time period and amplitude.**

Sol : $v = \omega\sqrt{A^2 - x^2}$

$0.13 = \omega\sqrt{A^2 - (0.12)^2}$ and $0.12 = \omega\sqrt{A^2 - (0.13)^2}$

Dividing the above,

$$\frac{0.13}{0.12} = \frac{\sqrt{A^2 - (0.12)^2}}{\sqrt{A^2 - (0.13)^2}}$$

Solving the above ,we get $A = 0.1769\ m$ and $\omega = 1 \Rightarrow T = 6.28\,s$

13. **Two particles executing SHM of the same amplitude and frequency on parallel lines side by side. They cross one another when moving in opposite directions each time their displacement $\frac{\sqrt{3}}{2}$ is times their amplitude. What is the phase difference between them?**

Sol: Let the two particles are initially at mean position where

$$\text{x = A } \sin\omega t \Rightarrow \frac{\sqrt{3}}{2}A = A\sin\omega t \Rightarrow \omega t = 60^0 \text{ or } 120^0$$

$\therefore$ One of the particles has phase 60^0 and the other has phase 120^0

$\therefore$ Phase difference between them is $120^0 - 60^0 = 60^0$

14. A 20 g particle moves in SHM with a frequency of 3 oscillations per second and amplitude of 5 cm. Through what total distance does the particle move during one oscillation? What is its average speed?

Sol: A = 5 cm

One oscillation = 4 A = 20 cm

3 oscillation = 1 sec

$\Rightarrow$ 1 oscillation= Time period = $\frac{1}{3}$ sec

$\therefore$ Average speed $= \frac{\text{Distance travelled in 1 oscillation}}{T} = \frac{20}{1/3} = 60\ cm/\text{sec}$

15. **Consider a simple harmonic oscillator of mass 0.5 kg, force constant 10 N/m and amplitude 3 cm. a) What is the maximum speed? b) What is the speed when x = 2 cm? c) If the oscillator is released from rest at x = A when t = O, determine the position and velocity of the oscillator at t = - 2 s.**

Sol: m = 0.5 kg ; $k = 10 N/m$; A = 3 cm = 0.03 m

$$T = 2\pi\sqrt{\frac{m}{K}} = 2\pi\sqrt{\frac{0.5}{10}} = 2\pi\sqrt{\frac{1}{20}} = 1.404\,\text{sec}$$

(a) Maximum speed = $A\omega = A.\left(\frac{2\pi}{T}\right) = 0.03\left[\frac{2\pi}{1.404}\right] = 0.1341\ m/\text{s}$

(b) $V = \omega\sqrt{A^2 - x^2}$

$$V = \frac{2\pi}{1.404}\sqrt{(0.03)^2 - (0.02)^2} = 0.1\ m/\text{s}$$

(c) $x = A\sin\left(\omega t + \frac{\pi}{2}\right) = +A\cos\omega t = -0.03 \times \cos\left[\sqrt{20}\,(2)\right] = -0.029\ m$

Here the particle is towards negative extreme

(d) x = -0.029

$V = \omega\sqrt{A^2 - x^2}$

$$V = \sqrt{20}\sqrt{(0.03)^2 - (-0.029)^2} = 0.05847\ m/\text{s}$$

16 **A simple harmonic oscillator is of mass 0.1 kg.It is oscillating with a frequency of $\frac{5}{\pi} Hz$. If its amplitude of vibration is 5 cm, find the force acting on the particle at this extreme position.**

Sol: The magnitude of force on the particle at extreme position is

$$F = ma_{\max} \text{ Or } F = mA\omega^2 = 0.1\times 0.05\times 4\pi^2\times\frac{25}{\pi^2} = 0.5 \text{ N}$$

17. **A body of mass 15 g oscillates about a fixed point with a SHM of amplitude 8 cm. If the body is attracted towards the fixed point, when at a distance of 4 cm from it with a force equal to the weight of 10 g, find (a) the period of oscillation of the mass about the fixed point and (b) the maximum KE of the mass.**

Sol: a) $g = 10 m/s^2$

$$\therefore F = mg = \left(\frac{10}{1000}\right)10 = \frac{1}{10} N$$

But, $F = m\omega^2 x$

$$\Rightarrow \frac{1}{10} = \frac{15}{1000}\times\omega^2\times\frac{4}{100} \Rightarrow \omega^2 = \frac{10^4}{60}$$

$$\Rightarrow T = \sqrt{\frac{60\times 4\pi^2}{10^4}} = \frac{\sqrt{240}\times\pi}{100} = 0.4863 \text{ sec}$$

b) Maximum $K.E = \frac{1}{2}m\omega^2A^2 = \frac{1}{2}.\frac{15}{1000}.\frac{10^4}{60}.\left(\frac{8}{100}\right).\left(\frac{8}{100}\right) = \frac{8}{1000} = 8\times 10^{-3} J$

18. **A block of 0.5 kg is placed on a horizontal platform. The system is making vertical oscillations about a fixed point with a frequency of 0.5 Hz. Find the maximum amplitude of oscillation if the block is not losing contact with the horizontal platform?**

Sol: The apparent weight at the extreme position = $mg - m\omega^2 A$

To lose the contact, the condition is $mg - m\omega^2 A = 0$

$$\Rightarrow mg = m\omega^2 A \text{ Or } A = \frac{g}{\omega^2}$$

$$A = \frac{10}{4\pi^2\times 0.5^2} = \frac{10}{\pi^2} \quad 1m$$

19. **If the above problem, what is the apparent weight of the block at the lower extreme position, when it is oscillating with maximum amplitude?**

Sol : The apparent weighy at lower extreme = $mg + m\omega^2 A$

or $W' = (0.5)10 + (0.5)4\pi^2(0.5)^2.1 \quad 10 N$

20. **A block is placed on a horizontal platform. The system is making horizontal oscillations about a fixed point with a frequency of 0.5 Hz. Find the maximum amplitude of oscillation if the block is not to slide on the horizontal platform? (Coefficient of friction between the block and the platform = 0.4).**

Sol : For the block, the limiting condition for not to slide is, $\mu mg = m\omega^2 A$

Frictional force = force in SHM $(0.4)10 = 4\pi^2 (0.5)^2 A \Rightarrow A \quad 0.4m$

21. A particle executing SHM has amplitude of 4 cm and its acceleration at a distance of 1cm from the mean position is 3 cms^{-2} .What will its velocity be when it is at a distance of 2 cm from its mean position?

Sol: A = 4 cm = 0.04 m

$a = 3cms^{-2}$ and x = 1 cm = 0.01m

$a = \omega^2 x \Rightarrow 0.03 = \omega^2 (0.01) \Rightarrow \omega^2 = 3 \Rightarrow \omega = \sqrt{3}$

$V = \omega\sqrt{A^2 - x^2}$

$\Rightarrow V = \sqrt{3}\sqrt{(0.04)^2 - (0.02)^2} = \sqrt{3}(0.034) = 5.99\ cm/\sec$

22. A particle executing SHM of period 12 seconds moves through a distance of 32 cm in one oscillation. Find the time it takes to travel 4 cm from the positive extremity of its oscillation.

Sol : T=12 sec ; 4 A = 32 cm $\Rightarrow A = 8cm = 0.08m$

x = A $\cos\omega t$

$0.04 = 0.08\cos\left(\frac{2\pi}{T}.t\right) \Rightarrow \frac{\pi}{6}.t = \frac{\pi}{3} \Rightarrow t = 2\sec$

23. A small body of mass 10 gram is making simple harmonic oscillations along a straight line with a time period of π/4 s and the maximum displacement is 10cm. Find the energy of oscillations. What is the average kinetic energy during one oscillation?

Sol : m = 10 g = $= 0.01kg$; $T = \frac{\pi}{4}\sec$; A = 10 cm = 0.1 m

Total energy $= \frac{1}{2}m\omega^2 A^2 = \frac{1}{2}(0.01)\left(\frac{4\pi^2}{\pi^2/16}\right)(0.1)^2 = 32\times10^{-4}J$

Average velocity $= \frac{Total\ dis\tan ce}{total\ time} = \frac{4A}{T} = \frac{4\times0.1}{\pi/4} = \frac{1.6}{\pi}m/\sec$

Average K.E = $\frac{1}{2}\times m\times v_{avg}^2$

$K.E_{avg} = \frac{1}{2}\times0.01\times\frac{1.6}{\pi}\times\frac{1.6}{\pi} = \frac{2.56\times10^{-2}}{19.7} = 12.99\times10^{-4}J$

24. At what displacement is the KE of a particle performing SHM of amplitude 10 cm, three times its PE?

Sol: K.E = 3 P.E and A = 10cm

$$\frac{1}{2}m\omega^2\left(A^2-x^2\right)=3\left(\frac{1}{2}m\omega^2x^2\right)$$

$$\therefore x=\frac{10}{2}=5cm$$

25. A particle of mass 2 g is executing SHM of frequency 10 Hz. Find its KE, PE and TE when its displacement is 1 cm. Its amplitude is 4cm.

Sol: m = 0.002 kg ; $n=10Hz$; $x=1cm=0.01m$; A = 0.04 m

$$K.E=\frac{1}{2}m\omega^2\left(A^2-x^2\right)$$

$$=\frac{1}{2}\times(0.002)\left(4\pi^2n^2\right)\left[(0.04)^2-(0.01)^2\right]=5.91\times10^{-3}J$$

$$\text{P.E}=\frac{1}{2}m\omega^2x^2=\frac{1}{2}(0.002)\left(4\pi^2\times100\right)(0.01)^2=0.394\times10^{-3}J$$

$$\therefore T.E=P.E+K.E=(5.91+0.394)\times10^{-3}J=6.318\times10^{-3}J$$

26) **A particle of mass 0.5 kg is executing SHM along a straight line. Its path length is 10 cm and time period is 8 s. Calculate its KE, PE and TE when its phase angle is $\frac{\pi}{6}$ radian.**

Sol : m = 0.5 kg ; 2A = 10 cm $\Rightarrow A=5cm=0.05m$; T = 8 sec; $\phi=\frac{\pi}{6}$

$$\text{x = A}\sin\phi=0.05\times\sin\left(\frac{\pi}{6}\right)=\frac{0.05}{2}=0.025m$$

$$\text{K.E}=\frac{1}{2}m\omega^2\left(A^2-x^2\right)=\frac{1}{2}\times0.5\times\left(\frac{2\pi}{8}\right)^2\left[(0.05)^2-(0.025)^2\right]$$

Or $\text{K.E}=2.88\times10^{-4}J$

$$\text{P.E}=\frac{1}{2}m\omega^2x^2=\frac{1}{2}\times0.5\times\frac{4\pi^2}{64}\times0.000625=0.962\times10^{-4}J$$

$$\text{T.E = P. E + K . E}=(2.88+0.962)\times10^{-4}=3.842\times10^{-4}J\text{ .}$$

ASSESS YOURSELF

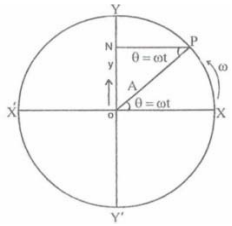
1. Can you think of a motion which is harmonic but not simple harmonic?

A. The motion of the seconds hand in a clock.

2. A falling ball undergoes elastic collision with the ground repeatedly and rebounds to the same height along the same path. Is the motion of the ball simple harmonic?

A. No, It is not simple harmonic motion since the acceleration is not varying with displacement. It is moving under acceleration due to gravity which is constant at the given place.

3. What happens to the acceleration of 'N' when 'P' moves from 'X' to 'Y' on the circle in Figure?



A. The acceleration increases from zero to a maximum $\left(A\omega^2\right)$.

4. How will the velocity of a particle in SHM vary as it moves from the mean position to an extreme position?

A. The velocity of the particle in SHM decreases from maximum at the mean position to zero at the extreme position according to the equation $v = \omega\sqrt{A^2 - y^2}$.

5. Which characteristic of SHM is given by the radius of a Reference Circle?

A. The radius of the reference circle gives 'Amplitude' of SHM.

6. What is the increase in the phase angle during 1/4th of an oscillation of SHM?

A. The phase angle increases by $\pi/2$

7. How does the energy of a simple harmonic oscillator depend on the amplitude?

A. The energy of the simple harmonic oscillator is directly proportional to the square of the amplitude $\left(E \propto A^2\right)$