MATRICES - 4

EXERCISE – 3(i)

Solve the following system of homogeneous equations.

1. **2x + 3y – z = 0**
x – y – 2z = 0
3x + y + 3z = 0.

Sol. The coefficient matrix is $\begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ 3 & 1 & 3 \end{bmatrix}$

$$\det \text{ of } \begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ 3 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ 3 & 1 & 3 \end{bmatrix}$$

$$= 2(-3+2) - 3(3+6) - 1(1+3)$$

$$= -2 - 27 - 4 = -33 \neq 0,\ \rho(A) = 3$$

Hence the system has the trivial solution
x = y = z = 0 only.

2. **3x + y – 2z = 0**
x + y + z = 0
x – 2y + z = 0

Hint : If the determinant of the coefficient matrix ≠ 0 then the system has trivial solution. i.e.ρ(A)= 3.

Sol. The coefficient matrix is $\begin{bmatrix} 3 & 1 & -2 \\ 1 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix}$

$$\begin{bmatrix} 3 & 1 & -2 \\ 1 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix} = 3(1+2) - 1(1-1) - 2(-2-1)$$

$$= 9 + 6 = 15 \neq 0,\ \rho(A) = 3$$

Hence the system has the trivial solutions
x = y = z = 0 only.

3. **x + y – 2z = 0**
2x + y – 3z = 0
5x + 4y – 9z = 0

Sol. The coefficient matrix is $\begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & 9 \end{bmatrix}$ =A (say)

$$\begin{vmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & 9 \end{vmatrix} = 1(-9+12) - 1(-18+15) - 2(8-5)$$

$$= 3 + 3 - 6 = 0$$

∴ Rank of A = 2 as the submatrix $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$ is non-singular, ρ(A) < 3.

Hence the system has non-trivial solution.

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & 9 \end{bmatrix}$$

The system of equation is equivalent to the given system of equations are

$x + y - 2z = 0$

$-y + z = 0$

Let $z = k \Rightarrow y = k, x = k$

$\therefore x = y = z = k$ for real number k.

4. **$x + y - z = 0$**
$x - 2y + z = 0$
$3x + 6y - 5z = 0$

Sol. Coefficient matrix $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -2 & 1 \\ 3 & 6 & -5 \end{bmatrix}$

$R_2 \to R_2 - R_1, R_3 \to R_3 - 3R_1$

$$A \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 2 \\ 0 & 3 & -2 \end{bmatrix}$$

$\Rightarrow$ det A = 0 as R_2, R_3 are identical.

and rank(A) = 2 as the submatrix $\begin{bmatrix} 1 & 1 \\ 0 & -3 \end{bmatrix}$ is non-singular. Hence the system has non-trivial solution, $\rho(A) < 3$.

The system of equations equivalent to the given system of equations are

$x + y - z = 0$

$3y - 2z = 0$

Let $z = k \Rightarrow y = \frac{2k}{3}, x = \frac{k}{3}$

$\therefore x = \frac{k}{3}, y = \frac{2k}{3}, z = k$ for any real number of k.

EXAMPLE PROBLEMS

1. **If $A = \begin{bmatrix} 2 & 3 & -1 \\ 7 & 8 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -4 & 1 \end{bmatrix}$ then find A + B.**

Sol. The matrices A and B both are of the same order 2×3. Hence A + B is defined. Now

$$A + B = \begin{bmatrix} 2 & 3 & -1 \\ 7 & 8 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 \\ 2 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 0 \\ 9 & 4 & 4 \end{bmatrix}$$

2. **If $\begin{bmatrix} x-1 & 2 & y-5 \\ 2 & 0 & 2 \\ 1 & -1 & 1+a \end{bmatrix} = \begin{bmatrix} 1-x & 2 & -y \\ 2 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$, then find the values of x, y, z and 'a'.**

Sol. From the equality of matrices

$x - 1 = 1 - x \Rightarrow 2x = 2 \Rightarrow x = 1$

$y - 5 = -y \Rightarrow 2y = 5 \Rightarrow y = 5/2$

$z = 2$

$1 + a = 1 \Rightarrow a = 1 - 1 = 0$

3. Find the trace of A if A = $\begin{bmatrix} 1 & 2 & -1/2 \\ 0 & -1 & 2 \\ -1/2 & 2 & 1 \end{bmatrix}$

Sol. The elements of the principal diagonal of A are 1, –1, 1.
Hence the trace of A is 1 + (–1) + 1 = 1.

4. If A = $\begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix}$ **then find –5A.**

Sol. $-5A = -5\begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -20 & 25 \\ 10 & -15 \end{bmatrix}$

5. Find the additive inverse of A, where A = $\begin{bmatrix} i & 0 & 1 \\ 0 & -i & 2 \\ -1 & 1 & 5 \end{bmatrix}$**.**

Sol. The additive inverse of A is –A = (–1)A

$\therefore$ Additive inverse of

$A = (-1)\begin{bmatrix} i & 0 & 1 \\ 0 & -i & 2 \\ -1 & 1 & 5 \end{bmatrix} = \begin{bmatrix} -i & 0 & -1 \\ 0 & i & -2 \\ 1 & -1 & -5 \end{bmatrix}$

6. If A = $\begin{bmatrix} 2 & 3 & 1 \\ 6 & -1 & 5 \end{bmatrix}$ **and B =** $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ **then find the matrix X such that A + B – X = 0. What is the order of the matrix X**

Sol. Given $A + B - X = 0 \Rightarrow X = A + B$

$= \begin{bmatrix} 2 & 3 & 1 \\ 6 & -1 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 0 \\ 6 & -2 & 8 \end{bmatrix}$

$\therefore X = \begin{bmatrix} 3 & 5 & 0 \\ 6 & -2 & 8 \end{bmatrix}$

$\therefore$ Order of X is 2×3.

7. If A = $\begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{bmatrix}$ **and B =** $\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 3 \end{bmatrix}$ **then find A – B and 4B – 3A.**

Sol. $A - B = \begin{bmatrix} 0-1 & 1+2 & 2-0 \\ 2-0 & 3-1 & 4+1 \\ 4+1 & 5-0 & 6-3 \end{bmatrix} = \begin{bmatrix} -1 & 3 & 2 \\ 2 & 2 & 5 \\ 5 & 5 & 3 \end{bmatrix}$

and $4B - 3A = 4\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 3 \end{bmatrix} - 3\begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{bmatrix}$

$$= \begin{bmatrix} 4 & -8 & 0 \\ 0 & 4 & -4 \\ -4 & 0 & 12 \end{bmatrix} - \begin{bmatrix} 0 & 3 & 6 \\ 6 & 9 & 12 \\ 12 & 15 & 18 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -11 & -6 \\ -6 & -5 & -16 \\ -16 & -15 & -6 \end{bmatrix}$$

8. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 8 \\ 7 & 2 \end{bmatrix}$ and $2X + A = B$ then find X.

Sol. $2X + A = B \Rightarrow 2X = B - A$

$$= \begin{bmatrix} 3 & 8 \\ 7 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 4 & -2 \end{bmatrix}$$

$$X = \frac{1}{2}\begin{bmatrix} 2 & 6 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$$

$$\therefore X = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$$

9. If $A = \begin{bmatrix} 9 & 1 \\ 4 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 5 \\ 6 & 11 \end{bmatrix}$, then find X. Such that $3A + 5B + 2X = 0$.

Sol. Given $3A + 5B + 2X = 0$

$2X = -3A - 5B$

$$= -3\begin{bmatrix} 9 & 1 \\ 4 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 5 \\ 6 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} -27 & -3 \\ -12 & -9 \end{bmatrix} - \begin{bmatrix} 5 & 25 \\ 30 & 55 \end{bmatrix} = \begin{bmatrix} -32 & -28 \\ -42 & -64 \end{bmatrix}$$

$$X = \frac{1}{2}\begin{bmatrix} -32 & -28 \\ -42 & -64 \end{bmatrix} = \begin{bmatrix} -16 & -14 \\ -21 & -32 \end{bmatrix}$$

10. If $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix}$ then find AB and BA.

Sol. The number of columns of A = 3 = the number of rows of B.

Hence AB is defined and

$$AB = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 5 & -5 \\ 7 & -8 \end{bmatrix}$$

Since the number of columns B = $2 \neq 3$ = the number of rows of A.

$\therefore$ BA is not defined.

11. If A = $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$ then examine whether A and B commute with respect to multiplication of matrices.

Sol. Both A and B are square matrices of order 3. Hence both AB and BA are define and are matrices of order 3.

$$AB = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -2 \\ 1 & 1 & 10 \\ -1 & 5 & -4 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 7 \\ -4 & 5 & 3 \\ 5 & 4 & 1 \end{bmatrix}$$

which shows that $AB \neq BA$

Therefore A and B do not commute with respect to multiplication of matrices.

12. If A = $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ then show that $A^2 = -I$ where $i^2 = -1$.

Sol. $$A^2 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$$= \begin{bmatrix} i^2 & 0 \\ 0 & i^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -1$$

13. If A = $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ then show that $A^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}$ for all positive integers n.

Sol. Let S(n) be the statement that

$$A^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}$$

Given $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$\Rightarrow A^1 = \begin{bmatrix} \cos 1\theta & -\sin 1\theta \\ \sin 1\theta & \cos 1\theta \end{bmatrix} \Rightarrow S(1) \text{ is true.}$$

Assume that S(k) is true.

$$\therefore A^k = \begin{bmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{bmatrix}$$

Now $A^{k+1} = A^k A$

$$= \begin{bmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos k\theta\cos\theta - \sin k\theta\sin\theta & -\cos k\theta\sin\theta - \sin k\theta\cos\theta \\ \sin k\theta\cos\theta + \cos k\theta\sin\theta & -\sin k\theta\sin\theta + \cos k\theta\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(k\theta+\theta) & -\sin(k\theta+\theta) \\ \sin(k\theta+\theta) & \cos(k\theta+\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(k+1)\theta & -\sin(k+1)\theta \\ \sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}$$

$\therefore$ S(k + 1) is true.

By principle of Mathematical induction S(n) is true for all $n \in N$.

$\therefore A^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}$ for all positive integers n.

14. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ then show that $A^2 - 4A - 5I = 0$.

Sol. $A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$4A = 4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix}$$

$$5I = 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$A^2 - 5A - 5I$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 9-4-5 & 8-8-0 & 8-8-0 \\ 8-8-0 & 9-4-5 & 8-8-0 \\ 8-8-0 & 8-8-0 & 9-4-5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

15. If $A = \begin{bmatrix} -2 & 1 & 0 \\ 3 & 4 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 4 & 3 \\ -1 & 5 \end{bmatrix}$ then find $A + B^T$.

Sol. $A + B^T = \begin{bmatrix} -2 & 1 & 0 \\ 3 & 4 & -5 \end{bmatrix} + \begin{bmatrix} 1 & 4 & -1 \\ 2 & 3 & 5 \end{bmatrix}$

$$= \begin{bmatrix} -1 & 5 & -1 \\ 5 & 7 & 0 \end{bmatrix}$$

16. If A = $\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$, then find AA^T. Do A and A^T commute with respect to multiplication of matrices

Sol. $A^T = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}$

$$AA^T = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$$

Since $AA^T \neq A^TA$, A and A^T do not commute with respect to multiplication of matrices.

17. (a) If A = $\begin{bmatrix} 0 & 4 & -2 \\ -4 & 0 & 8 \\ 2 & -8 & x \end{bmatrix}$ is a skew symmetric matrix, find the value of x.

Sol. A is a skew symmetric matrix and x is an element of the diagonal. Hence x = 0.

(b) For any $n \times n$ matrix A, prove that A can be uniquely expressed as a sum of a symmetric and a skew symmetric matrix.

Sol. For A square matrix of order n,

A + A′ is symmetric and A – A′ is a skew symmetric matrix and

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

To prove uniqueness, let B be a symmetric matrix and C be a skew-symmetric matrix, such that A = B + C.

Then A′ = (B + C)′ = B′ + C′

= B + (–C) = B – C

and hence $B = \frac{1}{2}(A + A'), C = \frac{1}{2}(A - A')$

18. Find the determinant of A = $\begin{bmatrix} 1 & -1 \\ -3 & 1 \end{bmatrix}$

Sol. det A = 1.1 – (–3)(–1) = 1 – 3 = –2

19. Find the minors of –1 and 3 in the matrix $\begin{bmatrix} 2 & -1 & 4 \\ 0 & -2 & 5 \\ -3 & 1 & 3 \end{bmatrix}$.

Sol. Minor of –1 = $\begin{vmatrix} 0 & 5 \\ -3 & 3 \end{vmatrix} = 0 + 15 = 15$

Minor of 3 = $\begin{vmatrix} 2 & -1 \\ 0 & -2 \end{vmatrix} = -4 - 0 = -4$

20. Find the cofactors of the elements 2, –5 in the matrix $\begin{bmatrix} -1 & 0 & 5 \\ 1 & 2 & -2 \\ -4 & -5 & 3 \end{bmatrix}$.

Sol. The element 2 is (2, 2)-th element of the given matrix.

Hence cofactor of 2 = $(-1)^{2+2}\begin{vmatrix} -1 & 5 \\ -4 & 3 \end{vmatrix}$

$= (-1)^4(-3+20) = 17$

The element –5 is (3, 2)-th element of the given matrix.

Hence cofactor of –5 = $= (-1)^{3+2}\begin{vmatrix} -1 & 5 \\ 1 & -2 \end{vmatrix}$

$= -1(2 - 5) = 3$

21. Find the determinant of the matrix $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 4 \\ -4 & -2 & 5 \end{bmatrix}$.

Sol. $1\begin{vmatrix} 0 & 4 \\ -2 & 5 \end{vmatrix} + 1\begin{vmatrix} 3 & 4 \\ -4 & 5 \end{vmatrix} + 2\begin{vmatrix} 3 & 0 \\ -4 & -2 \end{vmatrix}$

$= 1(0 + 8) + 1(15 + 16) + 2(-6 - 0)$

$= 8 + 31 - 12 = 27$

22. Show that $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$.

Sol. $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

We get $\begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$

$= 1\begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix}$

$= (b-a)(c-a)\begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix}$

$= (b-a)(c-a)(c+a-b-a)$

$= (b-a)(c-a)(c-b)$

$\therefore \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$

23. Show that $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

Sol. L.H.S. = $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$

By applying $R_1 \Rightarrow R_1 + R_2 + R_3$

$$= \begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

$$= 2\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

By applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$= 2\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ -b & -c & -a \\ -c & -a & -b \end{vmatrix}$$

By applying $R_1 \to R_1 + R_2 + R_3$

$$= 2\begin{vmatrix} a & b & c \\ -b & -c & -a \\ -c & -a & -b \end{vmatrix}$$

$$= (2)(-1)(-1)\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \text{R.H.S.}$$

24. Show that $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

Sol. L.H.S = $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$

By applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$= \begin{vmatrix} 1 & a^2 & a^3 \\ 0 & b^2-a^2 & b^2-a^3 \\ 0 & c^2-a^2 & c^3-a^3 \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & b+a & b^2+ba+a^2 \\ 0 & c+a & c^2+ca+a^2 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_3$

$$= -(a-b)(c-a)\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & b-c & b^2-c^2+a(b-c) \\ 0 & c+a & c^2+ca+a^2 \end{vmatrix}$$

$$= -(a-b)(c-a)(b-c)\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & 1 & b+c+a \\ 0 & c+a & c^2+ca+a^2 \end{vmatrix}$$

$$= -(a-b)(b-c)(c-a)$$
$$[(c^2+ca+a^2)-(b+c+a)(c+a)]$$
$$= -(a-b)(b-c)(c-a)$$
$$[c^2+ca+a^2-b(c+a)-(c+a)^2]$$
$$= -(a-b)(b-c)(c-a)$$
$$[c^2+ca+a^2-bc-ab-c^2-2ca-a^2]$$
$$= -(a-b)(b-c)(c-a)[-ab-bc-ca]$$
$$= (a-b)(b-c)(c-a)(ab+bc+ca)$$

$$\therefore \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$$

25. If ω is a complex cube root of 1 then show that $\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = 0$**.**

Sol. $\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$

$R_1 \to R_1 + R_2 + R_3$

$$= \begin{vmatrix} 1+\omega+\omega^2 & 1+\omega+\omega^2 & 1+\omega+\omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} [\because 1+\omega+\omega^2 = 0]$$

$$= 0$$

26. Show that

$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$$

Sol. $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

$R_1 \to R_1 + R_2 + R_3$

$$= \begin{vmatrix} a-b-c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$C_2 \to C_2 - C_1$

$C_3 \to C_3 - C_1$

$$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -(a+b+c) & 0 \\ 2c & 0 & -(a+b+c) \end{vmatrix}$$

$$= (a+b+c)^3$$

27. Show that the value of the determinant of skew-symmetric matrix of order three is always zero.

Sol. Let us consider a skew-symmetric matrix of order 3×3, say

$$A = \begin{bmatrix} 0 & -c & -b \\ c & 0 & -a \\ b & a & 0 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 0 & -c & -b \\ c & 0 & -a \\ b & a & 0 \end{vmatrix} = (-1)^3 \begin{vmatrix} 0 & c & b \\ -c & 0 & a \\ -b & -a & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -c & -b \\ c & 0 & -a \\ b & a & 0 \end{vmatrix} \because |B| = |B^T|$$

$= -|A| \Rightarrow 2|A| = 0$

Hence $|A| = 0$.

28. Find the value of x if $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$.

Sol. $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$

$R_2 \to R_2 - R_1$, $R_3 \to R_2 - R_1$ $\Rightarrow \begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ -2 & -6 & -12 \\ -6 & -24 & -60 \end{vmatrix} = 0$

$$(-2)(-6)\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ 1 & 3 & 6 \\ 1 & 4 & 10 \end{vmatrix} = 0$$

$$\Rightarrow (x-2)(30-24) - (2x-3)(10-6) + (3x-4)(4-3) = 0$$

$$\Rightarrow 6x - 12 - 8x + 12 + 3x - 4 = 0$$

$$x - 4 = 0 \Rightarrow x = 4$$

29. Find the adjoint and the inverse of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}$.

Sol. $|A| = \begin{vmatrix} 1 & 2 \\ 3 & -5 \end{vmatrix} = -5 - 6 = -11 \neq 0$

Hence A is invertible.

The cofactor matrix of $A = \begin{bmatrix} -5 & -3 \\ -2 & 1 \end{bmatrix}$

$$\therefore \text{Adj}A = \begin{bmatrix} -5 & -2 \\ -3 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{\text{Adj}A}{\det A} = -\frac{1}{11}\begin{bmatrix} -5 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{11} & \frac{2}{11} \\ \frac{3}{11} & -\frac{1}{11} \end{bmatrix}$$

30. Find the adjoint and the inverse of the matrix $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$.

Sol. $|A| = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

$$= 1(16-9) - 3(4-3) + 3(3-4)$$

$$= 7 - 3 - 3 = 1 \neq 0$$

$\therefore$ A is invertible.

The factor of A is $B = \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$

$\therefore$ Adj $A = B^T = \begin{bmatrix} 7 & -3 & 3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

$\therefore A^{-1} = \frac{\text{Adj}A}{\det A} = \begin{bmatrix} 7 & -3 & 3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, $\because |A| = 1$

31. Find the inverse of A = $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$**.**

Sol. Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$

Det A = 1(4 – 3) – 2(6 – 3) + 1(3 – 2)
= 1 – 6 + 1 = –4

The cofactors of elements of A are

$A_{11} = +(4-3) = 1$

$A_{12} = -(6-3) = -3$

$A_{13} = +(3-2) = 1$

$A_{21} = -(4-1) - 3$

$A_{22} = +(2-1) = 1$

$A_{23} = -(1-2) = 1$

$A_{31} = +(6-2) = 4$

$A_{32} = -(3-3) = 0$

$A_{33} = +(2-6) = -4$

$\therefore \text{Adj}\,A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 1 & -3 & 4 \\ -3 & 1 & 0 \\ 1 & 1 & -4 \end{bmatrix}$

$\therefore A^{-1} = \frac{\text{Adj}A}{\det A} = -\frac{1}{4}\begin{bmatrix} 1 & -3 & 4 \\ -3 & 1 & 0 \\ 1 & 1 & -4 \end{bmatrix}$

32. Find the rank of A = $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$ using elementary transformations.

Sol. $A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}$ (on interchanging R_1 and R_2)

$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -4 & -8 \end{bmatrix}$ $R_3 \to R_3 - 3R_1$

$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ $R_1 \to R_1 - 2R_2$, $R_3 \to R_3 + 4R_2$

The last matrix is singular and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a non-singular submatrix of it. Hence its rank is 2.

$\therefore$ Rank(A) = 2.

33. Find the rank A = $\begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix}$ using elementary transformations.

Sol. $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix}$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 + 2R_1$

$\sim \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & -2 & 1 & 5 \\ 0 & 7 & 2 & 3 \end{bmatrix}$

$R_1 \to R_1 + R_2, R_3 \to 2R_3 + 7R_2$

$\sim \begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & -2 & 1 & 5 \\ 0 & 0 & 11 & 41 \end{bmatrix}$

$R_1 \to 11R_1 - R_3, R_2 \to 11R_2 - R_3$

$\sim \begin{bmatrix} 11 & 0 & 0 & 3 \\ 0 & -22 & 0 & 13 \\ 0 & 0 & 11 & 41 \end{bmatrix}$

Now $\det \begin{bmatrix} 11 & 0 & 0 \\ 0 & -22 & 0 \\ 0 & 0 & 11 \end{bmatrix} = 11(-232) \neq 0.$

Hence ran (A) = 3.

34. a) Apply the test of rank to examine whether the following equations are consistent.
$2x - y + 3z = 8$, $-x + 2y + z = 4$, $3x + y = 4z = 0$ and, if consistent, find the complete solution.

Sol. The augmented matrix is $\begin{bmatrix} 2 & -1 & 3 & 8 \\ -1 & 2 & 1 & 4 \\ 3 & 1 & -4 & 0 \end{bmatrix}$

(on interchange R_1 and R_2)

$$\sim \begin{bmatrix} -1 & 2 & 1 & 4 \\ 2 & -1 & 3 & 8 \\ 3 & 1 & -4 & 0 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1, R_3 \to R_3 + 3R_1$

$$\sim \begin{bmatrix} -1 & 2 & 1 & 4 \\ 0 & 3 & 5 & 16 \\ 0 & 7 & -1 & 12 \end{bmatrix}$$

$R_3 \to 3R_3 - 7R_2$

$$\sim \begin{bmatrix} -1 & 2 & 1 & 4 \\ 0 & 3 & 5 & 16 \\ 0 & 0 & -38 & -76 \end{bmatrix}$$

Now $\det \begin{bmatrix} -1 & 2 & 1 \\ 0 & 3 & 5 \\ 0 & 0 & -38 \end{bmatrix} = (-1)(3)(-38) = 144$

Hence rank (A) = rank [AD] = 3

$\therefore$ The system has a unique solution.

We write the equivalent system of the equations from (F) : i.e.

$-x + 2y + z = 4$

$3y + 5z = 16$

$-38z = -76$

$\therefore$ $z = 2$, $y = 2$, $x = 2$ is the solution.

b) Show that the following system of equations is consistent and solve it completely $x + y + z = 3$, $2x + 2y - z = 3$, $x + y - z = 1$.

Sol. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & -1 \\ 1 & 1 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $D = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$

Then the given equations can be written as

$AX = D$

Augmented Matrix $[AD] = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 2 & -1 & 3 \\ 1 & 1 & -1 & 1 \end{bmatrix}$

$R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & -2 & -2 \end{bmatrix}$$

$R_3 \to 3R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Clearly all the submatrices of order 3×3 of the above matrix are singular.
Hence rank of A $\neq 3$, and rank of [AD] $\neq 3$
Now consider the submatrix $\begin{bmatrix} 1 & 1 \\ 0 & -3 \end{bmatrix}$ of both A and AD is non-singlar.
Hence Rank(A) = 2 = Rank [AD]
$\therefore$ The system of equations is consistent and has infinitely many solutions.
From the above [AD] matrix

$x + y + z = 3$

$-3z = -3 \Rightarrow z = 1$

and $x + y = 2$
Hence $x = k, y = 2 - k, z = 1, k \in R$ is a solution set.

35. Solve the following simultaneous linear equations by using Cramer's rule.
3x + 4y + 5z = 18
2x – y + 8z = 13
5x – 2y + 7z = 20

Sol. Let $A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $D = \begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix}$

i.e. AX = D

$$\Delta = \det A = \begin{vmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{vmatrix}$$

$$= 3(-7+16) - 4(14-40) + 5(-4+5)$$

$$= 27 + 104 + 5 = 136 \neq 0$$

$$\Delta_1 = \begin{vmatrix} 18 & 4 & 5 \\ 13 & -1 & 8 \\ 20 & -2 & 7 \end{vmatrix} = 408$$

$$\Delta_2 = \begin{vmatrix} 3 & 18 & 5 \\ 2 & 13 & 8 \\ 5 & 20 & 7 \end{vmatrix} = 136$$

$$\Delta_3 = \begin{vmatrix} 2 & 4 & 18 \\ 2 & -1 & 13 \\ 5 & -2 & 20 \end{vmatrix} = 136$$

Hence by Cramer's rule

$$x = \frac{\Delta_1}{\Delta} = \frac{408}{136} = 3$$

$$y = \frac{\Delta_2}{\Delta} = \frac{136}{136} = 1$$

$$z = \frac{\Delta_3}{\Delta} = \frac{136}{136} = 1$$

$\therefore$ The solution of the given system of equation is x = 3, y = 1, z = 1.

36. Solve 3x + 4y + 5z = 18, 2x – y + 8z = 13 and 5x – 2y + 7z = 20 by using Matrix inversion method.

Sol. Let $A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $D = \begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix}$

Given can be written as AX = 8

By matrix inversion method $X = A^{-1}B$ is the solution.

Det A = 3(–7 + 16) – 4(14 – 40) + 5(–4 + 5)

= 27 + 104 + 5 = 136

The cofactors of elements of A are

$$A_{11} = +(-7+16) = 9$$

$$A_{12} = -(-14-40) = 26$$

$$A_{13} = +(-4+5) = 1$$

$$A_{21} = -(28+10) = -38$$

$$A_{22} = +(21-25) = -4$$

$$A_{23} = -(-6-20) = 26$$

$$A_{31} = +(32+5) = 37$$

$$A_{32} = -(24-10) = -14$$

$$A_{33} = (-3-8) = -11$$

$$\therefore \text{Adj}\,A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj}\,A}{\det A} = \frac{1}{136}\begin{bmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix}$$

$$X = A^{-1}B = \frac{1}{136}\begin{bmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix}\begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix}$$

$$= \frac{1}{136}\begin{bmatrix} 162 & -494 & +740 \\ 468 & -52 & -280 \\ 18 & +338 & -220 \end{bmatrix} = \frac{1}{136}\begin{bmatrix} 408 \\ 136 \\ 136 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ x = 3, y = 1, z = 1 is the solution.

37. Solve the following equations by Gauss-Jordan method.

3x + 4y + 5z = 18

2x – y + 8z = 13

5x – 2y + 7z = 20

Sol. The augmented matrix is $\begin{bmatrix} 3 & 4 & 5 & 18 \\ 2 & -1 & 8 & 13 \\ 5 & -2 & 7 & 20 \end{bmatrix}$

$R_1 \to R_1 - R_2$

$= \begin{bmatrix} 1 & 5 & -3 & 5 \\ 2 & -1 & 8 & 13 \\ 5 & -2 & 7 & 20 \end{bmatrix}$

$R_2 \to R_2 - 2R_3, R_3 \to R_3 - 5R_1$

$= \begin{bmatrix} 1 & 5 & -3 & 5 \\ 0 & -11 & 14 & 3 \\ 0 & -27 & 22 & -5 \end{bmatrix}$

$R_2 \to -5R_2 + 2R_3$

$= \begin{bmatrix} 1 & 5 & -3 & 5 \\ 0 & 1 & -26 & 25 \\ 0 & -27 & 22 & -5 \end{bmatrix}$

$R_1 \to R_1 - 5R_2, R_3 \to R_3 + 27R_2$

$= \begin{bmatrix} 1 & 0 & 127 & 130 \\ 0 & 1 & -26 & -25 \\ 0 & 0 & -680 & -680 \end{bmatrix}$

$R_3 \to R_3 + (-680)$

$= \begin{bmatrix} 1 & 0 & 127 & 135 \\ 0 & 1 & -26 & -25 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

$R_1 \to R_1 - 127R_3, R_2 \to R_2 + 26R_3$

$= \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

Hence the solution is x = 3, y = 1, z = 1.

38. Solve the following system of equation by Gauss-Jordan method,
x + y + z = 3
2x + 2y – z = 3
x + y – z = 1.

Sol. The matrix equation is AX = D, where

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & -1 \\ 1 & 1 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $D = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$

The augmented matrix is

$$[AD] = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 2 & -1 & 3 \\ 1 & 1 & -1 & 1 \end{bmatrix}$$

$$R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & -2 & -2 \end{bmatrix}$$

$$R_3 \to R_3 - \frac{2}{3}R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence the following is the system of equations equivalent to the given system of equations :
x + y + z = 3, –3z = –3
Hence z = 1, x + y = 1
∴ The solution set is x = k, y = 2 – k, z = 1 where $k \in R$.

39. By using Gauss-Jordan method, show that 2x + 4y – z = 0, x + 2y + 2z = 5, 3x + 6y – 7z = 2 has no solution.

Sol. $\begin{bmatrix} 2 & 4 & -1 & 0 \\ 1 & 2 & 2 & 5 \\ 3 & 6 & -7 & 2 \end{bmatrix}$

$$R_2 \to 2R_2 - R_1, R_3 \to 2R_3 - 3R_1$$

$$\sim \begin{bmatrix} 2 & 4 & -1 & 0 \\ 0 & 0 & 5 & 10 \\ 0 & 0 & -11 & 4 \end{bmatrix}$$

$$R_3 \to 5R_3 + 11R_2$$

$$\sim \begin{bmatrix} 2 & 4 & -1 & 0 \\ 0 & 0 & 5 & 10 \\ 0 & 0 & 0 & 130 \end{bmatrix}$$ (Final matrix)

Hence the given system of equations is equivalent to the following system of equations 2x + 4y – z = 0, 5z = 10 ⇒ z = 2 and
0(x) + 0(y) + 0(z) = 130
Clearly no x, y, z satisfy the last equation

$\therefore$ Given system of equations has no solution.

40. Find the non-trivial solutions, if any, for the following equations.
$2x + 5y + 6z = 0$, $x - 3y - 8z = 0$
$3x - y - 4z = 0$

Sol. The coefficient matrix $A = \begin{bmatrix} 2 & 5 & 6 \\ 1 & -3 & -8 \\ 3 & 1 & -4 \end{bmatrix}$

On interchanging R_1 and R_2, we get

$$A \sim \begin{bmatrix} 1 & -3 & 8 \\ 2 & 5 & 6 \\ 3 & 1 & -4 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1, R_3 \to R_3 - 2R_1$ we get

$$A \sim \begin{bmatrix} 1 & -3 & -8 \\ 0 & 11 & 22 \\ 0 & 10 & 20 \end{bmatrix}$$

$R_2 \to R_2 - R_3$

$$A \sim \begin{bmatrix} 1 & -3 & -8 \\ 0 & 1 & 2 \\ 0 & 10 & 20 \end{bmatrix}$$

$R_3 \to R_3 + 10$

$$A \sim \begin{bmatrix} 1 & -3 & 8 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

Det A = 0 as R_2 and R_3 are identical.

Clearly rank (A) = 2, as the sub-matrix $\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$ is non-singular.

Hence the system has non-trivial solution.

The following system of equations is equivalent to the given system of equations.

$x - 3y - 8z = 0$

$y + 2z = 0$

On giving an arbitrary value k to z, we obtain the solution set is

$x = 2k, y = -2k, z = k, k \in R$ for $k \neq 0$.

We obtain non-trivial solutions.

41. Find whether the following system of linear homogeneous equations has a non-trivial solution.
$x - y + z = 0$
$x + 2y - z = 0$
$2x + y + 3z = 0$

Sol. The coefficient matrix is $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}$

Its determinant is $9 \neq 0$

Hence the system has the trivial solution

$x = y = z = 0$ only.

42. Theorem: Matrix multiplication is associative. i.e. if conformability is assured for the matrices A, B and C, then (AB)C = A(BC).

Sol. Proof :

Let $A = (a_{ij})_{m\times n}, B = (b_{jk})_{n\times p}, C = (c_{kl})_{p\times q}$

$AB = (d_{ik})_{m\times p}$, where $d_{ik} = \sum_{j=1}^{n} a_{ij}b_{jk}$

$(AB)C = (f_{il})_{m\times q}$, where $f_{il} = \sum_{k=1}^{p} d_{ik}c_{kl}$

$BC = (g_{il})_{n\times q}$, where $g_{il} = \sum_{k=1}^{p} b_{jk}c_{kl}$

$A(BC) = (h_{il})_{m\times q}$, where $h_{il} = \sum_{j=1}^{n} a_{ij}g_{jl}$

$$f_{il} = \sum_{k=1}^{p} d_{ik}c_{kl} = \sum_{k=1}^{p}\left(\sum_{j=1}^{n} a_{ij}b_{jk}\right)c_{kl}$$

$$= \sum_{j=1}^{n} a_{ij}\left(\sum_{k=1}^{p} b_{jk}c_{kl}\right) = \sum_{j=1}^{n} a_{ij}h_{jl} = h_{il}$$

(AB)C = A(BC)

43. Theorem : Matrix multiplication is distributive over matrix addition i.e. if conformability is assured for the matrices A, B and C, then

i) A(B + C) = AB + AC

ii) (B + C)A = BA + CA

Sol. Proof :

Let $A = (a_{ij})_{m\times n}$, $B = (b_{jk})_{n\times p}$, $C = (c_{jk})_{n\times p}$

$B + C = (d_{jk})_{n\times p}$, where $d_{jk} = b_{jk} + c_{jk}$

$A(B + C) = (x_{ik})_{m\times p}$, where $x_{ik} = \sum_{j=1}^{n} a_{ij}d_{jk}$

$AB = (f_{ik})_{m\times p}$, where $f_{ik} = \sum_{j=1}^{n} a_{ij}b_{jk}$

$AC = (g_{ik})_{m\times p}$, where $g_{ik} = \sum_{j=1}^{n} a_{ij}c_{jk}$

$AB + AC = (y_{ik})_{m\times p}$, where $y_{ik} = f_{ik} + g_{ik}$

$$x_{ik} = \sum_{j=1}^{n} a_{ij}d_{jk} = \sum_{j=1}^{n} a_{ij}(b_{jk} + c_{jk})$$

$$= \sum_{j=1}^{n} a_{ij}b_{jk} + \sum_{j=1}^{n} a_{ij}c_{jk} = f_{ik} + g_{ik} = y_{ik}$$

$\therefore A(B + C) = AB + AC$

Similarly we can prove that
(B + C) = BA + CA.

44. Theorem : If A is any matrix, then $(A^T)^T = A$.

Sol. Let $A = (a_{ij})_{m \times n}$

$A^T = (a'_{ji})_{n \times m}$, where $a'_{ji} = a_{ij}$

$(A^T)^T = (a''_{ji})_{m \times n}$, where $a''_{ij} = a_{ji}$

$a''_{ij} = a'_{ji} = a_{ij}$

$\therefore (A^T)^T = A$

45. Theorem : If A and B are two matrices o same type, then $(A + B)^T = A^T + B^T$.

Proof:

Let $A = (a_{ij})_{m \times n}, B = (b_{ij})_{m \times n}$

$A + B = (c_{ij})_{m \times n}$, where $c_{ij} = a_{ij} + b_{ij}$

$(A + B)^T = (c'_{ji})_{n \times m}, c'_{ji} = c_{ij}$

$A^T = (a'_{ji})_{n \times m}$, where $a'_{ji} = a_{ij}$

$B^T = (b'_{ji})_{n \times m}$, where, $b'_{kj} = b_{jk}$

$A^T + B^T = (d_{ji})_{n \times m}$, where $d_{ji} = a'_{ji} + b'_{ji}$

$c'_{ji} = c_{ij} = a_{ij} + b_{ij} = a'_{ji} + b'_{ji} = d_{ji}$

$\therefore (A + B)^T = A^T + B^T$

16. Theorem : If A and B are two matrices for which conformability for multiplication is assured, then $(AB)^T = B^T A^T$.

Sol. **Proof :** Let $A = (a_{ij})_{m \times n}, B = (b_{ji})_{n \times p}$

$AB = (c_{ik})_{m \times p}$, where $c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$

$(AB)^T = (c'_{ki})_{p \times m}$, where $c'_{ki} = c_{ik}$

$A^T = (a'_{ji})_{n \times m}$, where $a'_{ji} = a_{ij}$

$B^T = (b'_{kj})_{p \times n}$, where $b'_{kj} = b_{jk}$

$B^T \cdot A^T = (d_{ki})_{p \times m}$, where $d_{ki} = \sum_{j=1}^{n} b'_{kj} a'_{ji}$

$c'_{ki} = c_{ik} = \sum_{j=1}^{n} a_{ij} b_{ji} = \sum_{j=1}^{n} b'_{kj} a'_{ji} d_{ki}$

$\therefore (AB)^T = B^T A^T$

47. Theorem: If A and B are two invertible matrices of same type then AB is also invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Sol. Proof : A is invertible matrix $\Rightarrow A^{-1}$ exists and $AA^{-1} = A^{-1}A = I$.

B is an invertible matrix $\Rightarrow B^{-1}$ exists and

$BB^{-1} = B^{-1}B = I$

Now $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$

$= AIA^{-1} = AA^{-1} = I$

$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B$

$= B^{-1}IB = B^{-1}B = I$

$\therefore$ AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

$\therefore (AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1}) = (B^{-1}A^{-1})(AB) = 1$

48. Theorem : If A is an invertible matrix then A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$.

Sol. Proof : A is invertible matrix $\Rightarrow A^{-1}$ exists and $AA^{-1} = A^{-1}A = I$

$(AA^{-1})^T = (A^{-1}A)^T = I^T$

$\Rightarrow (A^{-1})^T A^T = A^T (A^{-1})^T = I$

$\Rightarrow$ By def. $(A^T)^{-1} = (A^{-1})^T$

49. Theorem : If A is a non-singular matrix then A is invertible and $A^{-1} = \frac{\text{Adj}A}{\det A}$.

Sol. Proof : Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ be a non-singular matrix.

$\therefore \det A \neq 0$.

$$\text{Adj}\,A = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

$$A \cdot \text{Adj}\,A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1A_1 + b_1B_1 + c_1C_1 & a_1A_2 + b_1B_2 + c_1C_2 & a_1A_3 + b_1B_3 + c_1C_3 \\ a_2A_1 + b_2B_1 + c_2C_1 & a_2A_2 + b_2B_2 + c_2C_2 & a_2A_3 + b_2B_3 + c_2C_3 \\ a_3A_1 + b_3B_1 + c_3C_1 & a_3A_2 + b_3B_2 + c_3C_2 & a_3A_3 + b_3B_3 + c_3C_3 \end{bmatrix}$$

$$= \begin{bmatrix} \det A & 0 & 0 \\ 0 & \det A & 0 \\ 0 & 0 & \det A \end{bmatrix} = \det A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$= \det A\, I$

$\therefore\ A \cdot \frac{\text{Adj}A}{\det A} = I$

Similarly we can prove that $A \cdot \frac{\text{Adj}A}{\det A} = I$

$\therefore\ A^{-1} = \frac{\text{Adj}A}{\det A}$

50. If A = $\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$**, B =** $\begin{bmatrix}3 & 1\\ 4 & 5\end{bmatrix}$**, prove that** $(AB)^T = B^T A^T$**.**

Sol. $A = \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} \Rightarrow A^T = \begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix}$

$B = \begin{bmatrix}3 & 1\\ 4 & 5\end{bmatrix} \Rightarrow B^T = \begin{bmatrix}3 & 4\\ 1 & 5\end{bmatrix}$

$AB = \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}3 & 1\\ 4 & 5\end{bmatrix}$

$= \begin{bmatrix}3+8 & 1+10\\ 9+16 & 3+20\end{bmatrix} = \begin{bmatrix}11 & 11\\ 25 & 23\end{bmatrix}$

$\therefore (AB)^T = \begin{bmatrix}11 & 25\\ 11 & 23\end{bmatrix}$

$B^T A^T = \begin{bmatrix}3 & 4\\ 1 & 5\end{bmatrix}\begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix}$

$= \begin{bmatrix}3+8 & 9+16\\ 1+10 & 3+20\end{bmatrix} = \begin{bmatrix}11 & 25\\ 11 & 23\end{bmatrix}$

$\therefore (AB)^T = B^T A^T$

51. If A = $\frac{1}{3}\begin{vmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ x & 2 & 7\end{vmatrix}$ **and** $AA^T = A^T A = I_3$**, find x and y.**

Sol. $A = \frac{1}{3}\begin{vmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ x & 2 & 7\end{vmatrix} \Rightarrow 3A = \begin{vmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ x & 2 & 7\end{vmatrix}$

$(3A)^T = 3A^T = \begin{bmatrix}1 & 2 & x\\ 2 & 1 & 2\\ 2 & -2 & y\end{bmatrix}$

$AA^T = I_3 \Rightarrow (3A)(3A)^T = 9I_3$

$\Rightarrow \begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ x & 2 & y\end{bmatrix}\begin{bmatrix}1 & 2 & x\\ 2 & 1 & 2\\ 2 & -2 & y\end{bmatrix} = 9\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

$\Rightarrow \begin{bmatrix}1+4+4 & 2+2-4 & x+4+2y\\ 2+2-4 & 4+1+4 & 2x+2-2y\\ x+4+2y & 2x+2-2y & x^2+4+y^2\end{bmatrix}$

$= \begin{bmatrix}9 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 9\end{bmatrix}$

$\Rightarrow x+4-2y=0 \quad ...(1)$

$2x+2-2y=0 \quad ...(2)$

$x^2+4+y^2=9$

$x+2y+4=0$

$x-y+1=0$

$3y+3=0 \Rightarrow y=-1$

$\therefore x=y-1=-1-1=-2$

$\therefore x=-2, y=-1$

52. Prove that $\begin{bmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{bmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

Sol. $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} a^2 & a^3 & abc \\ b^2 & b^3 & abc \\ c^2 & c^3 & abc \end{vmatrix}$

$= \frac{1}{abc}abc\begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$

$\underset{R_2 \to R_2-R_3}{=} \begin{vmatrix} 0 & a^2-c^2 & a^3-c^3 \\ 0 & b^2-c^2 & b^3-c^3 \\ 1 & c^2 & c^3 \end{vmatrix}$

$\underset{R_2 \to R_2-R_1}{=} \begin{vmatrix} 0 & a+c & a^2+ac+c^2 \\ 0 & b-c & b^2+bc+c^2 \\ 1 & c^2 & c^3 \end{vmatrix}$

$= (a-c)(b-c)\begin{vmatrix} 0 & a+c & a^2+ac+c^2 \\ 0 & b-c & b^2-a^2+bc-ac \\ 1 & c^2 & c^3 \end{vmatrix}$

$= (a-c)(b-c)(b-a)\begin{vmatrix} 0 & a+c & a^2+ac+c^2 \\ 0 & 1 & c+a+b \\ 1 & c^2 & c^3 \end{vmatrix}$

$= (a-c)(b-c)(b-a)\begin{vmatrix} a+c & a^2+ac+c^2 \\ 1 & a+b+c \end{vmatrix}$

$= (a-c)(b-c)(b-a)(ab+bc+ca)$

53. Prove that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$

Sol. $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$

$C_1 \to C_1 + C_2 + C_3$

$= abc\begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$

$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix}$

$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 0 & 0 \\ 1/b & 1 & 0 \\ 1/c & 0 & 1 \end{vmatrix}$ $\begin{matrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{matrix}$

$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$