MATRICES -2

EXERCISE – 3(D)

1. Find the determinants of the following matrices.

i) $\begin{bmatrix} 2 & 1 \\ 1 & -5 \end{bmatrix}$

Sol. det A = ad – bc = 2(–5) – 1(1) = –10 – 1 = –11

ii) $\begin{bmatrix} 4 & 5 \\ -6 & 2 \end{bmatrix}$

Sol. det A = $4 \cdot 2 - (-6) \cdot 5 = 8 + 30 = 38$

iii) $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

Sol. det A = $-i^2 - 0 = 1 - 0 = 1$

iv) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Sol. det A = 0(0 – 1) – 1(0 – 1) + 1(1 – 0)
= 1 + 1 = 2

v) $\begin{bmatrix} 1 & 4 & 2 \\ 2 & -1 & 4 \\ -3 & 7 & 6 \end{bmatrix}$

Sol. det A = 1(–6 – 28) – 4(12 + 12) + 2(14 – 3)
= –34 – 96 + 22 = –108

vi) $\begin{bmatrix} 1 & 4 & 2 \\ 2 & -1 & 4 \\ -3 & 7 & 6 \end{bmatrix}$

Sol. det A = 2(–3 –2) + 1(4 – 1) + 4(8 + 3)
= –10 + 3 + 44 = 37

vii) $\begin{bmatrix} 1 & 2 & -3 \\ a & -1 & 7 \\ 2 & 4 & -6 \end{bmatrix}$

Sol. det A = 0 since R_1 and R_3 are proportional.

viii) $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

Sol. $\det A = a(bc - f^2) - h(ch - fg) + g(hf - bg)$

$= abc - af^2 - ch^2 + fgh + fgh - bg^2$

$= abc + 2fgh - af^2 - bg^2 - ch^2$

ix) $\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$

Sol. $\det A = a(bc - a^2) - b(b^2 - ac) + c(ab - c^2)$

$= abc - a^3 - b^3 + abc + abc - c^3$

$= 3abc - a^3 - b^3 - c^3$

x) $\begin{bmatrix} 1^2 & 2^2 & 3^2 \\ 2^2 & 3^2 & 4^2 \\ 3^2 & 4^2 & 5^2 \end{bmatrix}$

Sol. $\det A = \begin{vmatrix} 1 & 4 & 9 \\ 4 & 9 & 16 \\ 9 & 16 & 25 \end{vmatrix}$

$= 1(225 - 256) - 4(100 - 144) + 9(64 - 81)$

$= -31 + 176 - 153 = -184 + 176 = -8$

2. If A= $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 4 \\ 5 & -6 & x \end{bmatrix}$ **and det A = 45, then find x.**

Sol. $\det A = 45 \Rightarrow \begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 4 \\ 5 & -6 & x \end{vmatrix} = 45$

$3x + 24 = 45 \Rightarrow 3x - 45 + 24 = 0$

$\Rightarrow 3x - 21 = 0 \Rightarrow x = \frac{21}{3} = 7$

II.

1. Show that $\begin{vmatrix} bc & b+c & 1 \\ ca & c+a & 1 \\ ab & a+b & 1 \end{vmatrix} = (a-b)(b-c)(c-a)$

Sol. L.H.S. = $\begin{vmatrix} bc & b+c & 1 \\ ca & c+a & 1 \\ ab & a+b & 1 \end{vmatrix}$

$= \begin{vmatrix} bc & b+c & 1 \\ c(a-b) & a-b & 0 \\ b(a-c) & a-c & 0 \end{vmatrix}$ by $\begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$

$= (a-b)(a-c)\begin{vmatrix} bc & b+c & 1 \\ c & 1 & 0 \\ b & 1 & 0 \end{vmatrix}$

= (a – b) (a – c) (c – b)

[expanding on 3rd column]

= (a – b) (b – c) (c – a) = R.H.S.

2. Show that $\begin{vmatrix} b+c & c+a & a+b \\ a+b & b+c & c+a \\ a & b & c \end{vmatrix}$ **= $a^3 + b^3 + c^3 - 3abc$.**

Sol. $\begin{vmatrix} b+c & c+a & a+b \\ a+b & b+c & c+a \\ a & b & c \end{vmatrix}$

$R_1 \to R_1 + R_3$

$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ a+b & b+c & c+a \\ a & b & c \end{vmatrix}$

$R_2 \to R_2 - R_1$

$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ -c & -a & -b \\ a & b & c \end{vmatrix}$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ -c & -a & -b \\ a & b & c \end{vmatrix}$$

$$= (a+b+c)[(-ac+b^2) - (-c^2+ab) + (-bc+a^2)]$$

$$= (a+b+c)(-ac+b^2+c^2-ab-bc+a^2)$$

$$= (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

$$= a^3+b^3+c^3-3abc$$

3. Show that $\begin{vmatrix} y+z & x & x \\ y & z+x & y \\ z & z & x+y \end{vmatrix} = 4xyz$ **.**

Sol. L.H.S. = (y + z)[(z + x) (x + y) – yz] – x[y(x + y) – yz] + x[yz – z(z + x)]

$$= (y+z)(zx+yz+x^2+xy-yz) - x(xy+y^2-yz) + x(yz-z^2-zx)$$

$$= (y+z)(zx+x^2+xy) - x(xy+y^2-yz) + x(yz-z^2-zx)$$

$$= xyz+x^2y+xy^2+xz^2+x^2z+xyz - x^2y-xy^2+xyz+xyz-xz^2-x^2z$$

$$= 4xyz$$

4. If $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$ **, then show that abc = –1.**

Hint : If each element in row (column) of a square matrix is the sum of two numbers, then its discriminant can be expressed as the sum of discriminants of two square matrices.

Sol. L.H.S = $\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$

$$= \begin{vmatrix} 1 & a^2 & a \\ 1 & b^2 & b \\ 1 & c^2 & c \end{vmatrix} + abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= (1+abc) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$$

$\therefore$ 1 + abc = 0 $\Rightarrow$ abc = –1

III.

1. Show that

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$$

Sol. L.H.S. $C_1 \to C_1 + (C_2 + C_3)$

$$= \begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix}$$

$$= 2(a+b+c) \begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

$(R_2 \to R_2 - R_1, R_3 \to R_3 - R_1)$

$$= 2(a+b+c) \begin{vmatrix} 1 & a & b \\ 0 & a+b+c & 0 \\ 0 & 0 & a+b+c \end{vmatrix}$$

$$= 2(a+b+c)(a+b+c)^2$$

$$= 2(a+b+c)^3 = \text{R.H.S}$$

2. Show that

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = (a^3+b^3+c^3-3abc)^2 .$$

Sol. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$= a(bc-a^2)-b(b^2-ac)+c(ab-c^2)$

$= abc-a^3-b^3+abc+abc-c^3$

$= -(a^3+b^3+c^3-3abc)$ ...(1)

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times (-) \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} -a & -b & -c \\ c & a & b \\ b & c & a \end{vmatrix}$$

$$= \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} \ldots(2)$$

From (1), (2) we get

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}$$

$$(a^3+b^3+c^3-3abc)^2$$

3. Show that $\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3$**.**

Sol. L.H.S. $= \begin{vmatrix} a^2-1 & a-1 & 0 \\ 2(a-1) & a-1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$ $\begin{matrix} R_1 \to R_1 - R_2 \\ R_2 \to R_2 - R_3 \end{matrix}$

$= (a-1)^2 \begin{vmatrix} a+1 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$

$= (a-1)^2[0(6-3) - 0[3(a+1)-3) + 1(a+1-2)$

$= (a-1)^2(a-1) = (a-1)^3 = \text{R.H.S.}$

4. Show that $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$ **= abc (a – b)(b – c)(c – a).**

Sol. L.H.S. $abc \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

$= abc \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^2-b^2 & b^2-c^2 & c^2 \end{vmatrix}$ $\begin{matrix} C_1 \to C_1 - C_2 \\ C_2 \to C_2 - C_3 \end{matrix}$

$= abc(a-b)(b-c) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a+b & b+c & c^2 \end{vmatrix}$

$= abc(a-b)(b-c)[0(c^2 - c(b+c) - 0(c^2 - c(a+b) + 1(b+c-a-b)]$

$= abc(a-b)(b-c)(c-a)$

5. Show that $\begin{vmatrix} -2a & a+b & c+a \\ a+b & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix}$ **= 4(a + b)(b + c)(c + a).**

Sol. Let $\Delta = \begin{vmatrix} -2a & a+b & c+a \\ a+b & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix}$

Let a + b = 0, then $\Delta = \begin{vmatrix} -2a & 0 & c+a \\ 0 & 2a & -a+c \\ c+a & c-a & -2c \end{vmatrix}$

Apply $R_1 \to R_1 + R_3$, $R_3 \to R_3 + R_2$

$\Delta = \begin{vmatrix} c-a & c-a & -c+a \\ 0 & 2a & -a+c \\ c+a & c+a & -c-a \end{vmatrix}$

$= (c-a)(c+a)\begin{vmatrix} 1 & 1 & -1 \\ 0 & 2a & c-a \\ 1 & 1 & -1 \end{vmatrix} = 0$ $(\because R_1 \equiv R_3)$

$\therefore$ (c + a) is a factor for Δ.

Similarly a + b, b + c are also factors of Δ.

$\because$ Δ is a third degree expression in a, b, c.

$\Delta \equiv k(a + b)(b + c)(c + a)$, where k is a non-zero scalar.

Put a = 1, b = 1, c = 1, then

$\begin{vmatrix} -2 & 2 & 2 \\ 2 & -2 & 2 \\ 2 & 2 & -2 \end{vmatrix}$ = k(1 + 1)(1 + 1)(1 + 1)

$\Rightarrow -2(4 - 4) - 2(-4 - 4) + 2(4 + 4) = 8k$

$\Rightarrow 16 + 16 = 8k \Rightarrow k = 4$

$\therefore \Delta = 4(a + b)(b + c)(c + a)$

Hence

$\begin{vmatrix} -2a & a+b & a+c \\ a+b & -2b & b+c \\ a+c & b+c & -2c \end{vmatrix}$ = 4(a + b)(b + c)(c + a).

6. Show that $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$.

Sol. L.H.S. $= \begin{vmatrix} 0 & 0 & 0 \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$

By $R_1 \to R_1 + (R_2 + R_3)$

7. Show that $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix} = 0$.

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_2$

Sol. L.H.S. $\begin{vmatrix} 1 & a & a^2-bc \\ 0 & b-a & b^2-a^2+bc-ca \\ 0 & c-b & c^2-b^2+ac-ab \end{vmatrix}$

$= (b-a)(c-b)\begin{vmatrix} 1 & a & a^2-bc \\ 0 & 1 & a+b+c \\ 0 & 1 & a+b+c \end{vmatrix}$

$= (b-a)(c-b)\cdot 0$ (R_2, R_3 are identical)

$= 0 =$ R.H.S.

8. Show that $\begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} = (x+2a)(x-a)^2$.

Sol. L.H.S. $= \begin{vmatrix} x+2a & a & a \\ x+2a & x & a \\ x+2a & a & x \end{vmatrix}$

By $C_1 \to C_1 + (C_2 + C_3)$

$= (x+2a)\begin{vmatrix} 1 & a & a \\ 1 & x & a \\ 1 & a & x \end{vmatrix}$

$$= (x+2a)\begin{vmatrix} 1 & a & a \\ 0 & x-a & 0 \\ 0 & 0 & x-a \end{vmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$= (x+2a)[1(x-a)^2 - a(0(x-a)-0)]$$
$$+a[0-0(x-a)]$$
$$= (x+2a)(x-a)^2 = \text{R.H.S}$$

EXERCISE – 3(E)

I.

1. Find the adjoint and inverses of the following matrices.

i) $\begin{bmatrix} 2 & -3 \\ 4 & 6 \end{bmatrix}$ **if A =** $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ **then**

Sol. Adj $A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ -4 & 2 \end{bmatrix}$

$|A| = 12 - (-12) = 24$

$$A^{-1} = \frac{AdjA}{DetA} = \frac{1}{24}\begin{bmatrix} 6 & 3 \\ -4 & 2 \end{bmatrix}$$

ii) $\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

Sol. Adj $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, det A = 1

$$A^{-1} = \frac{AdjA}{DetA} = \frac{1}{\cos^2\alpha + \sin^2\alpha}\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$
$$= \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

iii) $\begin{bmatrix} +1 & -0 & +2 \\ -2 & +1 & -0 \\ +3 & -2 & +1 \end{bmatrix}$

Sol. $A_1 = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1$

$$B_1 = -\begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} = -2$$

$$C_1 = \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 4 - 3 = 1$$

$$A_2 = -\begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix} = -(0 - 4) = 4$$

$$B_2 = \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 1 - 6 = 5$$

$$C_2 = -\begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix} = -(2 - 0) - 2$$

$$A_3 = \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} = 0 - 2 = -2$$

$$B_3 = -\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} = -(0 - 4) = 4$$

$$C_3 = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1 - 0 = 1$$

$$\text{AdjA} = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} = \begin{bmatrix} 1 & 4 & -2 \\ -2 & -5 & 4 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\det A = a_1A_1 + b_1B_1 + c_1C_1$$

$$= 1(1) + 0(-2) + 2(1) = 1 + 0 + 2 = 3$$

$$A^{-1} = \frac{\text{AdjA}}{\text{DetA}} = \frac{1}{3}\begin{bmatrix} 1 & 4 & -2 \\ -2 & -5 & 4 \\ 1 & -2 & 1 \end{bmatrix}$$

iv) $\begin{bmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}$

Sol. $A_1 = \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} = 0 - 2 = -2$

$$B_1 = -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -(1 - 2) = 1$$

$$C_1 = \begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix} = 2 - 0 = 2$$

$$A_2 = -\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -(1-4) = 3$$

$$B_2 = \begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} = 2-4 = -2$$

$$C_2 = -\begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} = -(4-2) = -2$$

$$A_3 = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1-0 = 1$$

$$B_3 = -\begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} = -(2-2) = 0$$

$$C_3 = \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = 0-1 = -1$$

$$\text{AdjA} = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} = \begin{bmatrix} -2 & 3 & 1 \\ 1 & -2 & 0 \\ 2 & -2 & -1 \end{bmatrix}$$

$$\text{DetA} = a_1A_1 + b_1B_1 + c_1C_1$$

$$= 2(-2) + 1(1) + 2(2) = -4 + 1 + 4 = 1$$

$$A^{-1} = \frac{\text{AdjA}}{\text{DetA}} = \begin{bmatrix} -2 & 3 & 1 \\ 1 & -2 & 0 \\ 2 & -2 & -1 \end{bmatrix}$$

2. If A = $\begin{bmatrix} a+ib & c+id \\ -c+id & a-ib \end{bmatrix}$, $a^2 + b^2 + c^2 + d^2 = 1$ then find inverse of A.

Sol. Det A = (a + ib)(a – ib) – (c + id)(–c + id)

$= a^2 - i^2 b^2 - (-c^2 + i^2 d^2)$

$= a^2 + b^2 + c^2 + d^2$ ($-i^2 = 1$) = 1

$$\text{AdjA} = \begin{bmatrix} a-ib & -c-id \\ c-id & a+ib \end{bmatrix}$$

$$A^{-1} = \frac{\text{AdjA}}{\text{DetA}} = \begin{bmatrix} a-ib & -c-id \\ c-id & a+ib \end{bmatrix}$$

3. If $A=\begin{bmatrix}1 & -2 & 3\\ 0 & -1 & 4\\ -2 & 2 & 1\end{bmatrix}$**, find** $(A^T)^{-1}$**.**

Sol. $A^T=\begin{bmatrix}1 & -2 & 3\\ 0 & -1 & 4\\ -2 & 2 & 1\end{bmatrix}^T=\begin{bmatrix}1 & 0 & -2\\ -2 & -1 & 2\\ 3 & 4 & 1\end{bmatrix}$

$A_1=\begin{vmatrix}-1 & 2\\ 4 & 1\end{vmatrix}=-1-8=-9$

$B_1=-\begin{vmatrix}-2 & 2\\ 3 & 1\end{vmatrix}=-(-2-6)=8$

$C_1=\begin{vmatrix}-2 & -1\\ 3 & 4\end{vmatrix}=(-8+3)=-5$

$A_2=-\begin{vmatrix}0 & -2\\ 4 & 1\end{vmatrix}=-(0+8)=-8$

$B_2=\begin{vmatrix}1 & -2\\ 3 & 1\end{vmatrix}=1+6=7$

$C_2=-\begin{vmatrix}1 & 0\\ 3 & 4\end{vmatrix}=-(4-0)=-4$

$A_3=\begin{vmatrix}0 & -2\\ -1 & 2\end{vmatrix}=0-2=-2$

$B_3=-\begin{vmatrix}1 & -2\\ -2 & 2\end{vmatrix}=-(2-4)=2$

$C_3=\begin{vmatrix}1 & 0\\ -2 & -1\end{vmatrix}=-1-0=-1$

$\text{Adj}A^T=\begin{bmatrix}-9 & -8 & -2\\ 8 & 7 & 2\\ -5 & -4 & -1\end{bmatrix}$

$\text{Det}A^T=1(-9)+0(8)-2(-5)=-9+10=1$

$(A^T)^{-1}=\dfrac{\text{Adj}(A^T)}{\det A^T}=\begin{bmatrix}-9 & -8 & -2\\ 8 & 7 & 2\\ -5 & -4 & -1\end{bmatrix}$

4. **If** $A = \begin{vmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{vmatrix}$, **then show that Adj A = 3A^T find A^{-1}.**

Sol. $A_1 = \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} = 1-4 = -3$

$B_1 = -\begin{vmatrix} 2 & -2 \\ 2 & 1 \end{vmatrix} = -(2+4) = -6$

$C_1 = \begin{vmatrix} 2 & 1 \\ 2 & -2 \end{vmatrix} = -4-2 = -6$

$A_2 = -\begin{vmatrix} -2 & -2 \\ -2 & 1 \end{vmatrix} = -(-2-4) = 6$

$B_2 = \begin{vmatrix} -1 & -2 \\ 2 & 1 \end{vmatrix} = -1+4 = 3$

$C_2 = -\begin{vmatrix} -1 & -2 \\ 2 & -2 \end{vmatrix} = -(2+4) = -6$

$A_3 = \begin{vmatrix} -2 & -2 \\ 1 & -2 \end{vmatrix} = 4+2 = 6$

$B_3 = -\begin{vmatrix} -1 & -2 \\ 2 & -2 \end{vmatrix} = -(2+4) = -6$

$C_3 = \begin{vmatrix} -1 & -2 \\ 2 & -2 \end{vmatrix} = -1+4 = 3$

$$\text{AdjA} = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} = \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix} \ldots(1)$$

$$A^T = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}^T = \begin{bmatrix} -1 & 2 & 2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}$$

$$3A^T = 3\begin{bmatrix} -1 & 2 & 2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix} \ldots(2)$$

From (1) and (2) we get Adj A = 3A^T

$\text{Det}A = a_1A_1 + b_1B_1 + c_1C_1$

$= (-1)(-3) + (-2)(-6) + (-2)(-6)$

$= 3+12+12 = 27$

$$A^{-1} = \frac{AdjA}{DetA} = \frac{1}{27}\begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} -1 & 2 & 2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}$$

5. If abc ≠ 0, find the inverse of $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$**.**

Sol. $A_1 = \begin{vmatrix} b & 0 \\ 0 & c \end{vmatrix} = bc$

$B_1 = -\begin{vmatrix} 0 & 0 \\ 0 & c \end{vmatrix} = 0 \qquad C_1 = \begin{vmatrix} 0 & b \\ 0 & 0 \end{vmatrix} = 0$

$A_2 = -\begin{vmatrix} 0 & 0 \\ 0 & c \end{vmatrix} = 0 \qquad B_2 = \begin{vmatrix} a & 0 \\ 0 & c \end{vmatrix} = ac$

$C_2 = -\begin{vmatrix} a & 0 \\ 0 & 0 \end{vmatrix} = 0 \qquad A_3 = \begin{vmatrix} 0 & 0 \\ b & 0 \end{vmatrix} = 0$

$B_3 = -\begin{vmatrix} a & 0 \\ 0 & 0 \end{vmatrix} = 0 \qquad C_3 = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$

Det A = $a_1A_1 + b_1B_1 + c_1C_1$

= 0(–1) + 1(1) + 1(1) = 1 + 1 = 2

$$A^{-1} = \frac{AdjA}{DetA} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} b+c & c-a & b-a \\ c-b & c+a & a-b \\ b-c & a-c & a+b \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} c-b+b-c & c+a+a-c & a-b+a+b \\ b+c+b-c & c-a+a-c & b-a+a+b \\ b+c+c-b & c-a+c+a & b-a+a-b \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 2a & 2a \\ 2b & 0 & 2b \\ 2c & 2c & 0 \end{bmatrix}$$

$$ABA^{-1} = \frac{1}{2}\begin{bmatrix} 0 & 2a & 2a \\ 2b & 0 & 2b \\ 2c & 2c & 0 \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 2a+2a & -2a+2a & 2a-2a \\ -2b+2b & 2b+2b & 2b-2b \\ -2c+2c & 2c-2c & 2c+2c \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4a & 0 & 0 \\ 0 & 4b & 0 \\ 0 & 0 & 4c \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = \text{Diagonal matrix.}$$

6. If $3A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$, then show that $A^{-1} = A^T$.

Sol. $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix} \Rightarrow A^T = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$

Now $A \cdot A^T = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}\frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$A.A^T = I$

$\therefore A^{-1} = A^T$

7. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, then show that $A^{-1} = A^3$.

Sol. $A^2 = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix}$

$$A^4 = A^2A^2 = \begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix}\begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1$$

$\therefore A^4 = I$

$\det A = 3(1) - 3(-2) + 4(-2) = 1$

$\because A \neq 0 \Rightarrow A^{-1}$exists

$\because A^4 = I$

Multiply with A^{-1}

$A^4(A^{-1}) = I(A^{-1})$

$\Rightarrow A^3(AA^{-1}) = A^{-1} \Rightarrow A^3(I) = A^{-1}$

$\therefore A^{-1} = A^3$

8. If AB = I or BA = I, then prove that A is invertible and B = A^{-1}.

Sol. Given AB = I $\Rightarrow$ |AB| = | I |

= |A| |B| = 1

= |A| $\neq$ 0

$\therefore$ A is a non-singular matrix.

and BA = I $\Rightarrow$ |BA| = | I |

$\Rightarrow$ |B| |A| = 1 $\Rightarrow$ |A| $\neq$ 0

$\therefore$ A is a non-singular matrix.

AB = I or BA = I, A is invertible.

$\therefore$ A^{-1} exists.

AB = I $\Rightarrow$ A^{-1} AB = A^{-1} I

$\Rightarrow$ IB = $A^{-1} \Rightarrow$ B = A^{-1}

$\therefore$ B = A^{-1}

EXERCISE – 3(F)

I. Find the rank of the following matrices.

1. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Sol. Det A = $\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0 - 0 = 0$

and $|a| = 1 \neq 0$

$\therefore \rho(A) = 1.$

2. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Sol. Det A = $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1 \neq 0$

$\therefore \rho(A) = 2$

3. $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

Sol. Det A = $\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} = 0 - 0 = 0$

$|1| = 1 \neq 0$

$\therefore \rho(A) = 1$

4. $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

Sol. Det A = $\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1 \neq 0$

$\therefore \rho(A) = 2$

5. $\begin{bmatrix} 1 & 0 & -4 \\ 2 & -1 & 3 \end{bmatrix}$

Sol. $\begin{vmatrix} 1 & -4 \\ 2 & 3 \end{vmatrix} = 3 + 8 = 11 \neq 0$

$\therefore \rho(A) = 2$

6. $\begin{bmatrix} 1 & 2 & 6 \\ 2 & 4 & 3 \end{bmatrix}$

Sol. $\begin{vmatrix} 2 & 6 \\ 4 & 3 \end{vmatrix} = 6 - 24 = -18 \neq 0$

$\therefore \rho(A) = 2$

II.

1. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Sol. Det A $= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$= 1(1-0) - 0(0-0) + 0(0-0)$

$= 1 - 0 + 0 = 1 \neq 0$

$\therefore \rho(A) = 3$

2. $\begin{bmatrix} 1 & 4 & -1 \\ 2 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

Sol. Det A $= \begin{vmatrix} 1 & 4 & -1 \\ 2 & 3 & 0 \\ 0 & 1 & 2 \end{vmatrix}$

$= 1(6-0) - 2(8+1) + 0(0+3)$

$= 6 - 18 = -12 \neq 0$

$\therefore \rho(A) = 3$

3. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$

Sol. Det A $= \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{vmatrix}$

$= 1(6-4) - 2(4-3) + 0(8-9)$

$= 2 - 2 + 0 = 0$

$\therefore \rho(A) \neq 3, \rho(A) < 3$

Take $\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3 - 4 = -1 \neq 0$

$\therefore \rho(A) = 2$

4. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, $\det A = 0$, $\rho(A) \neq 3$.

Sol. All 2×2 sub matrix det is zero.

$\therefore \rho(A) \neq 2$

$|1| = 1 \neq 0, \therefore \rho(A) = 1$

5. $\begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix}$

Sol. Take sub matrix $B = \begin{vmatrix} 1 & 2 & 0 \\ 3 & 4 & 1 \\ -2 & 3 & 2 \end{vmatrix}$

$= 1(8-3) - 2(6+2)$

$= 5 - 16 = -11 \neq 0$

Rank of the given matrix is 3.

6. $\begin{bmatrix} 0 & 1 & 1 & -2 \\ 4 & 0 & 2 & 5 \\ 2 & 1 & 3 & 1 \end{bmatrix}$

Sol. Take sub matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 4 & 0 & 2 \\ 2 & 1 & 3 \end{bmatrix}$

$= -1(12-4) + 1(4-0)$

$= -8 + 4 = -4 \neq 0$

$\therefore \rho(A) = 3$

EXERCISE – 3(G)

Examine whether the following systems of equations are consistent or inconsistent and if consistent find the complete solutions.

1. **x + y + z = 4**
2x + 5y – 2z = 3
X + 7y – 7z = 5

Sol. Augmented matrix = $A = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 2 & 5 & -2 & 3 \\ 1 & 7 & -7 & 5 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$, $R_2 \to R_3 - R_1$ we get

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 6 & -8 & 1 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$ we have $A \sim \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 0 & 0 & 11 \end{bmatrix}$

$\rho(A) = 2, \rho(AB) = 3$

$\rho(A) \neq \rho(AB)$

$\therefore$ The given system of equations are in consistent.

2. **x + y + z = 6**
x – y + z = 2
2x – y + 3z = 9

Sol. Augmented matrix $A = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -1 & 1 & 2 \\ 2 & -1 & 3 & 9 \end{bmatrix}$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$ we get

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -2 & 0 & -4 \\ 0 & -3 & 1 & -3 \end{bmatrix}$$

$\rho(A) = 3 = \rho(AB)$

$\therefore$ The given system of equations are consistent.

$\therefore$ $A \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -2 & 0 & -4 \\ 0 & -3 & 1 & -3 \end{bmatrix}$

By $R_2 \to R_2\left(-\frac{1}{2}\right)$, we obtain

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & -3 & 1 & -3 \end{bmatrix}$$

By $R_1 \to R_1 - R_2$, $R_3 \to R_3 + 3R_2$ we get

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

By $R_1 \to R_1 - R_3$, we have

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$\therefore$ Solution is x = 1, y = 2, z = 3.

3. **x + y + z = 1**
2x + y + z = 2
x + 2y + 2z = 1

Sol. Augmented matrix is $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 2 & 2 & 1 \end{bmatrix}$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$R_1 \to R_1 - R_3$, we obtain $A \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

$R_2 \to R_2 - R_1$, we have $A \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

$\rho(A) = 2 = \rho(AB) < 3$
The given system of equations are consistent and have infinitely many solutions.
The solutions are given by :
[(x, y, z)1x = y + z = 0]

4. **x + y + z = 9**
2x + 5y + 7z = 52
2x + y – z = 0

Sol. Augmented matrix is $A = \begin{bmatrix} 1 & 1 & 1 & 9 \\ 2 & 5 & 7 & 52 \\ 2 & 1 & -1 & 0 \end{bmatrix}$

By $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 2R_1$, we get

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 3 & 5 & 34 \\ 0 & -1 & -3 & -18 \end{bmatrix}$$

By $R_3 \to R_3 (-1)$, we obtain

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 3 & 5 & 34 \\ 0 & 1 & 3 & 18 \end{bmatrix}$$

By $R_1 \to R_1 - R_3$, $R_2 \to R_2 - 3R_3$, we have

$$A \sim \begin{bmatrix} 1 & 0 & -2 & -9 \\ 0 & 0 & -4 & -20 \\ 0 & 1 & 3 & 18 \end{bmatrix}$$

By $R_2 \to R_2\left(-\frac{1}{4}\right)$, we obtain

$$A \sim \begin{bmatrix} 1 & 0 & -2 & -9 \\ 0 & 0 & 1 & 5 \\ 0 & 1 & 3 & 18 \end{bmatrix}$$

By $R_1 \to R_1 + 2R_2$, $R_3 \to R_3 - 3R_2$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 5 \\ 0 & 1 & 0 & 3 \end{bmatrix}$$

By $R_2 \leftrightarrow R_3$, we obtain $A \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{bmatrix}$

$\therefore \rho(A) = \rho(AB) = 3$

The given system of equations are consistent have a unique solution.

$\therefore$ Solution is given by $x = 1$, $y = 3$, $z = 5$.

5. **x + y + z = 6**
x + 2y + 3z = 10
x + 2y + 4z = 1

Sol. Augmented matrix $A = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & 4 & 1 \end{bmatrix}$

By $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_2$, we obtain

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & -9 \end{bmatrix}$$

By $R_1 \to R_1 - R_3$, $R_2 \to R_2 - 2R_3$, we get

$$A \sim \begin{bmatrix} 1 & 1 & 0 & 15 \\ 0 & 1 & 0 & 22 \\ 0 & 0 & 1 & -9 \end{bmatrix}$$

By $R_1 \to R_1 - R_2$, we have

$$A \sim \begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & 22 \\ 0 & 0 & 1 & -9 \end{bmatrix}$$

$\rho(A) = \rho(AB) = 3$

The given system of equations are consistent. They have a unique solution.

$\therefore$ Solution is given by $x = -7$, $y = 22$, $z = -9$.

6. **x – 3y – 8z = –10**
3x + y – 4z = 0
2x + 5y + 6z = 13

Sol. The Augmented matrix

$$A = \begin{bmatrix} 1 & -3 & -8 & -10 \\ 3 & 1 & -4 & 0 \\ 2 & 5 & 6 & 13 \end{bmatrix}$$

By $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$, we get

$$A \sim \begin{bmatrix} 1 & -3 & -8 & -10 \\ 0 & 10 & 20 & 30 \\ 0 & 11 & 22 & 33 \end{bmatrix}$$

$R_2 \to R_2\left(\frac{1}{10}\right)$, $R_3 \to R_3\left(\frac{1}{11}\right)$ we have

$$A \sim \begin{bmatrix} 1 & -3 & -8 & -10 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

By $R_3 \to R_3 - R_2$, we obtain

$$A \sim \begin{bmatrix} 1 & -3 & -8 & -10 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By $R_1 \to R_1 + 4R_2$, we get

$$A \sim \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = \rho(AB) = 2 < 3$

$\therefore$ The given system of equations are consistent have infinitely many solutions.

$x + y = 2$ and $y + 2z = 3$

Taking $z = k$, $y = 3 - 2z = 3 - 2k$

$x = 2 - y = 2 - (3 - 2k) = 2 - 3 + 2k = 2k - 1$

$\therefore$ The solutions are given by $x = -1 + 2k$,

$y = 3 - 2k$, $z = k$ where k is any scalar.

7. **$2x + 3y + z = 9$**
$x + 2y + 3z = 6$
$3x + y + 2z = 8$

Sol. Augmented matrix $A = \begin{bmatrix} 2 & 3 & 1 & 9 \\ 1 & 2 & 3 & 6 \\ 3 & 1 & 2 & 8 \end{bmatrix}$

By $R_1 \leftrightarrow R_2$, we get $A = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 2 & 3 & 1 & 9 \\ 3 & 1 & 2 & 8 \end{bmatrix}$

By $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$, we have

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & -1 & -5 & -3 \\ 0 & -5 & -7 & -10 \end{bmatrix}$$

By $R_3 \to R_3 - 5R_2$, $R_1 \to R_1 + 2R_2$, we get

$$A \sim \begin{bmatrix} 1 & 0 & -7 & 0 \\ 0 & -1 & -5 & -3 \\ 0 & 0 & 18 & 5 \end{bmatrix}$$

By $R_3 \to R_3\left(\frac{1}{18}\right)$, $R_2 \to R_2(-1)$ we obtain

$$A \sim \begin{bmatrix} 1 & 0 & -7 & 0 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 1 & 5/18 \end{bmatrix}$$

By $R_1 \to R_1 + 7R_3$, $R_2 \to R_2 - 5R_3$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 35/18 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 1 & 5/18 \end{bmatrix}$$

By $R_2 \to R_2 - 5R_3$, we obtain

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 35/18 \\ 0 & 1 & 0 & 29/18 \\ 0 & 0 & 1 & 5/18 \end{bmatrix}$$

$\rho(A) = \rho(AB) = 3$

The given system of equations are consistent have a unique solution.

$\therefore$ Solution is given by $x = \frac{35}{18}, y = \frac{29}{18}, z = \frac{5}{18}$.

8. **Augmented matrix A =** $\begin{bmatrix} 1 & 1 & 4 & 6 \\ 3 & 2 & -2 & 9 \\ 5 & 1 & 2 & 13 \end{bmatrix}$

By $R_2 \to R_2 - 3R_1$, $R_3 \to R_2 - 5R_1$, we get

$$A \sim \begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & -1 & -14 & -9 \\ 0 & -4 & -18 & -17 \end{bmatrix}$$

By $R_3 \to R_3 - 4R_2$, we get

$$A \sim \begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & -1 & -14 & -9 \\ 0 & 0 & 38 & 19 \end{bmatrix}$$

By $R_2 \to R_2(-1)$, $R_3 \to R_3\left(\frac{1}{2}\right)$, we have

$$A \sim \begin{vmatrix} 1 & 1 & 4 & 6 \\ 0 & 1 & 14 & 9 \\ 0 & 0 & 1 & 1/2 \end{vmatrix}$$

By $R_1 \to R_1 - 4R_3$, $R_2 \to R_2 - 14R_3$, we obtain

$$A \sim \begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1/2 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1/2 \end{bmatrix}$$

$\rho(A) = \rho(AB) = 3$

$\therefore$ The given system of equations are consistent have a unique solution.
$\therefore$ Solution is given by $x = 2, y = 2, z = 1/2$.