# MATRICES

**1. Matrix:** The arrangement of numbers or letters in the horizontal and vertical lines so that each horizontal line contains same number of elements and each vertical row contains the same numbers of elements

**Ex:** $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & 3 \\ a & b & c \end{bmatrix}$

**2.** Horizontal lines are called rows. Vertical lines are called columns.

**3. Order of a matrix:** The number of rows and number of columns of a matrix is called order of matrix.

If a matrix contains 'm' rows and 'n' columns then its order is denoted by m x n {It is read as a m by n}

**4.** Generally matrix of order m x n is denoted by $(aij)_{m\times n}$

**5. Row matrix:** If a matrix contains only one row then the matrix is called row matrix

**Ex:** $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}_{1\times 3}$ general form of representing a row matrix is $(aij)_{1\times n}$

**6. Column matrix:** If a matrix contains only one column then the matrix is called column matrix

**Ex:** $\begin{bmatrix} a \\ b \\ c \end{bmatrix}_{3\times 1}$

General form of representing a column matrix is $(aij)_{m\times 1}$

**7. Rectangular matrix:** If the number of rows of a matrix in not equal to the number of columns of the matrix the matrix then the matrix is called rectangular matrix

**Ex :** $\begin{bmatrix} a & b & c \\ p & q & r \end{bmatrix}_{2\times 3}$

**8. Square matrix:** If the number of rows of a matrix is equal to the number of columns of the matrix then the matrix is called a square matrix.

**Ex:** $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

**9. Principal diagonal :** In a square matrix the diagonal joining the first row first column to the last row last column is called 'principal diagonal' (or) leading diagonal the principal diagonal is as shown below.

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

**10. Trace of a matrix:** The sum of the principal diagonal elements of a square matrix is called trace of a matrix

If $A=(aij)_{n\times n}$ is a square matrix then $a_{11}+a_{22}+a_{33}+.....+a_{nn}$ i.e $\sum_{i=1}^{n} aij$ is called trace of matrix A denoted by tr (A)

**11. Lower triangular matrix:** A square matrix $(aij)_{n\times n}$ is said to be a lower triangular matrix when aij = 0 $\forall\ i< j$

**Ex :** $\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{33} & a_{32} & a_{33} \end{bmatrix}$

**12. Upper triangular matrix:** A square matrix $(aij)_{n\times n}$ is said to be upper triangular matrix when aij = 0 $\forall\ i > j$

**Eg :** $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & 8 \end{bmatrix}$

**13. Triangular matrix:** A square matrix is said to be a triangular matrix if it is either upper triangular matrix or lower triangular matrix

**14. Diagonal matrix:** A square matrix is said to be a diagonal matrix if all the principal diagonal elements are non zero and all the remaining elements are zero

A square matrix $(aij)_{n\times n}$ is said to be diagonal matrix when $aij = 0 \;\; i \neq j \,, \neq 0$
i = j

**Ex:** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

The matrix $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ is also denoted by diag {a, b, c}

**15. Scalar matrix:** A square matrix $aij = k \quad (\neq 0)$ when i = j $\quad 0 \quad$ when $\quad i \neq j$

**Eg :** $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

**16. Unit matrix:** A square matrix $(aij)_{n\times n}$ is said to be unit matrix when

aij = 1 when $\quad$ i = j

$\quad = 0 \quad$ when $\quad i \neq j$

Eg : $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Generally unit matrix is denoted by I

**17. Null Matrix:** A matrix is said to be a null matrix if all the elements are zeros.

**18. Equality of matrices:** Two matrices A and B are said to be equal if

i) A and B are of same order and

ii) The corresponding elements of A and B are the same

if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$ are equal then

$a_{11} = b_{11}$ $a_{12} = b_{12}$ $a_{13} = b_{13}$

$a_{21} = b_{21}$ $a_{22} = b_{22}$ $a_{23} = b_{23}$

**19. Definition: {sum of two matrices}:** Let A and B be matrices of same order. Then the sum of A and B denoted by A + B is defined as the matrix of the same order in which each element is the sum of corresponding elements of A and B

If $A = (aij)_{m\times n}$; $B = (bij)_{m\times n}$ then $A + B = (cij)_{m\times n}$

$cij = aij + bij$

**20.** **(i)** matrix addition is commutative i.e. A + B = B + A

ii) Matrix addition is associative i.e. A + (B + C) = (A + B) + C

iii) If A is m x n matrix and O is the m x n null matrix A + O = O +A = A 'O' is called the additive identify

iv) If A is m x n matrix there is unique m x n matrix B such that A + B = B + A = O, O being the m x n null matrix.

This B is denoted by –A and is called the 'additive inverse' of A.

**21. Scalar multiplication of matrix:** Let A be a matrix of order m x n and k be a scalar then the m x n matrix obtained by multiplying each element of A by k is called a scalar multiple of A and is denoted by k A.

**Properties of scalar multiplication:** Let A and B be matrices of the same order and $\alpha, \beta$ be scalar then

**i)** $\alpha(\beta A) = (\alpha\beta) A = \beta(\alpha A)$

**ii)** $(\alpha + \beta) A = \alpha A + \beta A$

iii) OA = O

iv) $\alpha(A + \beta) = \alpha A + \alpha\beta$

**1. Multiplication of Matrices:** Two matrices A and B are confirmable for multiplication when the number of columns of A is equal to the number of rows of B.

**2. Definition: (Product of two matrices)** Let $A=(aij)_{m\times n}$ and $B=(bij)_{n\times p}$ where $cij=\sum_{k=1}^{n} aik\ bkj$ is called product of A and B and is denoted by AB.

**i)** If A, B are two matrices such that AB exist then BA need not exist

**ii)** If AB and BA both exist then they need not be equal

**iii)** Matrix product is not commutative

**Properties of Multiplication:**

1) If A, B, C are three matrices then A (BC) = (AB) C

**2)** i.e. matrix product is associative

**i)** Matrix product is distributive over addition i.e. A (B + C) = AB + AC

**ii)** If A, B are two matrices such that AB = 0 then it is not necessary that either A = 0 or B = 0 or both A and B are null matrices

**iii)** If A, B, C are three matrices such that AB = AC then it is not necessary that either a = 0 or B = C

**iv)** If A is a square matrix of order n and I is an identify matrix of order n then AI = IA = A

**3. Idem potent matrix:** A square matrix A is said to be an idempotent matrix if $A^2=A$

**4. Involutory matrix:** A square matrix A is said to be involutory matrix if $A^2=I$

**5. Nill Potent matrix:** A square matrix A is said to be nill point matrix when $A^n=0$ the least value of n is called the index of nill point matrix

**6. Transpose of a matrix:** If $A=(aij)_{m\times n}$ is a matrix then the matrix obtained by interchanging the rows into columns is called the transpose of A this is denoted by $A^T$ or $A'$

If $A=(aij)_{m\times n}$ then $A'=(aij)_{n\times m}$

**7. Properties of Transpose:**

i) $(A^T)^T = A$

ii) $(A+B)^T = A^T + B^T$

iii) $(A-B)^T = A^T - B^T$

iv) $(AB)^T = B^T A^T$

v) $(KA)^T = KA^T$

**8. Symmetric matrix:** A square matrix A is said to be symmetric matrix if $A^T = A$

**9. Skew symmetric matrix:** A square matrix A is said to be skew symmetric matrix if $A^T = -A$

**10.** Every square matrix can be uniquely expressed as the sum of symmetric and skew symmetric matrices

$$A = \frac{1}{2}\{A + A^T\} + \frac{1}{2}\{A - A^T\}$$

Where $\frac{1}{2}\{A + A^T\}$ is a symmetric matrix and $\frac{1}{2}\{A - A^T\}$ is skew symmetric matrix.

**3.4 Determinants**

**1. Determinant of 2 x 2 matrix**

**Def :** If $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a 2 x 2 matrix then ad – ac is called determinant of a matrix

**2. Minor of an element:** The determinant of a square matrix obtained by eliminating the row and column in which the element is present. This is denoted by Mij.

**3. Cofactor of an element:** The cofactor of an element in the $i^{th}$ row and $j^{th}$ defined as its minor multidied by $(-1)^{i+j}$ generally this is denoted by Aij

**4. Definition {determinant of 3 x 3 matrix}:** The sum of the products of the elements of a row or column with their cofactor is called determinant of the matrix

**Singular matrix :** If the determinant of a matrix is zero then the matrix is called singular matrix

**Non-singular matrix :** If the determinant of a matrix is non zero then the matrix is called non- singular matrix.

**Properties of Determinant :**

1. The sum of the products of the element of a row or column of a square matrix with their corresponding cofactors is the determinant of matrix this is denoted by $|A|$

2. The determinant of a matrix A is same as the determinant of its transpose i.e. $|A| = |A^T|$

3. If two rows or columns of a square matrix are interchanged then the determinant changes its sign.

4. It two rows or column of a square matrix are identical then the value of the determinant is zero

5. If all the elements of a row or column of a square matrix are multiplied by a constant k then the determinant is also multiplied by the same constant k

6. If A is a square matrix of order n then $|KA| = K^n |A|$

7. If all the elements of a row or column of a square matrix are k times the elements of any row or column then the value of the determinant is zero

8. If all the elements of a row are expressed as the sum of two elements then the determinant can also be expressed as the sum of two determinants.

9. If all the elements of a row or column of a square matrix are added to k times of the corresponding elements of any other row or column then the value of the determinant remains un altered

10. Sum of the product of the element of row with the corresponding cofactor of any other row or column is zero

11. If A, B are two square matrices of same order then $|AB| = |A||B|$

12. If all the elements of a determinant of a square are the polynomials of x and by wrify x = a if two rows are identical then x – a is a factor for the determinant

13. If A is a square matrix then $\det(A^n) = (det\ A)^n$ .

**1. Adjoint of a matrix: -** The transpose of a matrix obtained by replacing the elements of a matrix with their corresponding cofactors is called adjoint of the matrix,

2. If A is a matrix then its adjoint is denoted by adj A

3. If A is a matrix of order n x n then

i) $adj\,(KA) = K^{n-1}$ adj A

ii) $|adj\ A| = |A|^{n-1}$

iii) $adj\,(adj\ A) = |A|^{n-2}$

iv) $|Adj\ (adj\ A)| = |A|^{(n-1)^2}$

**4. Inverse of a matrix:** Let a be a non singular matrix if there exist a matrix B such that AB = BA = I then B is called the inverse of A denoted by $A^{-1}$

**5.** Matrix inverse if exist is unique

**6.** If A is a non singular matrix then $AA^{-1} = A^{-1}A = I$

**Theorem 1: If A is non singular matrix then $(A^T)^{-1} = (A^{-1})^T$ where $A^T$ is the transpose of matrix.**

**Sol:** $AA^{-1} = A^{-1}A = I$

$(AA^{-1})^T = (I)^I \Rightarrow A^T (A^{-1})^T = I \rightarrow (1)$

$(A^{-1}A)^T = (I)^I \Rightarrow A^T (A^{-1})^T = I \rightarrow (2)$

From (1) and (2) we have

$A^T\ (A^{-1})^T = (A^{-1})^T\ A^T = I$

This is in the form AB = BA = I

$\therefore B = A^{-1}$

Hence $(A^{-1})^T = (A^T)^{-1}$

**Theorem 2: If A, B are two non singular matrices then prove that** $(AB)^{-1} = B^{-1}A^{-1}$

**Sol:** A is a non singular matrix

$AA^{-1} = A^{-1}A = I \rightarrow (1)$

B is a non singular matrix

$BB^{-1} = BB^{-1} = I \rightarrow (2)$

Let AB = P; $B^{-1}A^{-1} = Q$

$PQ = (AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$

$= A(IA^{-1}) = AA^{-1} = I$

$QP = (B^{-1}A^{-1})(AB) = A(BB^{-1})A^{-1}$

$= B^{-1}(IB) = B^{-1}B = I$

$PQ = QP = I \Rightarrow Q = P^{-1}$

$\therefore B^{-1}A^{-1} = (AB)^{-1}$

**Theorem 3: If A is a non singular matrix then** $A^{-1} = \dfrac{adj\ A}{\det A}$

**Sol:** Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

$adj\ A = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C1 & C_2 & C_3 \end{bmatrix}$

Where $A_1, B_1, C_1$ are the cofactors of $a_1, b_1, c_1$

$A_2, B_2, C_2$ are the cofactors of $a_2, b_2, c_2$

$A_3, B_3, C_3$ are the cofactors of $a_3, b_3, c_3$

$$A(adj\ A)=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

$$\begin{bmatrix} a_1A_1+b_1B_1+c_1C_1 & a_1A_2+b_1B_2+c_1C_2 & a_1A_3+b_1B_3+c_1C_3 \\ a_2A_1+b_2B_1+c_2C_1 & a_2A_2+b_2B_2+c_2C_2 & a_2A_3+b_2B_3+c_2C_3 \\ a_3A_1+b_3B_1+c_3C_1 & a_3A_2+b_3B_2+c_3C_2 & a_3A_3+b_3B_3+c_3C_3 \end{bmatrix}$$

$$=\begin{bmatrix} def\ A & 0 & 0 \\ 0 & def\ A & - \\ 0 & 0 & def\ A \end{bmatrix}$$

$$\therefore A(adj\ A)=(\det A)\,I \Rightarrow A\left(\frac{adj\ A}{def\ A}\right)=I$$

Similarly we can prove

$$\left(\frac{adj\ A}{def\ A}\right)A=I$$

$$\therefore A\left(\frac{adj\ A}{\det A}\right)=\left(\frac{adj\ A}{\det A}\right)A=I$$

$$\therefore \frac{adj\ A}{\det A}=A^{-1}$$

**Sub Matrix:** A matrix obtained by eliminating some row or columns (or both) of a matrix is called sub matrix

**r – rowed minor :** The determinant of square sub matrix of order r is called r-rowed minor

**Rank of a matrix: A** positive integer r is said to be the rank of the matrix of there exist

i) At least one non-zero r-rowed minor

ii) Every (r + 1) rowed minor is zero

iii) rank of null matrix is ‘O”

**Elementary transformation :**

1. Interchanging any two row (or column)

2. Multiplication of elements of a row (or column)

3. Subtracting from (adding to) the elements of one row, the corresponding elements of any other row multiplied by a non zero number.

**Echelon form:** A matrix A is said to be in Echelon form if the number of zeros before the first non zero element in a row is less than the number of such zeros in the net row

**Ex:** $A=\begin{bmatrix} 2 & 3 & 1 & 5 & 7 \\ 0 & 3 & 5 & 0 & 2 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix}$

**Equivalent matrices :** Two matrices A and B are called equivalent if one can be btained from the other by a finite number of elementary transformation it is denoted by $A \sim B$

The equations

$a_1x+b_1y+ax=d_1;\ a_2x+b_2y+c_2z=d_2$

And $a_3x+b_3y+c_3z=d_3$ are called system of linear equations

The equations can be expressed in the matrix form as Ax = B

$$A=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\ x=\begin{bmatrix} x \\ y \\ z \end{bmatrix} B=\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

A is called coefficient matrix

X is called variable matrix

B is called constant matrix

The matrix $\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$

Is called augmented matrix denoted by k

If rank of A = rank of k = no. of variables

Then system of equations has unique solutions

Rank of A = rank of k $\neq$ no. of variables the system of equations have infinite solutions

If rank of A $\neq$ rank of k then the system has no solution.

A system of equations is said to be consistent if it has a solution A system of equations is said to be inconsistent if it has no solution

**Matrix inversion method of solving the equation**

The matrix form of equations is Ax = B

If $|A| \neq 0$ then

$x = A^{-1}B$

**Cramer's rule**

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta_1 = \begin{vmatrix} d_1 & d_1 & c_1 \\ d_2 & d_2 & c_2 \\ d_3 & d_3 & c_3 \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

$$\Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

$$x = \frac{\Delta_1}{\Delta}\ ;\ \ y = \frac{\Delta_2}{\Delta}$$

$$z = \frac{\Delta_3}{\Delta}$$

**Gauss – Jordan method**

Augmented matrix

$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$$

By applying finite no. of row transformations the matrix will be transformed into

$$\begin{bmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & \beta \\ 0 & 0 & 1 & \gamma \end{bmatrix}$$

$\therefore x = \alpha,\ y = \beta,\ z = \gamma$

# EXERCISE – 3(A)

**I.**

**1. Write the following as a single matrix.**

**Sol.** i) $[2\ \ 1\ \ 3] + [0\ \ 0\ \ 0]$

$= [2+0\ \ 1+0\ \ 3+0] = [2\ \ 1\ \ 3]$

ii) $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0-1 \\ 1+1 \\ -1+0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$

iii) $\begin{bmatrix} 3 & 9 & 0 \\ 1 & 8 & -2 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 2 \\ 7 & 1 & 4 \end{bmatrix}$

$= \begin{bmatrix} 3+4 & 9+0 & 0+2 \\ 1+7 & 8+1 & -2+4 \end{bmatrix} = \begin{bmatrix} 7 & 9 & 2 \\ 8 & 9 & 2 \end{bmatrix}$

iv) $\begin{bmatrix} -1 & 2 \\ 1 & -2 \\ 3 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ -2 & 1 \end{bmatrix}$

$= \begin{bmatrix} -1+0 & 2+1 \\ 1-1 & -2+0 \\ 3-2 & -1+1 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 0 & -2 \\ 1 & 0 \end{bmatrix}$

**2. If $A = \begin{bmatrix} -1 & 3 \\ 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 3 & -5 \end{bmatrix}$ and $X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$ and A + B = X, then find the values of $x_1$, $x_2$, $x_3$ and $x_4$.**

**Sol.** A + B = X

$$\begin{bmatrix} -1 & 3 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 \\ 7 & -3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$$

$\therefore x_1 = 1,\ x_2 = 4,\ x_3 = 7,\ x_4 = -3$

**3. If** $A=\begin{bmatrix}-1 & -2 & 3\\ 1 & 2 & 4\\ 2 & -1 & 3\end{bmatrix}$, $B=\begin{bmatrix}1 & -2 & 5\\ 0 & -2 & 2\\ 1 & 2 & -3\end{bmatrix}$ **and** $C=\begin{bmatrix}-2 & 1 & 2\\ 1 & 1 & 2\\ 2 & 0 & 1\end{bmatrix}$ **then find A + B + C.**

**Sol.** A + B + C =

$$\begin{bmatrix}-1 & -2 & 3\\ 1 & 2 & 4\\ 2 & -1 & 3\end{bmatrix}+\begin{bmatrix}1 & -2 & 5\\ 0 & -2 & 2\\ 1 & 2 & -3\end{bmatrix}+\begin{bmatrix}-2 & 1 & 2\\ 1 & 1 & 2\\ 2 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}-1+1-2 & -2-2+1 & 3+5+2\\ 1+0+1 & 2-2+1 & 4+2+2\\ 2+1+2 & -1+2+0 & 3-3+1\end{bmatrix}$$

$$=\begin{bmatrix}-2 & -3 & 10\\ 2 & 1 & 8\\ 5 & 1 & 1\end{bmatrix}$$

**4. If** $A=\begin{bmatrix}3 & 2 & -1\\ 2 & -2 & 0\\ 1 & 3 & 1\end{bmatrix}$, $B=\begin{bmatrix}-3 & -1 & 0\\ 2 & 1 & 3\\ 4 & -1 & 2\end{bmatrix}$ **and X = A + B then find the matrix X.**

**Sol.** $X = A + B = \begin{bmatrix}3 & 2 & -1\\ 2 & -2 & 0\\ 1 & 3 & 1\end{bmatrix}+\begin{bmatrix}-3 & -1 & 0\\ 2 & 1 & 3\\ 4 & -1 & 2\end{bmatrix}$

$$\therefore X=\begin{bmatrix}0 & 1 & -1\\ 4 & -1 & 3\\ 5 & 2 & 3\end{bmatrix}$$

**5. If** $\begin{bmatrix}x-3 & 2y-8\\ z+2 & 6\end{bmatrix}=\begin{bmatrix}5 & 2\\ -2 & a-4\end{bmatrix}$**, find the values of x, y, z and a.**

**Sol.** Given $\begin{bmatrix}x-3 & 2y-8\\ z+2 & 6\end{bmatrix}=\begin{bmatrix}5 & 2\\ -2 & a-4\end{bmatrix}$

$\therefore x - 3 = 5 \Rightarrow x = 3 + 5 = 8$

$2y - 8 = 2 \Rightarrow 2y = 8 + 2 = 10 \Rightarrow y = 5$

$z + 2 = -2 \Rightarrow z = -2 - 2 = -4$

$a - 4 = 6 \Rightarrow a = 4 + 6 = 10$

**II.**

**1. If** $\begin{bmatrix} x-1 & 2 & 5-y \\ 0 & z-1 & 7 \\ 1 & 0 & a-5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 7 \\ 1 & 0 & 0 \end{bmatrix}$ **then find the values of x, y, z and a.**

**Sol.** Given $\begin{bmatrix} x-1 & 2 & 5-y \\ 0 & z-1 & 7 \\ 1 & 0 & a-5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 7 \\ 1 & 0 & 0 \end{bmatrix}$

$\therefore x - 1 = 1 \Rightarrow x = 1 + 1 = 2$

$5 - y = 3 \Rightarrow y = 5 - 3 = 2$

$z - 1 = 4 \Rightarrow z = 4 + 1 = 5$

$a - 5 = 0 \Rightarrow a = 5$

**2. Find trace of A if A =** $\begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & 5 \\ 1 & 0 & 1 \end{bmatrix}$**.**

**Sol.** Trace of A = Sum of the diagonal elements

$= 1 - 1 + 1 = 1.$

**3. If A =** $\begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 5 & -6 \end{bmatrix}$ **and B =** $\begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ **find B – A and 4A – 5B.**

**Sol.** Given $A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 5 & -6 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$$B - A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 5 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} -1-0 & 2-1 & 3-2 \\ 0-2 & 1-3 & 0-4 \\ 0-4 & 0-5 & -1+6 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ -2 & -2 & -4 \\ -4 & -5 & 5 \end{bmatrix}$$

$$4A - 5B = 4\begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 5 & -6 \end{bmatrix} - 5\begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 4 & 8 \\ 8 & 12 & 16 \\ 16 & 20 & -24 \end{bmatrix} - \begin{bmatrix} -5 & 10 & 15 \\ 0 & 5 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 0+5 & 4-10 & 8-15 \\ 8-0 & 12-5 & 16-0 \\ 16-0 & 20-0 & -24+5 \end{bmatrix} = \begin{bmatrix} 5 & -6 & 7 \\ 8 & 7 & 16 \\ 16 & 20 & -19 \end{bmatrix}$$

**4. If A =** $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$ **and B =** $\begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ **find 3B – 2A.**

**Sol.** $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

$$3B - 2A = 3\begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} - 2\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 6 & 3 \\ 3 & 6 & 9 \end{bmatrix} - \begin{bmatrix} 2 & 4 & 6 \\ 6 & 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9-2 & 6-4 & 3-6 \\ 3-6 & 6-4 & 9-2 \end{bmatrix} = \begin{bmatrix} 7 & 2 & -3 \\ -3 & 2 & 7 \end{bmatrix}$$

## EXERCISE-3(B)

**I.**

**1. Find the following products wherever possible.**
**Hint: (1 × 3) by (3 × 1) = 1 × 1.**

**Sol. i)** $\begin{bmatrix} -1 & 4 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1\cdot 5 + 4\cdot 1 + 2\cdot 3 \end{bmatrix}$

$$= \begin{bmatrix} -5+4+6 \end{bmatrix} = \begin{bmatrix} 5 \end{bmatrix}$$

**ii)** $\begin{bmatrix} 2 & 1 & 4 \\ 6 & -2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2\cdot 1 + 1\cdot 2 + 4\cdot 1 \\ 6\cdot 1 + (-2)\cdot 2 + 3\cdot 1 \end{bmatrix}$

$$= \begin{bmatrix} 2+2+4 \\ 6-4+3 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$$

**iii)** $\begin{bmatrix} 3 & -2 \\ 1 & 6 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 12-4 & -3-10 \\ 4+12 & -1+30 \end{bmatrix}$

$= \begin{bmatrix} 8 & -13 \\ 16 & 29 \end{bmatrix}$

**iv)** $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} -2 & -3 & 4 \\ 2 & 2 & -3 \\ 1 & 2 & -2 \end{bmatrix}$

$= \begin{bmatrix} -4+4+1 & -6+4+2 & 8-6-2 \\ -2+0+2 & -3+0+4 & 4+0-4 \\ -4+2+2 & -6+2+4 & 8-3-4 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

**v)** $\begin{bmatrix} 3 & 4 & 9 \\ 0 & -1 & 5 \\ 2 & 6 & 12 \end{bmatrix}\begin{bmatrix} 13 & -2 & 0 \\ 0 & 4 & 1 \end{bmatrix}$

First matrix is a $3 \times 3$ matrix and second matrix is $2 \times 3$ matrix.
Number of columns in first matrix $\neq$ Number of rows in second matrix.
$\therefore$ Matrix product is not possible.

**vi)** $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 4 \\ 6 & -2 & 3 \end{bmatrix}$

Number of columns in first matrix = 1
Number of rows in second matrix = 2
Number of columns in first matrix $\neq$ Number of rows in second matrix
Multiplication of matrices is not possible.

vii) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1-1 & 1-1 \\ -1+1 & -1+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

viii) $\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}\begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$

$= \begin{bmatrix} 0+abc-abc & b^2c-b^2c & bc^2-bc^2 \\ -a^2c+a^2c & -abc+abc & -ac^2+ac^2 \\ a^2b-a^2b & ab^2-ab^2 & abc-abc \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

**2. If A = $\begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ and B = $\begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$, do AB and BA exist? If they exist, find them. Do A and B commutative with respect to multiplication of matrices.**

**Sol**. Given A = $\begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ and B = $\begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$

AB = $\begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 2-8+6 & 3-10+3 \\ -8+8+10 & -12+10+5 \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 10 & 3 \end{bmatrix}$

BA = $\begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$

$= \begin{bmatrix} 2-12 & -4+6 & 6+15 \\ 4-20 & -8+10 & 12+25 \\ 2-4 & -4+2 & 6+5 \end{bmatrix} = \begin{bmatrix} -10 & 2 & 21 \\ -16 & 2 & 37 \\ -2 & -2 & 11 \end{bmatrix}$ $AB \neq BA$

$\therefore$ A and B are not commutative with respect to multiplication of matrices.

**3. Find $A^2$ where A = $\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$**

**Sol.** $A^2 = A.A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 16-2 & 8+2 \\ -4-1 & -2+1 \end{bmatrix} = \begin{bmatrix} 14 & 10 \\ -5 & -1 \end{bmatrix}$

**4. If $A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$, find $A^2$.**

**Sol.** $A^2 = A, A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} i^2 & 0 \\ 0 & i^2 \end{bmatrix}$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

**5. If $A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$ then show that**

**(i) $A^2 = B^2 = C^2 = -I$,**

**(ii) $AB = -BA = -C$ ($i^2 = -1$ and I is the unit matrix of order 2)**

**Sol**. i) $A^2 = A.A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

$$= \begin{bmatrix} i^2 & 0 \\ 0 & i^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -1$$

$B^2 = B.B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -I$$

$C^2 = C.C = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$

$$= \begin{bmatrix} i^2 & 0 \\ 0 & i^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -I$$

$\therefore A^2 = B^2 = C^2 = -1$

ii) $AB = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} = -\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = -C$$

$BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = C$

$\therefore AB = -BA = -C.$

**6. If $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 0 & 4 \end{bmatrix}$, find AB. Find BA if exists.**

**Sol**. Given $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 0 & 4 \end{bmatrix}$

$$AB = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 & 0 \\ 1 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 6+1 & 4+0 & 0+4 \\ 3+3 & 2+0 & 0+12 \end{bmatrix} = \begin{bmatrix} 7 & 4 & 4 \\ 6 & 2 & 12 \end{bmatrix}$$

Order of AB is $2 \times 3$

BA does not exist since number of columns in B $\neq$ No.of rows in A.

**7. If $A = \begin{bmatrix} 2 & 4 \\ -1 & k \end{bmatrix}$ and $A^2 = 0$, then find the value of k.**

**Sol**. $A^2 = 0 \Rightarrow \begin{bmatrix} 2 & 4 \\ -1 & k \end{bmatrix}\begin{bmatrix} 2 & 4 \\ -1 & k \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} 4-4 & 8+4k \\ -2-k & -4+k^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow 8 + 4k = 0 \Rightarrow 4k = -8 \Rightarrow k = -2$

**II.**

**1. If $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ then find $A^4$.**

**Note** : A is diagonal matrix.

**Sol.** If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, then

$$A^n = \begin{bmatrix} a^n & 0 & 0 \\ 0 & b^n & 0 \\ 0 & 0 & c^n \end{bmatrix}, n \in N$$

$$A^4 = \begin{bmatrix} 3^4 & 0 & 0 \\ 0 & 3^4 & 0 \\ 0 & 0 & 3^4 \end{bmatrix} = \begin{bmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{bmatrix}$$

**2.** **If** $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ **then find** $A^3$.

**Sol.** $A^2 = A.A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$

$$= \begin{bmatrix} 1+5-6 & 1+2-3 & 3+6-9 \\ 5+10-12 & 5+4-6 & 15+12-18 \\ -2-5+6 & -2-2+3 & -6-6+9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+0 & 0+0+0 & 0+0+0 \\ 3+15-18 & 3+6-9 & 9+18-27 \\ -1-5+6 & -1-2+3 & -3-6+9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

**3.** **If** $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$**, then find** $A^3 - 3A^2 - A - 3I$.

**Sol** Given $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$

$$A^2 = A.A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+3 & -2-2-1 & 1+2+1 \\ 0+0-3 & 0+1+1 & 0-1-1 \\ 3-0+3 & -6-1-1 & 3+1+1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -5 & 4 \\ -3 & 2 & -2 \\ 6 & -8 & 5 \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} 4 & -5 & 4 \\ -3 & 2 & -2 \\ 6 & -8 & 5 \end{bmatrix}\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4+0+12 & -8-5-4 & 4+5+4 \\ -3+0-6 & 6+2+2 & -3-3-2 \\ 6+0+15 & -12-8-5 & 6+8+5 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & -17 & 13 \\ -9 & 10 & -7 \\ 21 & -25 & 19 \end{bmatrix}$$

Now $A^3 - 3A^2 - A - 3I$

$$= \begin{bmatrix} 16 & -17 & 13 \\ -9 & 10 & -7 \\ 21 & -25 & 19 \end{bmatrix} - 3\begin{bmatrix} 4 & -5 & 4 \\ -3 & 2 & -2 \\ 6 & -8 & 5 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 16-12-1-3 & -17+15+2+0 & 13-12-1-0 \\ -9+9+0-0 & 10-6-1-3 & -7+6+1+0 \\ 21-18-3+0 & -25+24+1+0 & 19-15-1-3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O_{3\times 3}$$

$\therefore A^3 - 3A^2 - A - 3I = 0$

**4. If $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, show that $(aI + bE)^3 = a^3I + 3a^2bE$.**

**Sol.** $aI + bE = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$

$$(aI + bE)^2 = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^2 & 2ab \\ 0 & a^2 \end{bmatrix}$$

$$(aI + bE)^3 = \begin{bmatrix} a^2 & 2ab \\ 0 & a^2 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^3 & 3a^2b \\ 0 & a^3 \end{bmatrix}$$

$$= \begin{bmatrix} a^3 & 0 \\ 0 & a^3 \end{bmatrix} + \begin{bmatrix} 0 & 3a^2b \\ 0 & 0 \end{bmatrix}$$

$$= a^3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 3a^2b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= a^3 I + 3a^2 bE$$

**III.**

**1. If** $A = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$**, then for any integer n≥1 show that** $A^n = \begin{bmatrix} a_1^n & 0 & 0 \\ 0 & a_2^n & 0 \\ 0 & 0 & a_3^n \end{bmatrix}$**.**

**Sol**. Given $A = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$

We shall prove the result by Mathematical induction.

$$A^n = \begin{bmatrix} a_1^n & 0 & 0 \\ 0 & a_2^n & 0 \\ 0 & 0 & a_3^n \end{bmatrix}$$

When n = 1

$$A^1 = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$$

The result is true for n = 1.

Suppose the result is true for n = k

i.e. $A^k = \begin{bmatrix} a_1^k & 0 & 0 \\ 0 & a_2^k & 0 \\ 0 & 0 & a_3^k \end{bmatrix}$

Now $A^{k+1} = A^k \cdot A$

$$= \begin{bmatrix} a_1^k & 0 & 0 \\ 0 & a_2^k & 0 \\ 0 & 0 & a_3^k \end{bmatrix} \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1^k \cdot a_1 + 0 + 0 & 0 + 0 + 0 & 0 + 0 + 0 \\ 0 + 0 + 0 & 0 + a_2^k \cdot a_2 + 0 & 0 + 0 + 0 \\ 0 + 0 + 0 & 0 + 0 + 0 & 0 + 0 + a_3^k \cdot a_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1^{k+1} & 0 & 0 \\ 0 & a_2^{k+1} & 0 \\ 0 & 0 & a_3^{k+1} \end{bmatrix}$$

$\therefore$ The given result is true for n = k + 1

By Mathematical induction, the given result is true for all positive integral values of n.

i.e. $A^n = \begin{bmatrix} a_1^n & 0 & 0 \\ 0 & a_2^n & 0 \\ 0 & 0 & a_3^n \end{bmatrix}$, for any integer $n \geq 1$.

**2. If $\theta - \phi = \frac{\pi}{2}$, show that**

$$\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix} = 0$$

**Sol.** Given $\theta - \phi = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{2} + \phi$

$$\cos\theta = \cos\left(\frac{\pi}{2} + \phi\right) = -\sin\phi$$

$$\sin\theta = \sin\left(\frac{\pi}{2} + \phi\right) = \cos\phi$$

$$\therefore \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \sin^2\phi & -\sin\phi\cos\phi \\ -\sin\phi\cos\phi & \cos^2\phi \end{bmatrix}$$

$$\therefore \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

$$\begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$$

$$= \begin{bmatrix} \sin^2\phi & -\sin\phi\cos\phi \\ -\sin\phi\cos\phi & \cos^2\phi \end{bmatrix} \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$$

$$= \begin{bmatrix} \sin^2\phi\cos^2\phi - \sin^2\phi\cos^2\phi & \sin^3\phi\cos\phi - \sin^3\phi\cos\phi \\ -\sin\phi\cos^3\phi + \sin\phi\cos^3\phi & -\sin^2\phi\cos^2\phi + \sin^2\phi\cos^2\phi \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

**3. If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ then show that $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$, n is a positive integer.**

**Sol.** We shall prove the result by Mathematical Induction.

$$A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$$

$$n = 1 \Rightarrow A' = \begin{bmatrix} 1+2 & -4 \\ 1 & 1-2 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

The result is true for n = 1

Suppose the result is true for n = k

$$A^k = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix}$$

$$A^{k+1} = A^k \cdot A = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3+6k-4k & -4-8k+4k \\ 3k+1-2k & -4k-1+2k \end{bmatrix}$$

$$= \begin{bmatrix} 2k+3 & -4k-4 \\ k+1 & -2k-1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2(k+1) & -4(k+1) \\ k+1 & 1-2(k+1) \end{bmatrix}$$

$\therefore$ The given result is true for n = k + 1

By Mathematical Induction, given result is true for all positive integral values of n.

**4. Give examples of two square matrices A and B of the same order for which AB = 0 and BA ≠ 0.**

**Sol.** $A = \begin{bmatrix} a & 0 \\ a & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ a & a \end{bmatrix}$

Then $AB = \begin{bmatrix} a & 0 \\ a & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ a & a \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} = 0$

$$BA = \begin{bmatrix} 0 & 0 \\ a & a \end{bmatrix} \begin{bmatrix} a & 0 \\ a & 0 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ a^2+a^2 & 0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 2a^2 & 0 \end{bmatrix} \neq 0$$

$\therefore$ AB = 0 and BA $\neq$ 0

## EXERCISE – 3(C)

**1. If A = $\begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & 5 \end{bmatrix}$ and B = $\begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & -2 \end{bmatrix}$ then find $(AB^T)^T$.**

**Sol.** $B^T = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & -2 \end{bmatrix}^T = \begin{bmatrix} -1 & 0 \\ 1 & 1 \\ 0 & -2 \end{bmatrix}$

$$AB^T = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & 5 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 1 \\ 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -2+0+0 & 0+0-2 \\ 1+1+0 & 0+1-10 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 2 & -9 \end{bmatrix}$$

$$(AB^T)^T = \begin{bmatrix} -2 & -2 \\ 2 & -9 \end{bmatrix}^T = \begin{bmatrix} -2 & 2 \\ -2 & -9 \end{bmatrix}$$

**2. If A = $\begin{bmatrix} -2 & 1 \\ 5 & 0 \\ -1 & 4 \end{bmatrix}$ and B = $\begin{bmatrix} -2 & 3 & 1 \\ 4 & 0 & 2 \end{bmatrix}$ find $2A + B^T$ and $3B^T - A$.**

**Sol.** $A = \begin{bmatrix} -2 & 1 \\ 5 & 0 \\ -1 & 4 \end{bmatrix} \Rightarrow 2A = 2\begin{bmatrix} -2 & 1 \\ 5 & 0 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 10 & 0 \\ -2 & 8 \end{bmatrix}$

$$B = \begin{bmatrix} -2 & 3 & 1 \\ 4 & 0 & 2 \end{bmatrix}$$

$$\Rightarrow B^T = \begin{bmatrix} -2 & 3 & 1 \\ 4 & 0 & 2 \end{bmatrix}^T = \begin{bmatrix} -2 & 4 \\ 3 & 0 \\ 1 & 2 \end{bmatrix}$$

$$2A + B^T = \begin{bmatrix} -4 & 2 \\ 10 & 0 \\ -2 & 8 \end{bmatrix} + \begin{bmatrix} -2 & 4 \\ 3 & 0 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -4-2 & 2+4 \\ 10+3 & 0+0 \\ -2+1 & 8+2 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 13 & 0 \\ -1 & 10 \end{bmatrix}$$

$$B^T = \begin{bmatrix} -2 & 3 & 1 \\ 4 & 0 & 2 \end{bmatrix}^T = \begin{bmatrix} -2 & 4 \\ 3 & 0 \\ 1 & 2 \end{bmatrix}$$

$$3B^T - A = 3\begin{bmatrix} -2 & 4 \\ 3 & 0 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} -2 & 1 \\ 5 & 0 \\ -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 12 \\ 9 & 0 \\ 3 & 6 \end{bmatrix} - \begin{bmatrix} -2 & 1 \\ 5 & 0 \\ -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -6+2 & 12-1 \\ 9-5 & 0-0 \\ 3+1 & 6-4 \end{bmatrix} = \begin{bmatrix} -4 & 11 \\ 4 & 0 \\ 4 & 2 \end{bmatrix}$$

**3. If $A = \begin{bmatrix} 2 & -4 \\ -5 & 3 \end{bmatrix}$ then find $A + A^T$ and $A.A^T$.**

**Sol.** $A = \begin{bmatrix} 2 & -4 \\ -5 & 3 \end{bmatrix}$

$$\Rightarrow A^T = \begin{bmatrix} 2 & -4 \\ -5 & 3 \end{bmatrix}^T = \begin{bmatrix} 2 & -5 \\ -4 & 3 \end{bmatrix}$$

$$A + A^T = \begin{bmatrix} 2 & -4 \\ -5 & 3 \end{bmatrix} + \begin{bmatrix} 2 & -5 \\ -4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2+2 & -4-5 \\ -5-4 & 3+3 \end{bmatrix} = \begin{bmatrix} 20 & -22 \\ -22 & 34 \end{bmatrix}$$

**4.** **If A =** $\begin{bmatrix} -1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & x & 7 \end{bmatrix}$ **is a symmetric matrix, then find x.**

**Sol.** A is a symmetric matrix $\Rightarrow A^T = A$

$$\begin{bmatrix} -1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & x & 7 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & x & 7 \end{bmatrix}$$

Equating $2^{nd}$ row, $3^{rd}$ column elements we get x = 6.

**5.** **If A =** $\begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & -2 \\ -1 & x & 0 \end{bmatrix}$ **is a skew symmetric matrix, find x.**

**Hint** : A is a skew symmetric matrix $\Rightarrow A^T = A$

**Sol**. A is a skew symmetric matrix

$\Rightarrow A^T = A$

$$\begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & x \\ 1 & -2 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & -2 \\ -1 & x & 0 \end{bmatrix} \qquad = \begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & 2 \\ 1 & -x & 0 \end{bmatrix}$$

Equating second row third column elements we get x = 2.

**6.** **Is** $\begin{bmatrix} 0 & 1 & 4 \\ -1 & 0 & 7 \\ -4 & -7 & 0 \end{bmatrix}$ **symmetric or skew symmetric 7.**

**Sol.** Let $A = \begin{bmatrix} 0 & 1 & 4 \\ -1 & 0 & 7 \\ -4 & -7 & 0 \end{bmatrix}$

$$A^T = \begin{bmatrix} 0 & 1 & 4 \\ -1 & 0 & 7 \\ -4 & -7 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & -1 & -4 \\ 1 & 0 & -7 \\ 4 & 7 & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & 1 & 4 \\ -1 & 0 & 7 \\ -4 & -7 & 0 \end{bmatrix} = -A$$

$\therefore$ A is a skew symmetric matrix.

**II.**

**1. If** $A=\begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}$**, show that** $A\cdot A^T = A^TA = I_2$**.**

**Sol.** $A\cdot A^T=\begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}$

$=\begin{bmatrix}\cos^2\alpha+\sin^2\alpha & -\sin\alpha\cos\alpha+\sin\alpha\cos\alpha\\ -\sin\alpha\cos\alpha+\cos\alpha\sin\alpha & \sin^2\alpha+\cos^2\alpha\end{bmatrix}$

$=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I_2$ ...(1)

$A^T\cdot A=\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}$

$=\begin{bmatrix}\cos^2\alpha+\sin^2\alpha & \cos\alpha\sin\alpha-\sin\alpha\cos\alpha\\ \sin\alpha\cos\alpha-\cos\alpha\sin\alpha & \sin^2\alpha+\cos^2\alpha\end{bmatrix}$

$=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I_2$ ...(2)

From (1), (2) we get $A\cdot A^T = A^T\cdot A = I_2$.

**2. If** $A=\begin{bmatrix}1 & 5 & 3\\ 2 & 4 & 0\\ 3 & -1 & -5\end{bmatrix}$ **and** $B=\begin{bmatrix}2 & -1 & 0\\ 0 & -2 & 5\\ 1 & 2 & 0\end{bmatrix}$ **then find** $\mathbf{3A-4B^T}$**.**

**Sol.** $B=\begin{bmatrix}2 & -1 & 0\\ 0 & -2 & 5\\ 1 & 2 & 0\end{bmatrix}$

$\Rightarrow B^T=\begin{bmatrix}2 & -1 & 0\\ 0 & -2 & 5\\ 1 & 2 & 0\end{bmatrix}^T=\begin{bmatrix}2 & 0 & 1\\ -1 & -2 & 2\\ 0 & 5 & 0\end{bmatrix}$

$3A-4B^T=3\begin{bmatrix}1 & 5 & 3\\ 2 & 4 & 0\\ 3 & -1 & -5\end{bmatrix}-4\begin{bmatrix}2 & 0 & 1\\ -1 & -2 & 2\\ 0 & 5 & 0\end{bmatrix}$

$$= \begin{bmatrix} 3 & 15 & 9 \\ 6 & 12 & 0 \\ 9 & -3 & -15 \end{bmatrix} - \begin{bmatrix} 8 & 0 & 4 \\ -4 & -8 & 8 \\ 0 & 20 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3-8 & 15-0 & 9-4 \\ 6+4 & 12+8 & 0-8 \\ 9-0 & -3-20 & -15-0 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 15 & 5 \\ 10 & 20 & -8 \\ 9 & -23 & -15 \end{bmatrix}$$

**3.** $A = \begin{bmatrix} 7 & -2 \\ -1 & 2 \\ 5 & 3 \end{bmatrix}$ **and** $B = \begin{bmatrix} -2 & -1 \\ 4 & 2 \\ -1 & 0 \end{bmatrix}$ **then find** $\mathbf{AB^T}$ **and** $\mathbf{BA^T}$**.**

**Sol**. $B = \begin{bmatrix} -2 & -1 \\ 4 & 2 \\ -1 & 0 \end{bmatrix}$

$$\Rightarrow B^T = \begin{bmatrix} -2 & -1 \\ 4 & 2 \\ -1 & 0 \end{bmatrix}^T = \begin{bmatrix} -2 & 4 & -1 \\ -1 & 2 & 0 \end{bmatrix}$$

$$AB^T = \begin{bmatrix} 7 & -2 \\ -1 & 2 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} -2 & 4 & -1 \\ -1 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -14+2 & 28-4 & -7+0 \\ 2-2 & -4+4 & 1+0 \\ -10-3 & 20+6 & -5+0 \end{bmatrix} = \begin{bmatrix} -12 & 24 & -7 \\ 0 & 0 & 1 \\ -13 & 26 & -5 \end{bmatrix}$$

$$A = \begin{bmatrix} 7 & -2 \\ -1 & 2 \\ 5 & 3 \end{bmatrix}$$

$$\Rightarrow A^T = \begin{bmatrix} 7 & -2 \\ -1 & 2 \\ 5 & 3 \end{bmatrix}^T = \begin{bmatrix} 7 & -1 & 5 \\ -2 & 2 & 3 \end{bmatrix}$$

$$BA^T = \begin{bmatrix} -2 & -1 \\ 4 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 7 & -1 & 5 \\ -2 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -14+2 & 2-2 & -10-3 \\ 28-4 & -4+4 & 20+6 \\ -7+0 & 1+0 & -5+0 \end{bmatrix} = \begin{bmatrix} -12 & 0 & -13 \\ 24 & 0 & 26 \\ -7 & 1 & -5 \end{bmatrix}$$

**4. For any square matrix A, show that $AA'$ is symmetric.**

**Sol.** A is a square matrix

$(AA')' = (A')'A' = A \cdot A'$

$\because (AA')' = AA'$

$\Rightarrow AA'$ is a symmetric matrix.