THEORY OF EQUATIONS -- 2

EXERCISE – 2(d)

I

1. Find the algebraic equation whose roots are 3 times the roots of $x^3 + 2x^2 - 4x + 1 = 0$

Sol: Given equation is $f(x) = x^3 + 2x^2 - 4x + 1 = 0$

We require an equation whose roots are 3 times the roots of $f(x) = 0$

i.e., required equation is $f\left(\frac{x}{3}\right) = 0$

$$\left(\frac{x}{3}\right)^3 + 2\left(\frac{x}{3}\right)^2 - \frac{4x}{3} + 1 = 0$$

$$\frac{x^3}{27} + \frac{2}{9}x^2 - \frac{4}{3}x + 1 = 0$$

Multiplying with 27, required equation is $x^3 + 6x^2 - 36x + 27 = 0$

2. Find the algebraic equation whose roots are 2 times the roots of $x^5 - 2x^4 + 3x^3 - 2x^2 + 4x + 3 = 0$

Sol: Given equation is

$f(x) = x^5 - 2x^4 + 3x^3 - 2x^2 + 4x + 3 = 0$

We require an equation whose roots are 2 times the roots of $f(x) = 0$

Required equation is $f\left(\frac{x}{2}\right) = 0$

$$\Rightarrow \left(\frac{x}{2}\right)^5 - 2\left(\frac{x}{2}\right)^4 + 3\left(\frac{x}{2}\right) - 2\left(\frac{x}{2}\right)^2 + 4\left(\frac{x}{2}\right) + 3 = 0$$

$$\Rightarrow \frac{x^5}{32} - 2\frac{x^4}{16} + 3.\frac{x^3}{8} - 2.\frac{x^2}{4} + 4.\frac{x}{2} + 3 = 0$$

Multiplying with 32, required equation is

$x^5 - 4x^4 + 12x^3 - 16x^2 + 64x + 96 = 0$

3. Find the transformed equation whose roots are the negative of the roots of $x^4 + 5x^3 + 11x + 3 = 0$

Sol: Given $f(x) = x^4 + 5x^3 + 11x + 3 = 0$

We want an equation whose roots are

$-\alpha_1, -\alpha_2, -\alpha_3, -\alpha_4$

Required equation $f(-x) = 0$

$$\Rightarrow (-x)^4 + 5(-x)^3 + 11(-x) + 3 = 0$$

$$\Rightarrow x^4 - 5x^3 - 11x + 3 = 0$$

4. Find the transformed equation whose roots the negative of the roots are of
$x^7+3x^5+x^3-x^2+7x+2=0$

Sol: Given $f(x)=x^7+3x^5+x^3-x^2+7x+2=0$

We want an equation whose roots are
$-\alpha_1,-\alpha_2,.....-\alpha_n$

Required equation is $f(-x)=0$

$(-x)^7+3(-x)^5+(-x)^3-(-x)^2+7(-x)+2=0$

$\Rightarrow x^7+3x^5+x^3+x^2+7x-2=0$

6. Find the polynomial equation whose roots the reciprocals of the roots are of
$x^4-3x^3+7x^2+5x-2=0$

Sol: Given equation is $f(x)$

$x^4-3x^3+7x^2+5x-2=0$

Required equation is $f\left(\frac{1}{x}\right)=0$

$\frac{1}{x^4}-\frac{3}{x^3}+\frac{7}{x^2}+\frac{5}{x}-2=0$

Multiplying by x^4

$1-3x+7x^2+5x^3-2x^4=0$

i.e., $2x^4+5x^3-7x^2+3x-1=0$

II

1. Find the polynomial equation whose roots are the squares of the roots of
$x^4+x^3+2x^2+x+1=0$

Sol: Given equation is

$f(x)=x^4+x^3+2x^2+x+1=0$

Required equation $f\left(\sqrt{x}\right)=0$

$\Rightarrow x^2+x\sqrt{x}+2x+\sqrt{x}+1=0$

$=\sqrt{x}(x+1)=-\left(x^2+2x+1\right)$

Squaring both sides,

$\Rightarrow x(x+1)^2=\left(x^2+2x+1\right)^2$

$\Rightarrow x\left(x^2+2x+1\right)$

$=x^4+4x^2+1+4x^3+4x+2x^2$

$\Rightarrow x^3+2x^2+x=x^4+4x^3+6x^2+4x+1$

i.e., $x^4+3x^3+4x^2+3x+1=0$

2. **Form the polynomial equation whose roots are the squares of the roots of** $x^3+3x^2-7x+6=0$

Sol: Given equation is

$f(x)=x^3+3x^2-7x+6=0$

Required equations is $f(\sqrt{x})=0$

$\Rightarrow x\sqrt{x}+3x-7\sqrt{x}+6=0$

$\Rightarrow \sqrt{x}(x-7)=-(3x+6)$

Squaring on both sides

$\Rightarrow x(x-7)^2=(3x+6)^2$

$\Rightarrow x(x^2-14x+49)=9x^2+36+36x$

$\Rightarrow x^3-14x^2+49x-9x^2-36x-36=0$

i.e., $x^3-23x^2+13x-36=0$

3. **Form the polynomial equation whose roots are cubes of the roots of** $x^3+3x^2+2=0$

Sol: Given equation is $x^3+3x^2+2=0$

Put $y=x^2$ so that $x=y^{1/3}$

$\therefore y+3y^{2/3}+2=0$

$3y^{2/3}=-(y+2)$

Cubing on both sides, $27y^2=-(y+2)^3$

$=-(y^3+6y^2+12y+8)$

$\therefore y^3+6y^2+12y+8=0$

$\Rightarrow y^3+33y^2+12y+8=0$

Required equation is $x^3+33x^2+12x+8=0$

III.

1. **Find the polynomial equation whose roots are translates of those of the equation** $x^4-5x^3+7x^2-17x+11=0$ *by* -2

Sol: Given equation is

$f(x)=x^4-5x^3+7x^2-17x+11=0$

The required equation is $f(x+2)=0$

$(x+2)^4-5(x+2)^3+7(x+2)^2-17(x+2)+11=0$

2	1	−5	7	−17	11
	0	2	−6	2	−30
	1	−3	1	−15	−19
	0	2	−2	−2	A_4
	1	−1	−1	−17	
	0	2	2	A_3	
	1	1	1		
	0	2	A_2		
	1	3			
	A_0	A_1			

Required equation is $x^4 + 3x^2 + x^2 - 17x - 19 = 0$

2. Find the polynomial equation whose roots are the translates of those of $x^5 - 4x^4 + 3x^2 - 4x + 6 = 0$ by - 3

Sol: Given equation is

$f(x) = x^5 - 4x^4 + 3x^2 - 4x + 6 = 0$

Required equation is $f(x+3) = 0$

$(x+3)^5 - 4(x+3)^4 + 3(x+3)^2 - 4(x+3) + 6 = 0$

3	1	−4	0	3	−4	6
	0	3	−3	−9	−18	−66
	1	−1	−3	−6	−22	−60
	0	3	6	9	9	A_5
	1	2	3	3	−13	
	0	3	15	54	A_4	
	1	5	18	57		
	0	3	24	A_3		
	1	8	42			
	0	3	A_3			
	1	11				
	A_0	A_1				

Required equation is

$x^5 + 11x^4 + 42x^3 + 57x^2 - 13x - 60 = 0$

3. Find the polynomial equation whose roots are the translates of the roots of the equation $x^4 - x^3 - 10x^2 + 4x + 24 = 0$ by 2

Sol: Given $f(x) = x^4 - x^3 - 10x^2 + 4x + 24 = 0$

Required equation is $f(x-2) = 0$

$(x-2)^4 + (x-2)^3 - 10(x-2)^2$

−2	1	−1	−10	4	24	
	0	−2	6	8	−24	
	1	−3	−4	12	0	A_4
	0	−2	10	−12		
	1	−5	6	0	A_3	
	0	−2	14			
	1	−7	20			
	0	−2	A_2			
	1	−9				
	A_0	A_1				

Required equation is

$x^4 - 9x^3 + 20x^2 = 0$

4. Find the polynomial equation whose roots are the translates of the equation $3x^5 + 5x^3 + 7 = 0$ by 4

Sol: Given $f(x) = 3x^5 - 5x^3 + 7 = 0$

Required equation is $f(x-4) = 0$

$3(x-4)^5 - 5(x-4)^3 + 7 = 0$

−4	3	0	−5	0	0	7
	0	−12	48	−172	688	−2752
	3	−12	43	−172	688	−2745
	0	−12	96	−556	2912	A_5
	3	−24	139	−728	3600	
	0	−12	144	−1132	A_4	
	3	−36	283	−1860		
	0	−12	192	A_3		
	3	−48	475			
	0	−12	A_2			
	3	60				
	A_0	A_1				

Required equation is $x^5 - 60x^4 + 475x^3$

$-1860x^2 + 3600x - 2745 = 0$

5. Transform each of the following equations into ones in which of the coefficients of the second highest power of x is zero and also find their transformed equations.

(i) $x^3 - 6x^2 + 10x - 3 = 0$

Sol: Given equation is $x^3 - 6x^2 + 10x - 3 = 0$

To remove the second term diminish the roots

By $-\frac{a_1}{na_0} = \frac{6}{3} = 2$

2	1	−6	10	−3
	0	2	−8	+4
	1	−4	2	+1
	0	2	−4	A_3
	1	−2	−2	
	0	+2	A_2	
	1	0		
	A_0	A_1		

Required equation is $x^3 - 2x + 1 = 0$

(ii) $x^4 + 4x^3 + 2x^2 - 4x - 2 = 0$

Sol: Given equation is $x^4 + 4x^3 + 2x^2 - 4x - 2 = 0$

Diminishing the roots by $-\frac{a_1}{na_0} = \frac{-4}{4} = -1$

−1	1	4	2	−4	−2
	0	−1	−3	1	3
	1	3	−1	−3	1
	0	−1	−2	3	A_4
	1	2	−3	0	
	0	−1	−1	A_3	
	1	1	−4		
	0	−1	A_2		
	1	0			
	A_0	A_1			

Required equation is $x^4 - 4x^2 + 1 = 0$

(iii) $x^3 - 6x^2 + 4x - 7 = 0$

Sol: Given equation is $x^3 - 6x^2 + 4x - 7 = 0$

Diminishing the roots by $-\frac{a_1}{na_0} = \frac{6}{3} = 2$

2	1	−6	4	−7
	0	2	−8	−8
	1	−4	−4	−15
	0	2	−4	A_3
	1	−2	−8	
	0	2	A_2	
	1	0		
	A_0	A_1		

Required equation is $x^3 - 8x - 15 = 0$

(iv) $x^3 + 6x^2 + 4x + 4 = 0$

Sol: Given equation is $x^3 + 6x^2 + 4x + 4 = 0$

To remove the second term diminish the roots by

−2	1	6	4	4
	0	−2	−8	8
	1	4	−4	12
	0	−2	−4	A_3
	1	2	−8	
	0	−2	A_2	
	1	0		
	A_0	A_1		

Required equation is $x^3 - 8x + 12 = 0$

6. Transform each of the following equations into ones in which the coefficients of the third highest power of x is zero

(i) $x^4 + 2x^3 - 12x^2 + 2x - 1 = 0$

Hint: To remove the rth term in an equation $f(x) = 0$ of degree n diminish the roots by 'h' such that $f^{(n-r+1)}(h) = 0$

Sol: Let $f(x) = x^4 + 2x^3 - 12x^2 + 2x - 1$

To remove the 3rd term, diminish the roots by h such that $f'(h) = 0$

$f(x) = 4x^3 + 6x^2 - 24x + 2$

$f'(x) = 12x^2 + 12x - 24$

$f''(h) = 0 \Rightarrow 12h^2 + 12h - 24 = 0$

$\Rightarrow h^2 + h - 2 = 0 \Rightarrow (h+2)(h-1) = 0$

$\Rightarrow h = -2 \text{ or } 1$

Case (i):

h = –2	1	2	– 12	2	– 1
	–	– 2	0	24	– 52
	1	0	– 12	26	– 53
	–	– 2	4	16	
	1	– 2	– 8	42	
	–	– 2	8		
	1	– 4	0		
	–	– 2			
	1	– 6			

Transformed equation is

$x^4 - 6x^3 + 42x - 53 = 0$

Case (ii):

h = 1	1	2	– 12	2	– 1
	–	1	3	– 9	– 7
	1	3	9	– 7	– 8
	–	1	4	– 5	
	1	4	– 5	–12	
	–	1	5		
	1	5	0		
	–	1			
	1	6			

Transformed equation is

$x^4 - 6x^3 - 12x - 8 = 0$

$\therefore$ The required equation is

$x^4 - 6x^3 + 42x - 53 = 0$

Or $x^4 + 6x^3 - 12x - 8 = 0$

(ii) $x^3 + 2x^2 + x + 1 = 0$

Sol: Let $f(x) = x^3 + 2x^2 + x + 1$

To remove the 3rd terms, diminish the roots by h such that

$f'(h) = 0,\ f'(x) = 3x^2 + 4x + 1\ \ f(h) = 0 \Rightarrow 3h^2 + 4h + 1 = 0$

$\Rightarrow (3h+1)(h+1) \Rightarrow h = 1, -\frac{1}{3}$

Case (i):

h = –1	1	2	–1	1
	–	–1	–1	0
	1	1	0	1
	–	–1	0	
	1	0	0	
	–	–1		
	1	–1		

Transformed equation is $x^3 - x^2 + 1 = 0$

Case (ii)

$h = -\frac{1}{3}$	1	2	1	1
	–	$-\frac{1}{3}$	$-\frac{5}{9}$	$-\frac{4}{27}$
	1	$\frac{5}{3}$	$\frac{4}{9}$	$\frac{23}{27}$
	–	$-\frac{1}{3}$	$-\frac{4}{9}$	
	1	$\frac{4}{3}$	0	
	–	$-\frac{1}{3}$		
	1	1		

Transformed equation is $x^3 + x^2 + \frac{23}{27} = 0$

$\Rightarrow 27x^3 + 27x^2 + 23 = 0$

$\therefore$ The required equation is $x^3 - x^2 + 1 = 0$ or $27x^3 + 27x^2 + 23 = 0$

7. Solve the following equations

(i) $x^4 - 10x^3 + 26x^2 - 10x + 1 = 0$

Sol: This is standard reciprocal equation

Dividing with x^2

$$x^2 - 10x + 26 - \frac{10}{x} + \frac{1}{x^2} = 0$$

$$\left(x^2 + \frac{1}{x^2}\right) - 10\left(x + \frac{1}{x}\right) + 26 = 0 \ldots\ldots(1)$$

Put $a = x + \frac{1}{x}$

$$x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2 = a^2 - 2$$

Substituting in (1)

$a^2 - 2 - 10a + 26 = 0$

$\Rightarrow a^2 = 10a + 24 = 0$

$\Rightarrow (a-4)(a-6) = 0$

$a = 4 \text{ or } 6$

Case (i) a = 4

$$x + \frac{1}{x} = 4$$

$\Rightarrow x^2 + 1 = 4x$

$\Rightarrow x^2 - 4x + 1 = 0$

$$\Rightarrow x = \frac{4 \pm \sqrt{16-4}}{2} = \frac{4 \pm 2\sqrt{3}}{2}$$

$= 2 \pm \sqrt{3}$

Case (ii) a = 6

$$x + \frac{1}{x} = 6$$

$x^2 + 1 = 6x$

$x^2 - 6x + 1 = 0$

$$x = \frac{6 \pm \sqrt{36-4}}{2} = \frac{6 \pm \sqrt{2}}{2}$$

$$x = \frac{2\left(3 \pm 2\sqrt{2}\right)}{2} = 3 \pm 2\sqrt{2}$$

$\therefore$ The roots are $3 \pm 2\sqrt{2}$, $2 \pm \sqrt{3}$

(ii) $2x^5 + x^4 - 12x^3 - 12x^2 + x + 2 = 0$

Sol: Given $f(x)$

$= 2x^5 + x^4 - 12x^3 - 12x^2 + x + 2 = 0$

This is an odd degree reciprocal equation of first type.

$\therefore -1$ is a root

Dividing $f(x)$ with $x+1$

x = –1	2	1	–12	–12	1	2
	0	–2	1	11	1	–2
	2	–1	–11	–1	2	0

Dividing $f(x)$ by $(x+1)$ we get

$24x^4 - x^3 - 11x^2 - x + 2 = 0$

Dividing by x^2

$$2x^2 - x - 11 - \frac{1}{x} + \frac{2}{x^2} = 0$$

$$2\left(x^2 + \frac{1}{x^2}\right) - \left(x + \frac{1}{x}\right) - 11 = 0 \text{ ---------- (1)}$$

Put $a = x + \frac{1}{x}$ so that $x^2 + \frac{1}{x^2} = a^2 - 2$

Substituting in (1), request equation is

$2\left(a^2 - 2\right) - a - 11 = 0$

$2a^2 - 4 - a - 11 = 0$

$2a^2 - a - 15 = 0$

$(a - 3)(2a + 5) = 0$

$a = 3 \text{ or } -\frac{5}{2}$

Case (i) a = 3

$\Rightarrow x + \frac{1}{x} = 3$

$\Rightarrow x^2 + 1 = 3x$

$\Rightarrow x^2 - 3x + 1 = 0$

$\Rightarrow x = \frac{3 \pm \sqrt{9 - 4}}{2} = \frac{3 \pm \sqrt{5}}{2}$

Case (ii) $a = -\frac{5}{2}$

$x + \frac{1}{x} = -\frac{5}{2}$

$\Rightarrow \frac{x^2 + 1}{x} = -\frac{5}{2}$

$\Rightarrow 2x^2 + 2 = -5x$

$\Rightarrow 2x^2 + 5x + 2 = 0$

$\Rightarrow (2x + 1)(x + 2) = 0$

$\Rightarrow x = -\frac{1}{2}, -2$

$\therefore$ The roots are $-1, -\frac{1}{2}, -2, \frac{3 \pm \sqrt{5}}{2}$

SOLVED PROBLEMS

1. Form the polynomial equation of degree 3 whose roots are 2, 3 and 6.

Sol: The required polynomial equation is

$(x-)(x-3)(x-6)=0$

$\Rightarrow x^3-11x^2+36x-36=0$

2. Find the relation between the roots and the coefficient of the cubic equation

$3x^2-10x^2+7x+10=0$

Sol; $3x^3-10x^2+7x+10=0$ --------- (1)

On comparing (1) with

$ax^3+bx^2+cx+d=0$

We have

$$\sum \alpha=\frac{-b}{a}=\frac{-(-10)}{3}=\frac{10}{3}$$

$$\sum \alpha\beta=\frac{c}{a}=\frac{7}{3}$$

And $\alpha\beta\gamma=\frac{-d}{a}=\frac{-10}{3}$

3. Write down the relations between the roots and the coefficients of the bi-quadratic equation

$x^4-2x^3+4x^2+6x-21=0$

Sol: Given equation is

$x^4-2x^3+4x^2+6x-21=0$ --------(1)

On comparing (1) with

$ax^3+bx^3+cx^2+dx+c=0$

We have

$a=1, b=-2, c=4, d=6, e=-21$

$$\sum \alpha=\frac{-b}{a}=\frac{-(-2)}{1}=2$$

$$\sum \alpha\beta=\frac{c}{a}=\frac{4}{1}=4$$

$$\sum \alpha\beta\gamma=\frac{-d}{a}=-6$$

And $\alpha\beta\gamma\delta=\frac{e}{a}=-21$

4. If 1, 2, 3 and 4 are the roots of $x^4+ax^3+bx^2+cx+d=0$, then find the values of a, b, c and d

Sol: Given that the roots of the given equation are 1, 2, 3 and 4 then

$x^4+ax^3+bx^2+cx+d$

$=(x-1)(x-2)(x-3)(x-4)=0$

$= \left(x^4 - 10x^3 + 35x^2 - 50x + 24 = 0 \right)$

On equating the coefficients of like of powers of x, we obtain

$a = -10, b = 35, c = -50, d = 24$

5. If a, b, c are the roots of $x^3 - px^2 + qx - r = 0$ *and* $r \neq 0$**, then find** $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$

Sol: Given that a, b, c are the roots of

$x^3 - px^2 + qx - r = 0$, then

$a + b + c = p, ab + bc + ca = q, abc = r$

Now $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{b^2c^2 + a^2c^2 + a^2b^2}{a^2b^2c^2}$

$= \frac{(ab + bc + ca)^2 - 2abc(a + b + c)}{a^2b^2c^2}$

$= \frac{q^2 - 2pr}{r^2}$

6. If α, β, γ **are roots of the equations** $x^3 - 10x^2 + 6x - 8 = 0$ **find** $\alpha^2 + \beta^2 + \gamma^2$

Sol: Given α, β, γ are the roots of the equation $x^3 - 10x^2 + 6x - 8 = 0$

$\alpha + \beta + \gamma = 10, \alpha\beta + \beta\gamma + \gamma\alpha = 6, \alpha\beta\gamma = 8$

Now

$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 = -2(\alpha\beta + \beta\gamma + \gamma\alpha)$

$= 100 - 12$

$\therefore \alpha^2 + \beta^2 + \gamma^2 = 88$

7. Find the sum of the squares and the sum of the cubes of the roots of the equation $x^3 - px^2 + qx - r = 0$

Sol: Let α, β, γ be the roots of the given equation then

$\alpha + \beta + \gamma = p, \alpha\beta + \beta\gamma + \gamma\alpha = q, \alpha\beta\gamma = r$

Sum of the squares of the roots is

$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) = p^2 - 2q$

Sum of the cubes of the roots is

$\alpha^3 + \beta^3 + \gamma^3 = (\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha) + 3\alpha\beta\gamma$

$= p(p^2 - 2q - q) + 3r$

$= p(p^2 - 3q) + 3r$

8. Obtain the cubic equation, whose roots are the squares of the roots of the equations, $x^3 + p_1x^2 + p_2x + p_3 = 0$

Sol: The required equation is $f(\sqrt{x}) = 0$

$\Rightarrow (\sqrt{x})^3 + p_1(\sqrt{x})^2 + p_2(\sqrt{x}) + p_3 = 0$

$\Rightarrow x\sqrt{x} + p_1 x + p_2\left(\sqrt{x}\right) + p_3 = 0$

$\Rightarrow \sqrt{x}\left(x + p_2\right) = -\left(p_1 x + p_3\right)$

Squaring on both sides

$\Rightarrow x\left(x + p_2\right)^2 = \left(p_1 x + p_3\right)^2$

$\Rightarrow x\left(x^2 + p_2^2 + 2p_2 x\right) = p_1^2 x^2 + {p_3}^2 + 2p_1 p_3 x$

$\Rightarrow x^3 + {p_2}^2 x + 2p_2 x^2 - {p_1}^2 x^2 - {p_3}^2 - 2p_1 p_3\ x = 0$

$\therefore x^2 + \left(2p_2 - p_1^2\right)x^2 + \left(p_2^2 - 2p_1 p_3\right)x - p_3^2 = 0$

9. **Let α, β, γ be the roots of $x^3 + px^2 + qx + r = 0$. Then find the following,**
$\alpha + \beta + \gamma = -p$, $\alpha\beta + \beta\gamma + \gamma\alpha = q, \alpha\beta\gamma = -r$

(i) $\sum \alpha^2$

Sol: $\sum \alpha^2 = \alpha^2 + \beta^2 + \gamma^2$

$= \left(\alpha + \beta + \gamma\right)^2 - 2\left(\alpha\beta + \beta\gamma + \gamma\alpha\right)$

$= p^2 - 2q$

(ii) $\sum \frac{1}{\alpha}$

Sol: $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \gamma\alpha + \alpha\beta}{\alpha\beta\gamma} = \frac{-q}{r}$

(iii) $\sum \alpha^3$

Sol: $\sum \alpha^3 = \alpha^3 + \beta^3 + \gamma^3$

$= \left(\alpha + \beta + \gamma\right)\left(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \gamma\alpha\right) + 3\alpha\beta$

$= \left(-p\right)\left(p^2 - 2q - q\right) - 3r$

$= -p\left(p^2 - 3q\right) - 3r$

$\therefore \sum \alpha^3 = -p^3 + 3pq - 3r = 3pq - p^3 - 3r$

(iv) $\sum \beta^2\gamma^2$

Sol: $\sum \beta^2\gamma^2 = \beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2$

$= \left(\beta\gamma + \gamma\alpha + \alpha\beta\right)^2 - 2\alpha\beta\gamma\left(\alpha + \beta + \gamma\right)$

$= q^2 - 2\left(-r\right)\left(-p\right)$

$= q^2 - 2pr$

(v) $(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)$

Sol: We know $\alpha+\beta+\gamma=-p$

$\Rightarrow \gamma+\alpha=-p-\beta$

$\therefore(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)$

$=(-p-\gamma)(-p-\alpha)(-p,-\beta)$

$=-p^3-p^2(\alpha+\beta+\gamma)-p(\alpha\beta+\beta\gamma+\gamma\alpha)-\alpha\beta\gamma$

$=-p^3+p^3-pq+r=r-pq$

10. Let α,β,γ be the roots of $x^3+ax^2+bx+c=0$ then find $\sum \alpha^2\beta^2\gamma^2+\sum \alpha\beta^2$

Sol: Since α,β,γ are roots of the given equation, we have

$\alpha+\beta+\gamma=-a, \alpha\beta+\beta\gamma+\gamma\alpha=b, \alpha\beta\gamma=-c$

Now $\sum \alpha^2\beta+\sum \alpha\beta^2=\sum \alpha\beta(\alpha+\beta)$

$=\sum \alpha\beta(-a-\gamma)$

$=-a\sum \alpha\beta-\sum \alpha\beta\gamma$

$=-a(\alpha\beta+\beta\gamma+\gamma\alpha)-3\alpha\beta\gamma$

$=-a(\ +b)+3c$

$=3c-ab$

$\therefore\sum \alpha^2\beta+\sum \alpha\beta^2=3c-ab$

11. If α,β,γ are the roots of $x^3+px^2+qx+r=0$, then form the cubic equation whose roots are $\alpha(\beta+\gamma),\beta(\gamma+\alpha),\gamma(\alpha+\beta)$

Sol: Let α,β,γ be the roots of the given equation,

We have, $\alpha+\beta+\gamma=-p, \alpha\beta+\beta\gamma+\gamma\alpha=q$

$\alpha\beta\gamma=-r$

Let $y=\alpha(\beta+\gamma)$

$=\alpha\beta+\alpha\gamma+\gamma\beta-\beta\gamma$

$y=q-\dfrac{\alpha\beta\gamma}{\alpha}$

$y=q+\dfrac{r}{a}\Rightarrow y\alpha=q\alpha+r$

$\alpha(y-q)=r\Rightarrow\alpha=\dfrac{r}{y-q}$

Since 'α' be the roots of $x^3+px^2+qx+r=0$

$$\frac{r^3}{(y-q)^3}+p\left(\frac{r^2}{y-q}\right)^2+q\left(\frac{r}{y-q}\right)+r\ =0$$

$\Rightarrow r^3+pr^2(y-p)+qr(y-q)^2+(y-q)^3\,r=0$

$r^3 + pr^2y - pr^2q + qr\left(y^2 + q^2 - 2qy\right) + r\left(y^3 - q^3 - 3y^2q + 3yq^2\right) = 0$

$r^3 + pr^2y - pr^2q + qry^2 + q^3r - 2q^2ry + ry^3 - rq^3 - 3y^2\ qr + 3yq^2\ r = 0$

$r\left(y^3 + r^2 + p^2rq + qy^2 + q^3 - 2q^2y\right) - q^3 - 3y^2q + 3yq^2) = 0$

$y^3 - 2qy^2 + y\left(pr - q^2\right) - p^2rq + r^2 = 0$

$\therefore x^3 - 2qx^2 + \left(q^2 + pr\right)x - r\left(pq - r\right) = 0$

12. **Solve** $x^3 - 3x^2 - 16x + 48 = 0$

Sol: Let $f(x) = x^3 - 3x^2 - 16x + 48$

By inspection, $f(3) = 0$

Hence 3 is a root of $f(x) = 0$

Now we divide $f(x)\ by\ (x-3)$

3	1	−3	−16	48
		3	0	−48
	1	0	−16	0

$\therefore x^2 - 16 = 0 \Rightarrow (x-4)(x+4) = 0$

$\Rightarrow x = -4, 4$

$\therefore$ The roots are $-4, 3, 4$

13. **Find the roots of** $x^4 - 16x^3 + 86x^2 - 176x + 105 = 0$

Sol: Let $f(x) = x^4 - 16x^3 + 86x^2 - 176x + 105$

Now, if $(1) = 1 - 16 + 86 - 176 + 105 = 0$

$\therefore$ 1 is a root of $f(x) = 0$

$\Rightarrow x - 1$ is a factor of $f(x)$

1	1	−16	86	−176	105
		1	−15	71	−105
	1	−15	71	−105	0

$\therefore f(x) = (x-1)\left(x^3 - 15x^2 + 71x - 105\right)$

$= (x-1)g(x)$ where

$g(x) = x^3 - 15x^2 + 71x - 105$

$g(1) = 1 - 15 + 71 - 105 = -48 \neq 0$

$g(2) = -15 \neq 0$

$g(3) = 27 - 135 + 213 - 105 = 0$

$\therefore 3$ is a root of $g(x) = 0$

$\Rightarrow x-3$ is a factor of $g(x)$

3	1	−15	71	−105
	−	3	−36	105
	1	−12	35	0

$\therefore g(x)=(x-3)(x^2-12x+35)$

$=(x-3)(x-5)(x-7)$

$\therefore f(x)=(x-1)(x-3)(x-5)(x-7)$

$\therefore 1, 3, 5, 7$ are the roots of $f(x)=0$

14. Solve $x^3-7x^2+36=0$, given one root being twice the other

Sol: Let α, β, γ be the root of the equation

$x^3-7x^2+36=0$ and

Let $\beta=2\alpha$

Now we have $\alpha+\beta+\gamma=7$

$\Rightarrow 3\alpha+\gamma=7$ ----------(1)

$\alpha\beta+\beta\gamma+\gamma\alpha=0$

$\Rightarrow 2\alpha^2+3\alpha\gamma=0$ ----------(2)

$\alpha\beta\gamma=-36\Rightarrow 2\alpha^2\gamma=-36$ ----------(3)

From (1) and (2), we have

$2\alpha^2+3\alpha(7-3\alpha)=0$

i.e., $\alpha^2-3\alpha=0\,(or)\,\alpha(\alpha-3)=0$

$\therefore \alpha=0$ *or* $\alpha=3$

Since $\alpha=0$ does not satisfy the given equation

$\therefore \alpha=3$, *so* $\beta=6$ *and* $\gamma=-2$

$\therefore$ The roots are 3, 6, -2

15. Given that 2 is a root of $x^3-6x^2+3x+10=0$, find the other roots

Sol: let $f(x)=x^3-6x^2+3x+10$

Since 2 is a root of $f(x)=0$, we divide $f(x)$ by $(x-2)$

	6	−13	−35	−1	3
4	−	24	44	12	−
−1	−	−	−6	−11	−3
	6	11	3	0	0

$\therefore x^3-6x^2+3x+10=(x-2)(x^2-4x-5)$

$=(x-2)(x+1)(x-5)$

$\therefore -1, 2$ and 5 are the roots of the given equation

16. Give that two roots of $4x^3 + 20x^2 - 23x + 6 = 0$ are equal, find all the roots of the given equation

Sol: Let α, β, γ are the roots of $4x^3 + 20x^2 - 23x + 6 = 0$

Given two roots are equal, let α, β

W have $\alpha + \beta + \gamma = \frac{-20}{4} = -5$

$\Rightarrow 2\alpha + \gamma = -5$

$\Rightarrow \alpha\beta + \beta\gamma + \gamma\alpha = \frac{-23}{4}$

$\Rightarrow \alpha^2 + 2\alpha\gamma = \frac{-23}{4}$ ---------- (2)

$\alpha\beta\gamma = \frac{-6}{4} = \frac{-3}{2} \Rightarrow \alpha^2\gamma = \frac{-3}{2}$ ---------- (3)

From (1) and (2) $\alpha^2 + 2\alpha(-5 - 2\alpha) = \frac{-23}{4}$

$\Rightarrow 12\alpha^2 + 40\alpha - 23 = 0$

$\Rightarrow (2\alpha - 1)(6\alpha + 23) = 0$

$\alpha = \frac{1}{2}, \alpha = \frac{-23}{6}$

On verification, we get that

$\alpha = \frac{1}{2}$ is a root of (1)

$(2) \Rightarrow \gamma = -6$

$\therefore$ Roots are $\frac{1}{2}, \frac{1}{2}, -6$

17. Given that the sum of two roots of $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$ is zero, find the roots of the equation

Sol: Let $\alpha, \beta, \gamma, \delta$ are the roots of given equation, since sum of two is zero

$\alpha + \beta = 0$

Now $\alpha + \beta + \gamma + \delta = 2 \Rightarrow \gamma + \delta = 2$

Let $\alpha\beta = p, \gamma\delta = q$

The equation having the roots α, β is

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

$\therefore x^2 + p = 0$

The equation having the roots γ, δ is

$x^2 - (\gamma + \delta)x + \gamma\delta = 0$

$\therefore x^2 - 2x + q = 0$

$\therefore x^4 - 2x^3 + 4x^2 + 6x - 21$

$= \left(x^2 + p\right)\left(x^2 - 2x + q\right)$

$= x^4 - 2x^3 + x^2\left(p + q\right) - 2px + pq$

Comparing the like terms

$p + q = 4, -2p = 6$

$-3 + q = 4 \; p = -3$

$q = 7$

$\therefore x^2 - 3 = 0 \Rightarrow x = \pm\sqrt{3}$ *and* $x^2 - 2x + 7 = 0 \quad \Rightarrow x = \dfrac{2 \pm \sqrt{4 - 28}}{2}$

$= \dfrac{2 \pm 2\sqrt{6i}}{2}$

$= 1 \pm \sqrt{6i}$

$\therefore$ Roots are $-\sqrt{3}, \sqrt{3}, 1 - i\sqrt{6}$ *and* $1 + i\sqrt{6}$

18. Solve $4x^3 - 24x^2 + 23x + 18 = 0$, given that the roots of this equation are in arithmetic progression

Sol: Let $a - d, a\; a + d$ are the roots of the given equation

Now, sum of the roots

$a - d + a + a + d = \dfrac{24}{4}$

$3a = 6$

$a = 2$

Product of the roots $\left(a - d\right)a\left(a + d\right) = \dfrac{-18}{4}$

$a\left(a^2 - d^2\right) = -\dfrac{9}{2}$

$2\left(4 - d^2\right) = -\dfrac{9}{2}$

$4\left(4 - d^2\right) = -9$

$16 - 4d^2 = -9$

$4d^2 = 25$

$d = \pm\dfrac{5}{2}$

$\therefore$ Roots are $-\dfrac{1}{2}, 2$ *and* $\dfrac{9}{2}$

19. **If the roots of $x^3+3px^2+3qx+r=0$, are in geometric progression, find the condition**

Sol: The roots are in GP

Suppose the roots be $\frac{a}{R}, a, aR$

Given $\left(\frac{a}{R}\right)(a)(aR)=-R$

$\Rightarrow a^3=-1$

$\Rightarrow a=(-r)^{1/3}$

$\because$ 'a' is a root of $x^3+3px^2+3qx+r=0$

$\Rightarrow\left(-r^{1/3}\right)^3+3p\left(-r^{1/3}\right)^2+3q\left(-r^{1/3}\right)+r=0$

$\Rightarrow -r+3pr^{2/3}-3qr^{1/3}+r=0$

$pr^{2/3}=qr^{1/3}$

$\Rightarrow pr^{1/3}=q$

$\Rightarrow p^3r=q$ is the required condition

20. **Solve $x^4-5x^3+5x^2+5x-6=0$ given that the product of two of its roots is 3**

Sol: Let $\alpha,\beta,\gamma,\delta$ be the root of the given equation

Product of the roots $\alpha\beta\gamma\delta 33=-6$

Given $\alpha\beta=3$ ($\because$ Products of two roots is 3)

$\therefore \alpha\beta\gamma\delta=-6$

$\gamma\delta=-2$

Let $\alpha+\beta=p, \gamma+\delta=q$

The equation having the roots α,β is

$x^2-(\alpha+\beta)x+\alpha\beta=0$

$x^2-px+3=0$

The equation having the root γ,δ is

$x^2-(\gamma+\delta)x+\gamma\delta=0\ x^2-qx-2=0$

$\therefore x^4-5x^3+5x^2+5x-6$

$=\left(x^2-px+3\right)\left(x^2-qx-2\right)$

$=x^4-(p+q)x^3+(1+pq)x^2$

$+(2p-3q)x-6$

Comparing the like terms,

$p+q=5, 2p-3q=5$

$\therefore 2p-3q=5$

$3p+3q=15$

$5p=20\Rightarrow p=4$

$\therefore p = 1$

Now $x^2 - 4x + 3 = 0 \Rightarrow (x-3)(x-1) = 0$

$\Rightarrow x = 1, 3$

$x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0$

$\Rightarrow x = -1, 2$

$\therefore$ The roots are -1, 2, 1, 3

21. Solve $x^4 + 4x^3 - 2x^2 - 12x + 9 = 0$. Given that it has two pairs of equal roots

Sol: Given equation is

$x^4 + 4x^3 - 2x^2 - 12x + 9 = 0$

Let the roots be $\alpha, \alpha, \beta, \beta$

Sum of the roots, $2(\alpha + \beta)$

$= -4 \Rightarrow \alpha + \beta = -2$

Let $\alpha\beta = p$

The equation having root α, β is

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

I.e., $x^2 + 2x + p = 0$

$\therefore x^4 + 4x^3 - 2x^2 - 12x + 9$

$= \left[x^2 - (\alpha + \beta)x + \alpha\beta\right]^2$

$= x^4 + 4x^3 + (2p + 4)x^2 + 4px + p^2$

Comparing coefficients of x on both sides

$4p = -12 \Rightarrow p = -3$

$x^2 + 2x + p = 0$

$\therefore x^4 + 4x^3 + (2p + 4)x^2 + 4px + p^2$

Comparing coefficients of x on both sides

$4p = -12 \Rightarrow p = -3$

$x^2 + 2x + p = 0 \Rightarrow x^2 + 2x - 3 = 0$

$\Rightarrow (x+3)(x-1) = 0$

$\Rightarrow x = -3, 1$

$\therefore$ The root s of the given equation are $-3, -3, 1, 1$

22. Form the polynomial equation of degree 4 whose roots we $4 + \sqrt{3}, 4 - \sqrt{3}, 2 + i$ and $2 - i$

Sol: The equation having roots $4 + \sqrt{3}, 4 - \sqrt{3}$ is

$x^2 - 8x + 13 = 0$

The equation having roots $2 + i, 2 - i$ is

$x^2 - 4x + 5 = 0$

The required equation is

$(x^2-8x+13)(x^2-4x+5)=0$

$\therefore x^4-12x^3+50x^2-92x+65=0$

23. **Solve** $6x^4-13x^3-35x^2-x+3=0$ **given that one of its root is** $2+\sqrt{3}$

Sol: $2+\sqrt{3}$ is a root $\Rightarrow 2-\sqrt{3}$ is also a root

The equation having roots

$2\pm\sqrt{3}$ *is* $x^2-4x+1=0$

$\therefore x^2-4x+1$ is a factor of

$6x^4-13x^3-53x^2-x+3=0$

	6	−13	−35	−1	3
4	–	24	44	12	–
−1	–	–	−6	−11	−3
	6	11	3	0	0

$6x^2+11x+3=0$

$\Rightarrow 6x^2+9x+2x+3=0$

$\Rightarrow 3x(2x+3)+1(2x+3)=0$

$\Rightarrow (3x+1)(2x+3)=0$

$\Rightarrow x=-\frac{1}{3},\frac{-3}{2}$

$\therefore$ The roots of the given equation are $-\frac{1}{3},\frac{-3}{2},2\pm\sqrt{3}$

24. **Find the polynomial equation of degree 4 whose roots is the negatives root of** $x^4-6x^3+7x^2-2x+1=0$

Sol: Let $f(x)\equiv x^4-6x^3+7x^2-2x+1$

The required equation is $f(-x)=0$

i.e., $(-x)^4-6(-x)^3+7(-x)^2-2(-x)+1=)$

$\therefore x^4+6x^3+7x^2+2x+1=0$

25. **Find the algebraic equation of the degree 4 whose roots are 3 times the roots of the equation**

$6x^4-7x^3+8x^2-7x+2=0$

Sol: Let $f(x)\equiv 6x^4-7x^3+8x^2-7x+2$

The required equation is $f\left(\frac{x}{3}\right)=0$

i.e., $6\left(\frac{x}{3}\right)^4 - 7\left(\frac{x}{3}\right)^3 + 8\left(\frac{x}{3}\right)^2 - 7\left(\frac{x}{3}\right) + 2 = 0$

i.e., $\frac{6x^4}{81} - \frac{7x^3}{27} + \frac{8x^2}{9} - \frac{7x}{3} + 2 = 0$

$\therefore 6x^4 - 21x^3 + 72x^2 - 189x + 162 = 0$

26. Form the equation whose roots are m times the roots of the equation $x^3 + \frac{x^2}{4} - \frac{x}{16} + \frac{1}{72} = 0$ and deduce the case when $m = 12$

Sol: Let $f(x) \equiv x^3 + \frac{x^2}{4} - \frac{x}{16} + \frac{1}{72}$

The required equation is $f\left(\frac{x}{m}\right) = 0$

i.e., $\left(\frac{x}{m}\right)^3 + \frac{1}{4}\left(\frac{x}{m}\right)^2 - \frac{1}{16}\left(\frac{x}{m}\right) + \frac{1}{72} = 0$

i.e., $\frac{x^3}{m^3} + \frac{x^2}{4m^2} - \frac{x}{16m} + \frac{1}{72} = 0$

$\therefore x^3 + \frac{m}{4}x^2 - \frac{m^2}{16} \times + \frac{m^2}{72} = 0$

When m = 12

$x^3 + \frac{12}{4}x^2 - \frac{144}{16} \times + \frac{144 \times 12}{72} = 0$

$\therefore x^3 + 3x^2 - 9x + 24 = 0$

27. Find the algebraic equation of degree 5 whose roots are the translates of the root of $x^5 + 4x^3 - x^2 + 11 = 0$ by -3

Sol: Let $f(x) \equiv x^5 + 4x^3 - x^2 + 11$

The required equation is $f(x+3) = 0$

3	1	0	4	−1	0	11
		−	3	9	39	114	342
3	1	3	13	38	114	353
		−	3	18	93	393
3	1	6	31	131	507
		−	3	27	174
3	1	9	58	305
		−	3	36
3	1	12	94
		−	3
	1	15

The required equation is

$x^5 + 15x^4 + 94x^3 + 305x^2 + 507x + 353 = 0$

28. Find the algebraic equation of degree 4 whose roots are $4x^4 + 32x^3 + 83x^2 + 76x + 21 = 0$ by 2

Sol: Let $f(x) \equiv 4x^4 + 32x^3 + 83x^2 + 76x + 21$

The required equation is $f(x-2) = 0$

−2	4	32	83	76	21
		−8	−48	−70	−21
−2	4	24	35	6	9
		−8	−32	−6	
−2	4	16	3	0	
		−8	−16		
−2	4	8	−13		
		−8			
	4	0			

The required equation is

$4x^3 - 13x^2 + 9 = 0$

29. Find the polynomial equation whose roots are the reciprocals of the roots of the equation $x^4 + 3x^3 - 6x^2 + 2x - 4 = 0$

Sol: Let $f(x) \equiv x^4 + 3x^3 - 6x^2 + 2x - 4$

The required equation is $f\left(\frac{1}{x}\right) = 0$

i.e., $\left(\frac{1}{x}\right)^4 + 3\left(\frac{1}{x}\right)^3 - 6\left(\frac{1}{x}\right)^2 + 2\left(\frac{1}{x}\right) - 4 = 0$

i.e., $\frac{1}{x^4} + 3\frac{1}{x^3} - 6\frac{1}{x^2} + 2\frac{1}{x} - 4 = 0$

i.e., $1 + 3x - 6x^2 + 2x^3 - 4x^4 = 0$

30. Find the polynomial equation whose roots are the squares of the roots of $x^3 - x^2 + 8x - 6 = 0$

Sol: Let $f(x) \equiv x^3 - x^2 + 8x - 6$

The required equation is $f\left(\sqrt{x}\right) = 0$

i.e., $\left(\sqrt{x}\right)^3 - \left(\sqrt{x}\right)^2 + 8\sqrt{x} - 6 = 0$

$\Rightarrow x\sqrt{x} + x + 8\sqrt{x} - 6 = 0$

$\Rightarrow \sqrt{x}(x+8) = x + 6$

Squaring on both sides

$$\Rightarrow x\left(x^2+16x+64\right)=x^2+12x+36$$

$$\Rightarrow x^3+16x^2+64x-x^2-12x-36=0$$

$$\therefore x^3+15x^2+52x-36=0$$

31. Show that $2x^3+5x^2+5x+2=0$ is a reciprocal equation of class one

Sol: Given equation is $2x^3+5x^2+5x+2=0$

$P_0=2,\ p_1=5,\ P_2=5,\ p_3=2$

Here $p_0=p_3,\ p_1=p_2$

$\therefore$ The equation $2x^3+5x^2+5x+2=0$ is a reciprocal equation of class one

32. Solve the equation $4x^3-13x^2-13x+4=0$

Sol: $4x^3-13x^2-13x+4=0$ is a reciprocal equation of first class and of odd degree. Thus -1 is a root of the given equation

–1	4	–13	–13	4
	–	–4	17	–4
	4	–17	4	0

$$4x^2-17x+4=0\Rightarrow 4x^2-16x-x+4=0$$

$$\Rightarrow 4x\left(x-4\right)-1\left(x-4\right)=0$$

$$\Rightarrow \left(x-4\right)\left(4x-1\right)=0$$

$$\Rightarrow x=4\ or\ \frac{1}{4}$$

$\therefore$ The roots are $-1,\ 4,\ \frac{1}{4}$

33. Solve $x^4+x^3-4x^2+x+1=0$

Sol: Given equation is $x^4+x^3-4x^2+x+1=0$

This is a reciprocal equation of even degree and of first class

Given equation can be written as

$$x^2+x-4+\frac{1}{x}+\frac{1}{x^2}=0$$

i.e., $\left(x^2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)-4=0$ ----------(1)

Put $x+\frac{1}{x}=y\Rightarrow x^2+\frac{1}{x^2}=y^2-2$

$\therefore$ (1) becomes $\left(y^2-2\right)+\left(y\right)-4=0$

$$\Rightarrow y^2+y-6=0$$

$\Rightarrow (y+3)(y-2)=0$

$\Rightarrow y=-3, 2$

Case (i) $y=-3 \Rightarrow x+\frac{1}{x}=-3$

$\Rightarrow \frac{x^2+1}{x}=-3$

$\Rightarrow x^2+1=-3x$

$\Rightarrow x^2+3x+1=0$

$\therefore x=\frac{-3\pm\sqrt{9-4}}{2}=\frac{-3\pm\sqrt{5}}{2}$

Case (ii) $y=2 \Rightarrow x+\frac{1}{x}=2$

$\Rightarrow x^2+1=2x$

$\Rightarrow x^2-2x+1=0 \Rightarrow (x-1)^2=0$

$\therefore x=1,1$

$\therefore$ The roots are $1, 1, \frac{-3\pm\sqrt{5}}{2}$

34. **Solve** $x^5-5x^4+9x^3-9x^2+5x-1=0$

Sol: Given equation is $x^5-5x^4+9x^3-9x^2+5x-1=0$

Is a reciprocal equation of odd degree and of class two

$\therefore 1$ is a root of the given equation

$\Rightarrow (x-1)$ is a factor of

$x^5-5x^4+9x^3-9x^2+5x-1=0$

1	1	−5	9	−9	5	−1
	−	1	−4	5	−4	1
	1	−4	5	−4	1	0

$x^4+4x^3+5x^2-4x+1=0$

$\Rightarrow x^2-4x+5-\frac{4}{x}+\frac{1}{x^2}=0$

$\Rightarrow \left(x^2+\frac{1}{x^2}\right)-4\left(x+\frac{1}{x}\right)+5=0$ ----------(1)

Put $x+\frac{1}{x}=y \Rightarrow x^2+\frac{1}{x^2}=y^2-2$

$\therefore$ (1) becomes $(y^2-2)-4y+5=0$

$\Rightarrow y^2-4y+3=0$

$\Rightarrow (y-1)(y-3)=0$

$\Rightarrow y = 1, 3$

Case (i) $y = 1 \Rightarrow x + \frac{1}{x} = 1$

$\Rightarrow \frac{x^2 + 1}{x} = 1$

$\Rightarrow x^2 + 1 = x$

$\Rightarrow x^2 - x + 1 = 0$

$\therefore x = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{3}i}{2}$

Case (ii) $y = 3 \Rightarrow x + \frac{1}{x} = 3$

$\Rightarrow \frac{x^2 + 1}{x} = 3 \Rightarrow x^2 + 1 = 3x$

$\Rightarrow x^2 - 3x + 1 = 0$

$\therefore x = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2}$

$\therefore$ The roots of the given equation are $1, \frac{1 \pm \sqrt{3}i}{2}, \frac{3 \pm \sqrt{5}}{2}$

35. Solve the equation $6x^6 - 25x^5 + 31x^4 - 31x^2 + 25x - 6 = 0$

Sol: Given equation is $6x^6 - 25x^5 + 31x^4 - 31x^2 + 25x - 6 = 0$ is a reciprocal equation of second class and of even degree

$\therefore x^2 - 1$ is a factor of

$6x^6 - 25x^5 + 31x^4 - 31x^2 + 25x - 6 = 0$

1	6	−25	31	0	−31	25	−6
	−	6	−19	12	12	−19	6
−1	6	−19	12	12	−19	6	0
	−	−6	25	−37	25	−6	
	6	−25	37	−25	6	0	

$6x^4 - 25x^3 + 37x^2 - 25x + 6 = 0$

$\Rightarrow 6x^2 - 25x + 37 - \frac{25}{x} + \frac{6}{x^2} = 0$

$\Rightarrow 6\left(x^2 + \frac{1}{x^2}\right) - 25\left(x + \frac{1}{x}\right) + 37 = 0$ ---------- (1)

Put $x + \frac{1}{x} = y \Rightarrow x^2 + \frac{1}{x^2} = y^2 - 2$

$\therefore$ (1) Becomes $6\left(y^2 - 2\right) - 25\left(y\right) + 37 = 0$

$\Rightarrow 6y^2 - 12 - 25y + 37 = 0$

$\Rightarrow 6y^2 - 25 + 25 = 0$

$\Rightarrow 6y^2 - 15y - 10y + 25 = 0$

$\Rightarrow 3y(2-5) - 5(2y-5) = 0$

$\Rightarrow (2y-5)(3y-5) = 0$

$\Rightarrow y = \frac{5}{2}, \frac{5}{3}$

Case (i) : $y = \frac{5}{2} \Rightarrow x + \frac{1}{x} = \frac{5}{2} \Rightarrow \frac{x^2+1}{x} = \frac{5}{2}$

$\Rightarrow 2x^2 + 2 = 5x$

$\Rightarrow 2x^2 - 5x + 2 = 0, \Rightarrow (2x-1)(x-2) = 0$

$\therefore x = \frac{1}{2}, 2$

Case (ii): $y = \frac{5}{2} \Rightarrow x + \frac{1}{x} = \frac{5}{3} \Rightarrow \frac{x^2+1}{x} = \frac{5}{3}$

$\Rightarrow 3x^2 + 3 = 5x$

$\Rightarrow 3x^2 - 5x + 3 = 0$

$\Rightarrow x = \frac{5 \pm \sqrt{25-36}}{6}$

$\therefore x = \frac{5 \pm \sqrt{11}i}{6}$

$\therefore$ The roots of the given equation are

$\pm 1, \frac{1}{2}, 2, \frac{5 \pm \sqrt{11}i}{6}$