RATE OF CHANGE

TOPICS: Derivate as Rate of change

RATE OF CHANGE

Let y = f(x) be defined on an interval (a,b). let δy be change in y corresponding to a change δx in x.

Then, $\frac{\delta y}{\delta x}$ is called average rate of change of y. if $\underset{\delta x \to 0}{Lt} \frac{\delta y}{\delta x}$ exists finitely, the this limit (i.e., $\frac{dy}{dx}$) is called rate of change of y with respect to x .

Note: $\left(\frac{dy}{dx}\right)_{at\ x=c}$ represents the rate of change of y with respect to x at x=c.

RECTILINEAR MOTION

Let a particle start at a point *O* on a line *L* and move along the line. After a time of *t* units, let the particle be at *P* and *OP*=*s*. Since *s* is dependent on time *t*, we write *s*=*s*(*t*). *s* is called the displacement of the particle during time *t*.

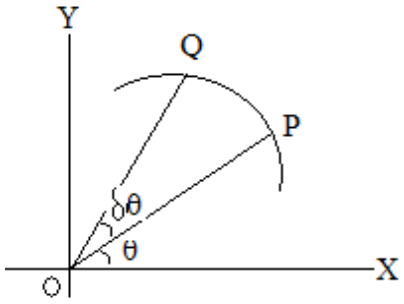
The rate of change of displacement s is called the velocity of the particle and it is denoted by $v=\frac{ds}{dt}$.

The rate of change of velocity is called acceleration. It is denoted by *a*.

$$\therefore a=\frac{dv}{dt}=\frac{d}{dt}\left(\frac{ds}{dt}\right)=\frac{d^2s}{dt^2}$$

ANGULAR VELOCITY

At time t, let P be the position of a moving point, Q be the position of the point after an interval δt. Let $\angle xop=\theta$ and $\angle POQ=\delta\theta$



The angular velocity of P at O is $\underset{\delta t \to 0}{Lt} \frac{\delta\theta}{\delta t}=\frac{d\theta}{dt}$ rad/sec.

Angular velocity $w\frac{d\theta}{dt}$

The angular acceleration of P at O is $\frac{d\omega}{dt}=\frac{d}{dt}\left(\frac{d\theta}{dt}\right)=\frac{d^2\theta}{dt^2}$

EXERCISE – 10 (e)

I.

1. At time t, the distance s of a particle moving in a straight line given by $s = -4t^2 + 2t$. Find the velocity and acceleration when $t = \frac{1}{2}$ seconds.

Sol: displacement $s = -4t^2 + 2t$

Velocity $v = \frac{ds}{dt} = \frac{d}{dt}\left(-4t^2 + 2t\right) = -8t + 2$

$\therefore$ Velocity at $t = \frac{1}{2}$ is $v = \left(\frac{ds}{dt}\right)_{t=\frac{1}{2}}$ $= -8\left(\frac{1}{2}\right) + 2 = -4 + 2 = -2$ units/sec.

Acc. $a = \frac{dv}{dt} = \frac{d}{dt}(-8t + 2) = -8$

Acceleration $= a = \left(\frac{dv}{dt}\right)_{t=\frac{1}{2}} = -8$ units.sec^2.

2. The displacement s of a particle travelling in a straight line in t seconds is given by $s = 45t + 11t^2 - t^3$. Find the time when the particle comes to rest.

Sol: displacement $s = 45t + 11t^2 - t^3$

Velocity $v = \frac{ds}{dt} = 45 + 22t - 3t^2$

Particle comes to rest if v= 0.

$\Rightarrow 45 + 22t - 3t^3 = 0$ $\Rightarrow 3t^2 - 22t - 45 = 0$ $\Rightarrow 3t^2 - 27t + 5t - 45 = 0$

$\Rightarrow (3t + 5)(t - 9) = 0$ $\therefore t = 9$ or $t = -\frac{5}{3}$

$\therefore$ t = 9

$\therefore$ The particle comes to rest at t = 9 seconds.

3. A particle is moving along a straight line so that at time t its, displacement s from a fixed point in the line is given by s = a sin (bt + c) where a, b c are constants. If v and f are the velocity and acceleration of the particle respectively at time t prove that

i) $\mathbf{v^2 - fs}$ is constant ii) $\mathbf{\frac{df}{ds}}$ is constant and iii) $\mathbf{s.\frac{df}{ds} = v.f}$

Sol: displacement $s = a\sin(bt + c)$

Velocity $v = \frac{ds}{dt} = ab\cos(bt + c)$

Acc $f = \frac{d^2s}{dt^2} = -ab^2\sin(bt + c)$

i) $v^2 - fs = a^2b^2\cos^2(bt + c)$

$-\left[-ab^2\sin(bt + c)\right]\left[\left(a\sin(bt + c)\right)\right] = a^2b^2\cos^2(bt + c) + a^2b^2\sin^2(bt + c)$

$= a^2b^2$

ii) $f = -ab^2\sin(bt + c) = -b^2s$

$\frac{df}{ds} = -b^2 = \text{constant}$

iii) $s.\frac{df}{ds} = a\sin(bt + c).\left(-b^2\right) = -ab^2.\sin(bt + c) = f$

4. A particle moves in a straight line in such a manner that $\mathbf{s = \frac{1}{2}vt}$, s being the distance travelled in time t and v the velocity at the end of time t. Prove that the acceleration is a constant.

Sol: displacement $s = \frac{1}{2}vt$

Differentiating w.r.to t

Velocity $v = \frac{ds}{dt} = \frac{1}{2}\left[v + t\frac{dv}{dt}\right]$

$\Rightarrow 2v = v + t\frac{dv}{dt} \Rightarrow v = t\frac{dv}{dt}$

$\frac{dv}{dt} = \frac{v}{t}$ = acceleration =a

$$a = \frac{v}{t}$$

Diff. w.r.t t,

$\frac{da}{dt} = \frac{t.\frac{dv}{dt} - v.1}{t^2} = \frac{t\frac{v}{t} - v}{t^2} = 0$ $\Rightarrow$ a is constant.

5. **The distance – time formula for the motion of a particle along a straight line is $s = t^2 - 9t^2 + 24t - 18$. Find when and where the velocity is zero.**

Sol: displacement $s = t^2 - 9t^2 + 24t - 18$

Velocity $v = \frac{ds}{dt} = 3t^2 - 18t + 24$

$v = 0 \Rightarrow 3\left(t^2 - 6t + 8\right) = 0$

$\therefore \left(t-2\right)\left(t-4\right) = 0$

$\therefore$ t = 2 or 4

The velocity is zero after 2 and 4 seconds.

Case (i): t = 2

$s = t^3 - 9t^2 + 24t - 18$

= 8 – 36 + 48 – 18 = 56 – 54 = 2

Case (ii): $t = 4; s = t^3 - 9t^2 + 24t - 18$

= 64 – 144 + 96 – 18 = 160 – 162 = –2

The particle comet to rest at a distance of 2 units from the starting point ‘O’ on either side.

6. **A point P is moving on the curve $y = 2x^2$. The x co-ordinate of P is increasing at the rate of 4 units per second. Find the rate at which y co-ordinate is increasing when the point is (2, 8).**

Sol: equation of the curve $y = 2x^2$

Diff. w.r.t. t, $\frac{dy}{dt} = 4x.\frac{dx}{dt}$

Given $x = 2$ and $\frac{dx}{dt} = 4.$

$\frac{dy}{dt} = 4(2).4 = 32$

y co-ordinate is increasing at the rate of 32 units/sec.

7. **Show that the rate of increase in the slope of the tangent at a point on the curve $y = 2x^3$ is 12 times at the rate of increase of x co-ordinate of the point when the point is at $(1,2)$.**

Sol: Given curve is $y = 2x^3$

Slope $m = \frac{dy}{dx} = 6x^2$

Diff.w.r to 't' , $\frac{dm}{dt} = 12x.\frac{dx}{dt}$

At the point $p(1,2)$, $\frac{dm}{dt} = 12(1)\frac{dx}{dt} = 12.\frac{dx}{dt}$

$\therefore$ The rate of increase in the slope of the tangent at (1, 2) on the curve $y = 2x^3$ is 12 times the rate of increase in x-co-ordinate of the point.

8. **The radius of a circular plate is increasing in length at 0.01 cm/sec. What is the rate at which the area is increasing when the radius is 12 cm?**

Sol: let r be the radius and A be the area of the circular plate. Then $A = \pi r^2$

Diff w.r.t.t, $\frac{dA}{dt} = 2\pi r\left(\frac{dr}{dt}\right)$

Given $r = 12$ and $\frac{dr}{dt} = 0.01$

$\frac{dA}{dt} = 2\pi(12)(0.01) = 0.24\pi$ sq.cm / sec

The area is increasing at the rate of 0.24π sq.cm / sec.

9. A stone is thrown into still water causes a series of concentric ripples. If the radius of the outer ripple is increasing at the rate of 3 meters per second, find the rate at which the area of the disturbed water is increasing when the outer ripple has a radius of 10 meters.

Sol: let r be the radius and A be the area of the circular plate. Then $A = \pi r^2$

Diff w.r.t.t, $\frac{dA}{dt} = 2\pi r.\frac{dr}{dt}$

Given $r = 10, \frac{dr}{dt} = 3$

$\frac{dA}{dt} = 2\pi(10).3 = 60\pi$ sq.metres / sec.

$\therefore$ The area of disturbed water is increasing at the rate of 60π sq.metres / sec.

III.

1. Sand is poured from a pipe at the rate of 12 cc/sec. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand-cone increasing when the height is 4cm?

Sol: Let h be the height, r be the radius and V be the volume of a conical shape of sand .

Given $h = \frac{1}{6}r, \frac{dV}{dt} = 12$cc / sec and $h = 4$cm

Now $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(6h)^2.h = 12\pi h^3$

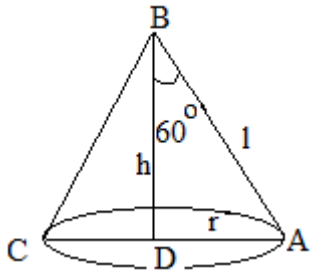
Diff w.r.t .t, $\frac{dv}{dt} = 36\pi^2\frac{dh}{dt}$

$\therefore \frac{dh}{dt} = \frac{\frac{dv}{dt}}{36\pi h^2} = \frac{12}{36\pi(4)^2} = \frac{1}{48\pi}$ cm / sec

$\therefore$ The height of the sand cone increasing at the rate of $\frac{1}{48\pi}$ cm / sec.

2. Water is dripping out from a conical funnel, at a uniform rate of 2 cm^2/sec, through a tiny hole at the vertex at the bottom. When the slant height of the water is 4cm, find the rate of decrease of the slant height of the water given that the vertical angle of the funnel is 120º.

Sol: Let r be the radius, h be the height, l be the slant height of the cone.



$\frac{dV}{dt} = -2$ cubic cms / sec

l = 4cms

Vertical angle = 120º => Semi vertical angle =60º

Let $\angle ADB = 60^\circ$, AD = h, AB = r

From , triangle ABD, $\cos 60^\circ = \frac{h}{l} \Rightarrow h = l\cos 60^\circ = l.\frac{1}{2} = \frac{l}{2}$

$\sin 60^\circ = \frac{r}{l} \Rightarrow r = l\sin 60^\circ \Rightarrow r = \frac{\sqrt{3}}{2}l$

Volume $V = \frac{1}{3}\pi r^2 h$ => $V = \frac{\pi}{3}.\frac{3}{4}l^2.\frac{l}{2} = \frac{\pi}{8}l^3$

$\frac{dV}{dt} = \frac{\pi}{8}3l^2.\frac{dl}{dt}$ => $-2 = \frac{\pi}{8}3(4)^2 \times \frac{dl}{dt}$

$\frac{dl}{dt} = \frac{-1}{3\pi}$ cm / sec.

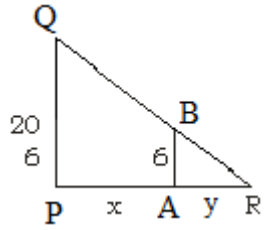
The slant height decreases at the rate of $\frac{1}{3\pi}$ cm / sec .

3. A man 6 ft height walks at a uniform rate of 4 miles per hour away from a lamp 20 feet high. Find the rate at which the length of his shadow increases. (1 mile = 5280 ft).

Sol: let PQ be the lamp post and AB be the position of the man.

BR is his shadow.

Suppose PA = x, AR = y



From figure, ΔPQR, ΔABR are similar,

Therefore, $\frac{PQ}{AB}=\frac{PR}{AR}$ => $\frac{20}{6}=\frac{x+Y}{Y}=\frac{x}{y}+1 => \frac{x}{y}=\frac{10}{3}-1=\frac{7}{3}$

$y=\frac{3}{7}.x =>$ $\frac{dy}{dt}=\frac{3}{7}.\frac{dx}{dt}=\frac{3}{7}\times 4\times\frac{22}{5}=\frac{88}{75}$ ft/sec.

The shadow of the man is increasing at the rate of $\frac{88}{35}$ ft / sec .

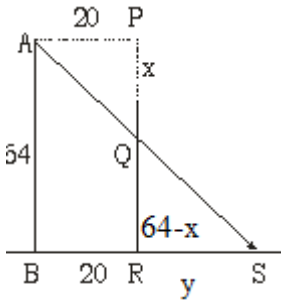
4. A man 180 cm height walks at a uniform rate of 12 km per hour away from a lamp post of 450 cm high. Find the rate which the length of his shadow increases.

Ans: 8 km/hour.

5. A light is at the top of a pole 64 m high. A ball is dropped from the same height 64m from a point 20m from the light. Assuming, that the ball falls according to the law $s = 5t^2$, how fast is the shadow of the ball moving along the ground, 2 second later.

Sol: let A be the light, Q be the ball, S be its shadow.

Let RS = y , PQ = x then QR = 64-x



ΔRQS, VAQP are similar.

$\frac{QR}{PQ}=\frac{RS}{AP}\Rightarrow\frac{64-x}{x}=\frac{y}{20}$

$$\Rightarrow \frac{64}{x}-1=\frac{y}{20}, diff \text{ w.r.t.t}, \Rightarrow -\frac{64}{x^2}\cdot\frac{dx}{dt}=\frac{1}{20}.\frac{dy}{dt}$$

$x = 5t^2$ and t=2. Then x= 5.4 =20 and $\frac{dx}{dt} = 5.2.t = 5.2.2 = 20$

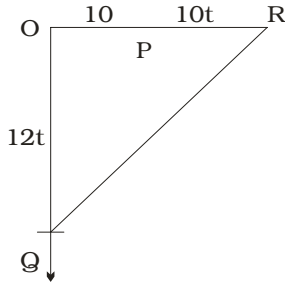
$$\therefore \frac{-64}{20.20}.20=\frac{1}{20}\frac{dy}{dt}\Rightarrow \frac{dy}{dt}=-64$$

$$\frac{dy}{dt} = -64 m/sec.$$

$\therefore$ The shadow of the ball is moving towards the pole at the rate of 64m/sec.

6. A ship starts moving from a point at 10 a.m. and heading due east at 10 km/hour. Another ship starts moving from the same point at 11 a.m. and heading due to south at 12 km/hour. How fast is the distance between the ships increasing at 12 noon?

Sol: Suppose the two ships starts from O and are at R and Q at 12 noon



Let R, Q be the position of the ships at 12 noon.

Let OR = x and OQ = y. then x = 20 and y = 12

$\Rightarrow \frac{dx}{dt}=10, \frac{dy}{dt}=12$. Let RQ = l then $l^2 = 20^2 + 12^2 = 544 \Rightarrow l = \sqrt{544}$

$QR^2 = OR^2 + OQ^2$

$$l^2 = x^2 + y^2 \Rightarrow 2l\frac{dl}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\Rightarrow l\frac{dl}{dt} = 20.10 + 12.12 \Rightarrow \sqrt{544}.\frac{dl}{dt} = 344 \quad \Rightarrow \frac{dl}{dt} = \frac{344}{\sqrt{544.}} = = \frac{86}{\sqrt{34}}$$

The distance between the ships is increasing at the rate of $\frac{86}{\sqrt{34}}$ km / hour.

PROBLEMS FOR PRACTICE

1 A point is moving on $\mathbf{y = 4 - 2x^2}$ The x coordinate of the point is decreasing at the rate of 5 units per second. Find the rate at which y coordinate of the point is changing when the point is at (1, 2).

2. A particle moving along a straight line has the relation $\mathbf{s - t^2 + 2t + 3}$ connecting the distance s described by the particle in time t. Find the velocity and acceleration of the particle at time t = 3 seconds.

3. A balloon is in the shape of an inverted cone surmounted by hemisphere. Diameter of the sphere is equal to the height of the cone. If h is the total height of the balloon, then how does the volume of the balloon changes with? What is the rate of change in volume when h = 9 units?

Sol: Let r be the radius of the sphere, so that h = 3r and height of the cone is 2r.

Volume of the cone $= \frac{1}{3}\pi r^2 (2r) = \frac{2}{3}\pi r^2$

Volume of the hemisphere $= \frac{2}{3}\pi r^3$

$\therefore$ V, the total volume of the balloon is equal to the sum of the two volumes.

$\therefore V = \frac{4}{3}\pi r^3;\ r = \frac{h}{3}$

$\therefore V = \frac{4}{3}\pi\left(\frac{h}{3}\right)^3 = \frac{4}{81}\pi r^3$

Differentiating both sides w.r. to. h,

$\frac{dv}{dh} = \frac{4\pi}{81}.3h^2 = \frac{4}{27}\pi h^2$

When h = 9

$\left(\frac{dV}{dh}\right)_{h=9} = \frac{4}{27}\times\pi\times 81 = 12\pi$

Thus, V changes at the rate of 12π times the rate at which h changes.