SECOND ORDER DERIVATIVES

If y = f(x) is a diffrentiable function of x then its derivative f '(x) is a function of x. If f '(x) is a differentiable function then its derivative is called the second order derivatie of f(x). It is denoted by f ''(x) and $f''(x) = \underset{h\to 0}{Lt} \frac{f'(x+h) - f'(x)}{h}$

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f "(x) is also denoted by $f''(x) = \underset{h\to 0}{Lt} \frac{f'(x+h) - f'(x)}{h}$ or $\frac{d^2 f}{dx^2}$ or D^2y or y" or y_2.

EXERCISE – 9 (d)

I. **If** $y = \frac{2x+3}{4x+5}$ **then find y"**

Sol : $y = \frac{2x+3}{4x+5}$

Differentiating w. r. to x

$$\frac{d}{dx}y = \frac{d}{dx}\frac{2x+3}{4x+5}$$

$$\frac{dy}{dx} = \frac{(4x+5).2 - (2x+3).4}{(4x+5)^2} = \frac{8x+10-8x-12}{(4x+5)^2}$$

$$= \frac{-2}{(4x+5)^2} = 2(4x+5)^{-2}$$

Again diff w.r.t x,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}2(4x+5)^{-2}$$

$$\frac{d^2y}{dx^2} = (-2)(-2)(4x+5)^{-3}.4$$

$$y'' = \frac{16}{(4x+5)^3}$$

2. $y = ae^{nx} + be^{-nx}$ **then prove that** $y'' = n^2 y$

Sol : $y = ae^{nx} + be^{-nx}$

Diff with respect to x

$$y_1 = na\,e^{nx} - nb\,e^{-nx}$$

Diff with respect to x

$$y_2 = n^2.ae^{nx} + n^2\ be^{-nx}$$

$$y_2 = n^2\left(ae^{nx} + b.e^{-nx}\right) = n^2 y$$

II.

1. Find the second order derivatives of the following functions $f(x)$

i) $\cos^3 x$

sol : $y = \cos^3 x = \frac{1}{4}[\cos 3x + 3\cos x]$

diff w.r.t x, we get

$$y_1 = \frac{1}{4}[-3\sin 3x - 3\sin x]$$

Diff w.r.t x, we get

$$y_2 = \frac{1}{4}(-9\cos 3x - 3\cos x)$$

$$= -\frac{1}{4}(\cos x + 3\cos 3x)$$

ii) $y = \sin^4 x$

sol : $y = \sin^4 x = \left(\sin^2 x\right)^2 = \left(\frac{(1-\cos 2x)^2}{2}\right)$

$= \frac{1}{4}\left[1 - 2\cos 2x + \cos^2 2x\right]$

$= \frac{1}{4}\left[1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right]$

$= \frac{1}{8}[2 - 4\cos 2x + 1 + \cos\ 4x]$

$= \frac{1}{8}[3 - 4\cos 2x + \cos 4x]$

Diff w.r.t. x , then

$y_1 = \frac{1}{8}(8\cos 2x - 4\sin 4x)$

$y_2 = \frac{1}{8}(16\cos 2x - 16\ \cos\ 4x) = 2(\cos 2x - \cos 4x)$

iii) $y = \log\left(4x^2 - 9\right)$

$= \log(2x-3)(2x+3) = \log(2x-3) + \log(2x+3)$

Diff with respect to x, then

$y_1 = \frac{2}{2x-3} + \frac{2}{2x+3}$

Again diff w.r.t. x, then

$y_2 = \frac{2(-1)^{-2}}{(2x-3)^2} + \frac{2(-1)^2}{(2x+3)^2}$

$$= -4\frac{\left((2x+3)^2+(2x+3)^2\right)}{\left(4x^2-9\right)^2} = \frac{-4\left(2\left\{4x^2+9\right\}\right)}{\left(4x^2-9\right)^2}$$

$$= -8\frac{\left(4x^2+9\right)}{\left(4x^2-9\right)^2}$$

iv) $y = e^{-2x}.\sin^3 x$

sol : $y = e^{-2x}.\sin^3 x$

Diff with respect to x

$$y' = e^{-2x}\left(3\sin^2 x.\cos x\right) + \sin^3 x\left(e^{-2x}\right)(-2)$$

$$= e^{-2x}\left[3\sin^2 x.\cos x - 2\sin^3 x\right]$$

Diff with respect to x

$$\frac{d^2y}{dx^2} = e^{-2x}[3\sin^2 x(-\sin x) + 3\cos x(2\sin x)\cos x - 6\sin^2 x\cos x) - 2.e^{-2x}$$

$$\left[3\sin^2 x.\cos x - 2\sin^3 x\right]$$

$$= e^{-2x}\left[6\sin x.\cos^2 x - 6\sin^2 x.\cos x - 3\sin^3 x\right] - 6\sin^2 x.\cos x + 4\sin^3 x)$$

$$= e^{-2x}\left[\sin^3 x - 12\sin^2 x.\cos x + 6\sin x.\cos^2 x\right]$$

v) $e^x.\sin x.\cos 2x$

sol : $y = e^x.\sin x.\cos 2x = \frac{e^x}{2}(2\cos 2x.\sin x)$

$$= \frac{e^x}{2}(\sin 3x - \sin x)$$

diff w.r.t. x, then

$$y_1 = \frac{1}{2}\left[e^x(3\cos 3x - \cos x) + e^x(\sin 3x - \sin x)\right]$$

Again diff w.r.t. x, then

$$y_2 = \frac{1}{2}\left[e^x(-9\sin 3x + \sin x) + e^x(3\cos 3x - \cos x)\right] + e^x[3\cos 3x - \cos x] + e^x(\sin 3x - \sin x)$$

$$= \frac{e^x}{2}[-9\sin 3x + \sin x + 3\cos 3x - \cos x + 3\cos 3x - \cos x + \sin x 3x - \sin x]$$

$$= \frac{e^x}{2}[6\cos 3x - 8\sin 3x - 2\cos x]$$

$$= e^x[3\cos 3x - 4\sin 3x - \cos x]$$

vi) $Tan^{-1}\left(\frac{1+x}{1-x}\right)$

sol : $y = \tan^{-1}\left(\frac{1+x}{1-x}\right)$

Put $x = \tan\theta$ **then θ = tan^{-1}x**

$$y = \tan^{-1}\left(\frac{\tan\frac{\pi}{4} + \tan\theta}{1 - \tan\frac{\pi}{4}.\tan\theta}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4} + \theta\right)\right) = \frac{\pi}{4} + \theta$$

$$\therefore\ f(x) = \frac{\pi}{4} + \tan^{-1}(x)$$

Diff. w. r. to x

$$f'(x) = 0 + \frac{1}{1+x^2}$$

Again diff w.r.t. x,

$$f''(x) = (-1)\left(1+x^2\right)^{-2}(2x)$$

$$\therefore\ f''(x) = \frac{-2x}{\left(1+x^2\right)^2}$$

vii) $\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

sol : $f(x) = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

$Put\ x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$$f(x) = \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\right)$$

$$= \tan^{-1}(\tan 3\theta) = 3\theta$$

$$\therefore\ f(x) = 3\tan^{-1}(x)$$

Diff. w. r. to x

$$f'(x) = 3\left(\frac{1}{1+x^2}\right) = \frac{3}{1+x^2}$$

Again Diff. w. r. to x

$$f''(x) = (3)(-1)\left(1+x^2\right)^{-2}(2x) \Rightarrow\ f''(x) - \frac{-6x}{\left(1+x^2\right)^2}$$

II. Prove the following .

1. If $y = ax^{n+1} + bx^{-n}$, **then** $x^2 y'' = n(n+1)y.$

Sol : $y = ax^{n+1} + bx^{-1}$

Diff w.r.t x, $y_1 = (n+1).ax^n - nbx^{-n-1}$

Again diff with respect x,

$y_2 = n(n+1).ax^{n-1} + n(n+1)bx^{-n-2}$

$\therefore x^2 y_2 = n(n+1)ax^{n+1} + n(n+1)bx^{-n}$

$= n(n+1)\left(ax^{n+1} + bx^{-n}\right) = n(n+1)y$

$\therefore x^2 y'' = n(n+1)y$

ii) If $y = a\cos x + (b+2x)\sin x,$ **then show that** $y'' + y = 4\cos x$

iii) If $y = 6(x+1) + (a+bx)e^{3x}$ **then** $y'' = 6y' + 9y = 54x + 18$

iv) If $ay^4 = (a+b)^5$, *then* $5y\ y'' = (y')^2$

sol : $ay^4 = (x+b)^5 \Rightarrow y^4 = \dfrac{(x+b)^5}{a}$

$\Rightarrow y = \dfrac{(x+b)^{5/4}}{a^{1/4}}$

Diff w.r.t x, we get

$y_1 = \dfrac{1}{a^{1/4}}.\dfrac{5}{4}(x+b)^{\left(\frac{5}{4}-1\right)} = \dfrac{5}{4a^{1/4}}(x+b)^{1/4}.1$ **Again diff with respect x,**

$$y_2 = \frac{5}{4a^{1/4}}.\frac{1}{4}(x+b)^{\frac{1}{4}-1} = \frac{5}{16a^{1/4}}(x+b)^{-3/4}$$

L.H.S. $= 5y\,y'' = 5.\frac{(x+b)^{5/4}}{a^{1/4}}.\frac{5}{16a^{1/4}}(a+b)^{-3/4}$

$$= \frac{25}{16a^{1/2}}(x+b)^{2/4} = \left[\frac{5}{4a^{1/4}}(x+b)^{1/4}\right] = (y')^2$$

v) **If** $y = a\cos(\sin x) + b\sin(\sin x)$, **then** $y''+(\tan x)y'+y\cos^2 x = 0$ **try yourself.**

III.

1. **i) If** $y = 128\sin^3 x\cos^4 x$, **then find** y''.

Sol : $y = 128\sin^3 x.\cos^4 x$

D.w.r. to x

$$y_1 = 128\left[\sin^3 x\left\{4\cos^3 x(-\sin x)\right\}\right] + \cos^4 x\left\{3\sin^2 x.\cos x\right\}$$

$$= 128\left[3\sin^2 x\cos^5 x - 4\sin^4 x\cos^3 x\right]$$

Again D. w. r to x

$$y_2 = 128\left\{3[\sin^2 x\ 5\cos^4 x.(-\sin x) + \cos^5 x\right\}\ 2\sin x\cos x] - 4[\sin^4 x.3.\cos^2 x(-\sin x)$$

$$+\cos^3 x.4\sin^3 x\cos x]\}$$

$$= 128[-15\sin^3 x.\cos^4 x + 6\sin x\cos^6 x + 12\sin^5 x\cos^2 x - 16\sin^3 x\cos^4 x]$$

$$f''(x) = 128[6\sin x\cos^6 x + 12\sin^5 x.\cos^2 x - 31\sin^3 x.\cos^4 x]$$

ii) **If** $y = \sin 2x \sin 3x \sin 4x$, **then find** y''.

sol : $y = \sin 2x \sin 3x \sin 4x$

$$= \frac{1}{2}\sin 2x[2\sin\ 4x.\ \sin 3x]$$

$$= \frac{1}{2}\sin 2x[\cos x - \cos 7x]$$

$$= \frac{1}{2}[\sin 2x.\cos x - \cos 7x.\ sin\ 2x]$$

$$= \frac{1}{2} \times \frac{1}{2}[2\sin 2x.\cos x - 2\cos 7x.\sin 2x]$$

$$= \frac{1}{4}[(\sin 3x + \sin x) - (\sin 9x - \sin 5x)]$$

$$= \frac{1}{4}[-\sin 9x + \sin 5x + \sin 3x + \sin x]$$

D.w.r to x

$$y_1 = \frac{1}{4}[-9\cos 9x + 5\cos 5x + 3\cos 3x + \cos x]$$

D. w. r. to x

$$y_2 = \frac{1}{4}[81\sin 9x - 25\sin 5x - 9\sin 3x - \sin x]$$

iii) **If** $ax^2 + 2hxy + by^2 = 1$, **then prove that** $\frac{d^2y}{dx^2} = \frac{h^2 - ab}{(hx + by)^3}$

sol : **Given** $ax^2 + 2hxy + by^2 = 1$

Differentiating w. r. to x

$$\frac{d}{dx}\left(ax^2 + 2hxy + by^2\right) = 0$$

$$\Rightarrow a.2x + 2h\left(x.\frac{dy}{dx} + y\right) + b.2y\frac{dy}{dx} = 0$$

$$\Rightarrow 2ax + 2hx.\frac{dy}{dx} + 2hy + 2by.\frac{dy}{dx} = 0$$

$$\Rightarrow 2(hx + by).\frac{by}{dx} = -2(ax + hy)$$

$$\frac{dy}{dx} = \frac{-2(ax + hy)}{2(hx + by)} = -\frac{(ax + hy)}{(hx + by)} \quad .....(1)$$

Differentiating again w. r. to x,

$$\frac{d^2y}{dx^2} = -\frac{d}{dx}\frac{(ax + hy)}{(hx + by)}$$

$$\frac{d^2y}{dx^2} = \frac{\left[(hx + by)\left(a + h\frac{dy}{dx}\right) - (ax + hy)\left(h + b.\frac{by}{dx}\right)\right]}{(hx + by)^2}$$

$$= \frac{(ax + hy)\left[h - b.\frac{(ax + hy)}{hx + by}\right] - (hx + by)\left[\frac{(ax + hy)}{hx + by}\right]}{(hx + by)^2}\,(ax + hy)\left(h^2x + bhy - abx - bhy\right)$$

$$= \frac{-(hx + by)\left(ahx + aby - ahx - h^2y\right)}{(hx + by)^3}$$

$$= \frac{\left(h^2 - ab\right)[x(ax + hy)] + \left(h^2 - ab(y(hx + by))\right)}{(hx + by)^3} = \frac{\left(h^2 - ab\right)[x(ax + hy) + y(hx + by)]}{(hx + by)^3}$$

$$= \frac{\left(h^2 - ab\right)\left[ax^2 - 2hxy + by^2\right]}{(hx + by)^3}$$

$$= \frac{h^2 - ab}{(hx + by)^3} \left[\because\ ax^2 + 2hxy + by^2 = 1 \right]$$

iv) **If** $y = ae^{-bx}\cos(cx + d)$ **then prove that** $y'' + 2by' + \left(b^2 + c^2\right)y = 0$**.**

v) **If** $y = e^{\frac{-k}{2}x}(a\cos nx + b\sin nx)$**. Then prove that** $y'' + ky' + \left(n^2 + \frac{k^2}{4}\right)y = 0$

sol : **Given** $y = e^{\frac{-k}{2}x}(a\cos nx + b\sin nx)$ .....(1)

diff w.rt. x, we get,

$$y_1 = e^{\frac{-k}{2}x}(-n.a\sin nx + n.b\cos nx) + (a\cos nx + b\sin nx).e^{\frac{-k}{2}x}\left(-\frac{k}{2}\right)$$

$$= -\frac{k}{2}.y - n.e^{-kx/2}(a\sin nx + b\cos nx)$$

$$\therefore\ y_1 + \frac{k}{2}y = -n\,e^{-kx/2}(a\sin nx + b\cos nx) \quad .....(2)$$

Differentiating w. r. to x

$$y_2 + \frac{k}{2}y_1 = -n\left[\left\{e^{-kx/2}(-na\cos nx - nb\sin nx)\right\}\right] + \left\{(a\sin nx + b\cos nx).e^{-kx/2}\left(-\frac{k}{2}\right)\right\}$$

$$= -n^2 - e^{-kx/2}(a\cos nx + b\sin nx) - \frac{k}{2}\left\{-n.e^{-kx/2}(a\sin nx + b\cos nx)\right\}$$

$$= -n^2.y - \frac{k}{2}\left[y_1 + \frac{k}{2}y\right] by\ (1),(2)$$

$$= -n^2 y - \frac{k}{2}y_1 - \frac{k^2}{4}y$$

$$= -\frac{k}{2}y_1 - \left(n^2 + \frac{k^2}{4}\right)y$$

$$\therefore\ y_2 + k.y_1 + \left(n^2 + \frac{k^2}{4}\right)y = 0$$

PROBLEMS FOR PRACTICE

1. **If** $f(x) = x^2\,(x \in R),$ **Prove that f is differentiable on R and find its derivative.**

2. **Suppose** $f(x) = \sqrt{x}\,(x > 0)$**, prove that f is differentiable on** $(0,\infty)$ **and find** $f'(x)$**.**

3. **If** $f(x) = \frac{1}{x^2+1}(x \in R)$**, prove that f is differentiable and find** $f'(x)$

4. **If** $f(x) = \sin x\,(x \in R),$ **then show that f is differentiable on R and** $f'(x) = \cos x$

5. **Show that** $f(x) = |x|\,(x \in R)$ **is not differentiable at zero and is differentiable at any** $x \neq 0$

6. **Check whether the following function is differentiable at zero**

$$f(x) = \begin{cases} 3 + x & if \quad x \geq 0 \\ 3 - x & if \quad x < 0 \end{cases}$$

Sol : Suppose $h > 0$ and $\frac{f(0+h) - f(0)}{h} = \frac{f(h) - 3}{h}$

$$= \frac{3 + h - 3}{h} = \frac{h}{h} = 1$$

$$\underset{h \to 0}{Lt} \frac{f(0+h) - f(0)}{h} = 1$$

f has the right hand derivative zero and f $'\left(0^+\right)$

$$\underset{h\to 0}{Lt}\frac{f(0+h)-f(0)}{h}=\underset{h\to x}{Lt}=\frac{3-h-3}{h}=\underset{h\to 0}{Lt}-\frac{h}{h}=-1$$

F has the left hand derivative at zero and $f'\left(0^{-}\right)=-1$

$\therefore\ f'\left(0^{+}\right)\neq f'\left(0^{-}\right)$

$f(x)$ is not differentiable at zero

7. Show that the derivative of a constant function on an interval is zero.

8. Suppose for all $x, y\in R\ f(x+y)=f(x).\ f(y)$ *and* $f'(0)$ **exists. Then show that** $f'(x)$ **exists and equal to** $f(x)f'(0)$ **for all** $x\in R$.

Sol : Let $x\in R$, *for* $h\neq 0$, we have

$$\frac{f(x+h)-f(x)}{h}=\frac{f(x)f(h)-f(x)}{h}$$

$$=f(x)\frac{[f(h)-1]}{h}\ \ .....(1)$$

$f(0)=f(0+0)=f(0)f(0)\Rightarrow f(0)(1-f(0))=0$

$\therefore\ f(0)=0\ f(0)=1$

Case (i) : Suppose $f(0)=0$

$f(x)=f(x+0)=f(x)f(0)=0\ \forall x\in R$

$\therefore\ f(x)$ is a constant function $\Rightarrow f'(x)=0$

For all $x\in R$

$\therefore\ f'(x)=0=f(x).f'(0)$

Case (ii) : Suppose $f(0) = 1$

$$\underset{h\to 0}{Lt}\frac{f(x+h)-f(x)}{h} = \underset{h\to 0}{Lt}\frac{f(x)f(h)-f(x)}{h} = f(x)\underset{h\to 0}{Lt}\frac{f(h)-1}{h}$$

$$f(x)\underset{h\to 0}{Lt}\frac{f(h)-f(0)}{h} \ (\because f(0)=0)$$

$$= f(x)f'(0)$$

$\therefore f$ is differential and $f'(x) = f(x)f'(0)$

9. **If** $f(x) = (ax+b)^n\left(x > -\frac{b}{a}\right)$, **then find** $f'(x)$

10. **Find the derivative of** $f(x) = ex\left(x^2+1\right)$,

11. **If** $y = \frac{a-x}{a+x}(x \neq -a)$, **find** $\frac{dy}{dx}$

12. **If** $f(x) = e^{2x}.\log x(x>0)$, **then find** $f'(x)$

13. **If** $f(x) = \frac{\sqrt{1+x^2}}{\sqrt{1-x^2}}(|x|<1)$, **then find** $f'(x)$

14. **If** $f(x) = x^2, 2^x \log x(x>0)$, **find** $f'(x)$.

15. **if** $y = \begin{vmatrix} f(x) & g(x) \\ \varphi(x) & \psi(x) \end{vmatrix}$ **then show that** $\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) \\ \varphi(x) & \psi(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) \\ \varphi'(x) & \psi'(x) \end{vmatrix}$

Sol : **Given is** $= \begin{vmatrix} f(x) & g(x) \\ \varphi(x) & \psi(x) \end{vmatrix} = f(x)\ \psi'(x) + \psi(x)g(x)$

$$\frac{dy}{dx} = f(x)\psi'(x) + \psi(x)f'(x) - [\varphi(x).g'(x) + g(x).\varphi'(x)]$$

$$= [f(x)\psi'(x) - g(x)\varphi'(x)] + [f'(x)\psi(x) - \psi(x) - \varphi(x).g'(x)]$$

$$= \begin{vmatrix} f(x) & g(x) \\ \varphi'(x) & \psi'(x) \end{vmatrix} + \begin{vmatrix} f'(x) & g'(x) \\ \varphi(x) & \psi(x) \end{vmatrix}$$

16. **If** $f(x) = 7^{x^3+3x}\,(x > 0)$, **then find** $f'(x)$.

17. **If** $f(x) = x.e^x \sin x$, **then find** $f'(x)$.

18. **If** $f(x) = \sin(\log x)(x > 0)$, **find** $f'(x)$.

19. **If** $f(x) = \left(x^3 + 6x^2 + 12x - 13\right)^{100}$; **find** $f'(x)$.

20. **Find the derivative of** $f(x) = \dfrac{x\cos x}{\sqrt{1+x^2}}$.

21. **If** $f(x) = \log(\sec x + \tan x)$ **find** $f'(x)$

22. **If** $y = \sin^{-1}\sqrt{x}$, **find** $\dfrac{dy}{dx}$.

23. **If** $y = \sec\left(\sqrt{\tan x}\right)$, **find** $\dfrac{dy}{dx}$

24. **If** $y = \dfrac{x\sin^{-1}x}{\sqrt{1+x^2}}$, **find** $\dfrac{dy}{dx}$

25. **If** $y = \log(\cos h\, 2x)$, **find** $\dfrac{dy}{dx}$.

26. **If** $y = \log(\sin(\log x))$, **find** $\dfrac{dy}{dx}$.

27. **If** $y = \left(\cot^{-1}x^3\right)^2$, **find** $\dfrac{dy}{dx}$.

28. **If** $y = \operatorname{cosec}^{-1}\left(e^{2x+1}\right)$ **find** $\dfrac{dy}{dx}$.

29. **If** $y = \tan^{-1}\left(\cos\sqrt{x}\right)$, **find** $\frac{dy}{dx}$

Sol : $v = \sqrt{x}$ and $u = \cos v, y = \tan^{-1} u$

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}}, \frac{du}{dv} = -\sin u \frac{dy}{du} = \frac{1}{1+u^2}$$

$$= -\sin\sqrt{x} = \frac{1}{1+\cos^2\left(\sqrt{x}\right)}$$

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dv}.\frac{du}{dx}$$

$$= \frac{1}{1+\cos^2\left(\sqrt{x}\right)} - \sin\sqrt{x}\frac{1}{2\sqrt{x}}$$

$$= \frac{-\sin\sqrt{x}}{2\sqrt{x}\left[1+\cos^2\left(\sqrt{x}\right)\right]}$$

30. **If** $y = Tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1+x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$ **for** $0 < |x| < 1$, **find** $\frac{dy}{dx}$**.**

31. **If** $y = x^2.e^x \sin x$, **find** $\frac{dy}{dx}$**.**

32. **If** $y = x^{\tan x} + (\sin x)^{\cos x}$, **find** $\frac{dy}{dx}$

Sol : **Let** $u = x^{\tan x}$ *and* $v = (\sin x)^{\cos x}$

$$\log u = \log x^{\tan x} = (\tan x)\log x$$

$$\frac{1}{u}.\frac{dy}{dx} = \tan x\frac{1}{x} + (\log x)\sec^2 x.$$

$$\frac{du}{dx} = u\left(\frac{\tan x}{x} + (\log x).\sec^2 x\right)$$

$$= x^{\tan x}\left(\frac{\tan x}{x} + (\log x).\sec^2 x\right)$$

$$\log v = \log(\sin x)\cos x] = \cos x.\log\sin x$$

$$\frac{1}{v}.\frac{dv}{dx} = \cos x.\frac{1}{\sin x}\cos x + (\log\sin x)(-\sin x)$$

$$= \frac{\cos^2 x}{\sin x} - \sin x\log\left(\sin^x\right)^{\cos x}$$

$$\frac{dv}{dx} = v\left(\frac{\cos^2 x}{\sin x} - \sin x\log\sin x\right)$$

$$= (\sin x)^{\cos x}\left(\frac{\cos^2}{\sin x} - \sin x\log(\sin x)\right)$$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} = x^{\tan x}$$

$$\left(\frac{\tan x}{x} + (\log x)\left(\sec^2 x\right)\right) + (\sin x)^{\cos x}\left(\frac{\cos^2 x}{\sin x} - \sin x.\log(\sin x)\right)$$

33. **If** $x = a\left(\cos t + \log\tan\left(\frac{t}{2}\right)\right), y = a\sin t,$ **find** $\frac{dy}{dx}$

34. **If** $x^y = e^{x-y}$, **then show that** $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$

Sol : $x^y = e^{x-y} \Rightarrow \log x^y = \log e^{x-y}$

$$\Rightarrow y\log x = x - y(\log e = 1)$$

$$y(1+\log x) = x,\ y = \frac{x}{1+\log x}\ \mathbf{c}$$

$$\frac{dy}{dx} = \frac{(1+\log x).1 - x.\frac{1}{x}}{(1+\log x)^2}$$

$$= \frac{1+\log x - 1}{(1+\log x)^2} = \frac{\log x}{(1+\log x)^2}$$

35. **If** $\sin y = x\sin(a+y)$, **then show that** $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$ **(a is not a multiple of** π **)**

Sol : $x = \frac{\sin y}{\sin(a+y)}$

$$\frac{dy}{dx} = \frac{\sin(a+y)\cos y - \sin y\cos(a+y)}{\sin^2(a+y)}.\frac{dy}{dx}$$

$$= \frac{\sin(a+y-y)}{\sin^2(a+y)}\frac{\sin a}{\sin^2(a+y)}$$

$$\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} = \frac{\sin^2(a+y)}{\sin a}$$

36. **If** $y = x^4 + \tan x,$ **then find y".**

37. **If** $f(x) = \sin x, \sin 2x, \sin 3x$ **find** $f''(x)$**.**

Sol :

$$f(x) = \frac{1}{2}\sin 2x(2\sin 3x\sin x) = \frac{1}{2}(\sin 2x)(\cos 2x - \cos 4x)$$

$$= \frac{1}{4}(2\sin 2x\cos 2x - 2\sin 2x\cos 4x)$$

$$= \frac{1}{4}(\sin 2x + \sin 4x - \sin 6x)$$

Therefore,

$$f'(x) = \frac{1}{4}[2\cos 2x + 4\cos 4x - 6\cos 6x]$$

Hence,

$$f''(x) = \frac{1}{4}(-4\sin 2x - 16\sin 4x + 36\sin 6x) = 9\sin 6x - 4\sin 4x - \sin 2x.$$

38. **Show that** $y = x + \tan x$ **satisfies** $\cos^2 x\frac{dy^2}{dx^2} + 2x = 2y.$

39. **If** $x = a(t - \sin t), y = a(1 + \cos t),$ **find** $\frac{d^2y}{dx^2}.$

40. **Find the second order derivative of** $y = \tan^{-1}\left(\frac{2x}{1 - x^2}\right)$

41. **If** $y = \sin(\sin x),$ **show that** $y'' + (\tan x)y' + y\cos^2 x = 0.$

$$y'' + ky^1 + \left(n^2 + \frac{k^2}{4}\right)y = 0$$