CHAPTER 9
DIFFERENTIATION

TOPICS

1.Derivative of a function
2. Derivative of sum/difference of two or more functions
3. Product Rule.
4. Quotient Rule.
5. The derivative of a composite function and chain rule.
6.The derivatives of algebraic functions
7. Derivative of inverse function.
8. Differentiation from the first principle.
9.The derivatives of trigonometric functions
10.Logorithmic differentiation
11.Implicit differentiation
12. Substitution method.
13.parametric differentiation
14.Derivative of a function w.r.t another function
15. Second order derivatives.

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DERIVATIVE OF A FUNCTION

Let f be a function defined in a nbd of a point a. If $\underset{x\to a}{Lt}\frac{f(x)-f(a)}{x-a}$ exists finitely, then f is said to be differentiable or derivable at a. In this case, the limit is called the derivative or differential coefficient of f. It is denoted by f '(a). f '(a) = $\underset{x\to a}{Lt}\frac{f(x)-f(a)}{x-a}$.

Note: In the above definition, by the substitution x – a=h, we get another equivalent definition for f '(a). i.e., f '(a) = $\underset{x\to a}{Lt}\frac{f(x)-f(a)}{x-a}$

Left hand derivative:- Let f be a function defined in a nbd of a point a. If $\underset{x\to a-}{Lt}\frac{f(x)-f(a)}{x-a}$ exists finitely then f is said to be differentiable from the left at a. This limit is called the left hand derivative (LHD) or left derivative at a . It is denoted by L f '(a) or f '(a–) or f '(a–0).

L f '(a) f '(a–) = $\underset{x\to a-}{Lt}\frac{f(x)-f(a)}{x-a}$

Right hand derivative:- Let f be a function defined in a nbd of a point a. If $\underset{x\to a+}{Lt}\frac{f(x)-f(a)}{x-a}$ exists finitely then f is said to be differentiable from the right at a. This limit is called the right hand derivative (RHD) or right derivative at a . It is denoted by R f '(a) or f '(a+) or f '(a + 0).

R f '(a) f '(a+) = $\underset{x\to a+}{Lt}\frac{f(x)-f(a)}{x-a}$

Note: f is derivable at a iff f is both left and right derivable at a and f '(a+) = f '(a–)

Derivability of a function in (a,b):

Let f(x) be a function defined on [a,b]. If

f is derivable for all c ∈ (a,b) ,then we say that f is derivable on (a,b).

Derivability of a function in [a,b]:

A function f, defined on [a, b] is said to be differentiable on [a, b] if f is differentiable on (a, b), f is right differentiable at a and f is left differentiable at b.

Note: the process of finding derivative of a function using the definition is called derivative from the first principles.

2. Every differentiable function is continuous but every continuous function is not differentiable.

THEOREM

If a function f is a differentiable at a then it is continuous at a.

Proof:

f is a differentiable at a

=> f '(a) = $\underset{x\to a}{Lt}\ \frac{f(x)-f(a)}{x-a}$

$\underset{x\to a}{Lt}\ (f(x)-f(a)) = \underset{x\to a}{Lt}\ \frac{f(x)-f(a)}{x-a}$(x–a)$(\because x \neq a)$

= $\underset{x\to a}{Lt}\left(\frac{f(x)-f(a)}{x-a}\right)\underset{x\to a}{Lt}\ (x-a)$

= f ' (a) .0 = 0

=> $\underset{x\to a}{Lt}\ f(x) = f(a)$

$\because$ f is continuous at a.

The converse of the above theorem need not be true. That is, if a function f is continuous at a then it need not be differentiable at a.

e.g., The function f(x) = | x | is continuous but not differentiable at x = 0. For,

$\underset{x\to 0}{Lt}\ f(x) = \underset{x\to 0}{Lt}\ |x| = 0 = f(0)$

f(x) is continuous at x = 0

Now, L f '(0) = $\underset{x\to 0-}{Lt}\ \frac{f(x)-f(0)}{x-0} = \underset{x\to 0-}{Lt}\ \frac{|x|-0}{x-0} = \underset{x\to 0-}{Lt}\ \frac{-x}{x}$ =-1

R f '(0) = $\underset{x\to 0+}{Lt}\ \frac{f(x)-f(0)}{x-0} = \underset{x\to 0+}{Lt}\ \frac{|x|-0}{x-0} = \underset{x\to 0+}{Lt}\ \frac{+x}{x}$ =1

$\because$ L f '(0)= –1 1 = R f '(0)

$\because$ f(x) is not differentiable at x = 0.

DERIVATIVES OF SOME REAL FUNCTIONS FROM IST PRINCIPLE

1. DERIVATIVE OF CONSTANT

A constant function is differentiable on R and its derivative is equal to zero for all x$\in$ R.

Proof:

Let f (x) = c (c is constant),for all x

f '(x) = $\underset{h\to 0}{Lt}\ \frac{f(x+h)-f(x)}{h} = \underset{h\to 0}{Lt}\ \frac{c-c}{h}$ =0

$\therefore$ f (x) = c is differentiable and is zero.

Hence the derivative of a constant function is 0.

2. If f(x) = x ,x ∈ R then f '(x) = 1

Proof:

$$f'(x) = \underset{h\to 0}{Lt}\frac{f(x+h)-f(x)}{h} = \underset{h\to 0}{Lt}\frac{x+h-x}{h} = \underset{h\to 0}{Lt}\frac{h}{h} = 1$$

3. If f(x) = x^n, x∈ R then f '(x) = nx^{n-1}

Proof

$$f'(x) = \underset{h\to 0}{Lt}\frac{f(x+h)-f(x)}{h} = \underset{h\to 0}{Lt}\frac{(x+h)^n - x^n}{h}$$

$$= \underset{h\to 0}{Lt}\frac{(x+h)^n - x^n}{(x+h)-x} = n \cdot x^{n-1}$$

4. If f(x)= e^x ,x∈ Rthen f '(x) = e^x

$$f'(x) = \underset{h\to 0}{Lt}\frac{f(x+h)-f(x)}{h} = \underset{h\to 0}{Lt}\frac{e^{x+h}-e^x}{h}$$

$$= e^x \underset{h\to 0}{Lt}\left(\frac{e^h-1}{h}\right) == e^x .1 = e^x$$

5. If f(x) = a^x (a > 0, a≠ 1) x∈ R then

f '(x) = a^x.ln a.

i.e., $\frac{d}{dx}(a^x) = a^x \ln a,\ (x\in R, a>0, a\neq 1)$

6. If f(x) = ln x ll x > 0 then

$$f'(x) = \frac{1}{x}$$

$$f'(x) = \underset{h\to 0}{Lt}\frac{f(x+h)-f(x)}{h}$$

$$= \underset{h\to 0}{Lt}\frac{\ln(x+h)-\ln(x)}{h} = \underset{h\to 0}{Lt}\frac{1}{h}\ln\left(\frac{x+h}{x}\right)$$

$$= \underset{h\to 0}{Lt}\ell n\left(1+\frac{h}{x}\right)^{\frac{1}{h}} = \left[\underset{h\to 0}{Lt}\left(1+\frac{h}{x}\right)^{\frac{x}{h}}\right]^{\frac{1}{x}}$$

$= \ln (e^{1/x}) = \dfrac{\mathbf{1}}{\boldsymbol{x}}$

7. If f(x) = sin x x ∈ R, then f '(x) = cos x.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0} \frac{2\cos\left[x+\frac{h}{2}\right]\sin\left(\frac{h}{2}\right)}{h}$$

$$= \lim_{h\to 0} \cos\left(x+\frac{h}{2}\right).\lim_{h\to 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

$= \cos (x + 0) .1 = \cos x.$

Therefore, $\frac{d}{dx}(\sin x) = \cos x$

8. If f(x) = cos x x ∈ R,then f '(x) = –sin x.

$$\mathbf{f'(x)} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\cos(x+h)-\cos x}{h}$$

$$= \lim_{h\to 0} \frac{-2\sin\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}$$

$$= -\lim_{h\to 0} \sin\left(x+\frac{h}{2}\right) \lim_{h\to 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

$= – \sin (x + 0) . 1= = – \sin x.$

$\frac{d}{dx}(\cos x) =– \sin x$

9. If f(x) = tan x x ∈ R –{(2n + 1), nZ} then f '(x) = $\sec^2 x$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\tan(x+h)-\tan x}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{\sin(x+h-x)}{\cos x.\cos(x+h)}\right)$$

$$= \lim_{h\to 0} \frac{\sinh}{h} \lim_{h\to 0} \frac{1}{\cos x.\cos(x+h)}$$

$$=1.\ \frac{1}{\cos^2 x} = \sec^2 x.$$

Therefore, $\frac{d}{dx}(\tan x) = \sec^2 x.$

10.f(x) = cot x is differentiable on R –{n, nZ} and f '(x) = – cosec^2 x.

i.e., $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x.$

11. If f(x) = sec x x ∈ R – {(2n + 1) , nZ} then f '(x) = secx. tanx.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sec(x+h)-\sec x}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{\cos(x+h)}-\frac{1}{\cos x}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{\cos x-\cos(x+h)}{\cos x.\cos(x+h)}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{2.\sin\left(x+\frac{h}{2}\right).\sin\left(\frac{h}{2}\right)}{\cos x\cos(x+h)}\right]$$

$$= \lim_{h\to 0} \frac{1}{\cos x.\cos(x+h)} \lim_{h\to 0} \sin\left(x+\frac{h}{2}\right) \lim_{h\to 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

$= \frac{1}{\cos^2 x}$ sin x. 1 = sec x. tan x.

$\frac{d}{dx}(\sec x) = \sec x.\tan x$

12. If f(x) = cosec x then f '(x) = – cosec x. cot x.

i.e., $\frac{d}{dx}(\operatorname{cosec} x) = -\cos ecx.\cot x$

Theorem

THE DERIVATIVE OF SUM AND DIFFERENCE OF TWO FUNCTIONS

If f and g are two differentiable functions at x, then f + g is differentiable at x and (f + g)' (x) = f '(x) + g'(x).

Proof:

Since f and g are differentiable at x,therefore

f '(x) = $\underset{h\to 0}{Lt}\frac{f(x+h)-f(x)}{h}$

and $\underset{h\to 0}{Lt}\frac{g(x+h)-g(x)}{h}$ = g '(x)

Let = f + g in a sufficiently small neighborhood of x. Then

$\underset{h\to 0}{Lt}\frac{\phi(x+h)-\phi(x)}{h} = \underset{h\to 0}{Lt}\frac{(f+g)(x+h)-(f+g)(x)}{h}$

$= \underset{h\to 0}{Lt}\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h}$

$= \underset{h\to 0}{Lt}\left[\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right]$

$= \underset{h\to 0}{Lt}\frac{f(x+h)-f(x)}{h}+\underset{h\to 0}{Lt}\frac{g(x+h)-g(x)}{h}$

= f '(x) + g'(x) which exists and is finite is differentiable at x and

$\phi'(x) = \underset{h\to 0}{Lt}\frac{\phi(x+h)-\phi(x)}{h}$

Hence φ = f + g is differentiable at x and (f + g)' (x) = f '(x) + g'(x).

Similarly,

If f and g are two differentiable functions of x then f – g is differentiable and

(f – g)' (x) = f '(x) – g'(x).

Note : 1.If u and v are two differentiable functions of x then u + v is a differentiable and

$$\frac{d}{dx}(u+v) = \frac{du}{dx}+\frac{dv}{dx}$$

2. $\frac{d}{dx}(u_1+u_2+...+u_n)=\frac{du_1}{dx}+\frac{du_2}{dx}+....+\frac{du_n}{dx}$

THEOREM

If f is a differentiable function at x and k is a constant then kf is differentiable and (kf)'(x) = k.f '(x)

Proof:

Since f is differentiable at x,therefore,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{(kf)(x+h)-(kf)(x)}{h}$$

$$= \lim_{h\to 0}\frac{kf(x+h)-kf(x)}{h}$$

$$= k\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = k.f'(x)$$

THEOREM

If f and g are two differentiable functions at x then the product function f.g is differentiable at x and (fg)' (x) = f '(x). g(x) + f(x) . g'(x)

Proof:

Since f and g are differentiable at x,therefore

f '(x) and g'(x) exist and

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

and $\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = g'(x)$

$$\lim_{h\to 0}\frac{(fg)(x+h)-(fg)(x)}{h} =$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}[f(x+h).g(x+h)-f(x)g(x+h)] + \lim_{h\to 0}\frac{1}{h}[f(x)g(x+h)-f(x)g(x)]$$

$$= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]\lim_{h\to 0}g(x+h) + f(x).\lim_{h\to 0}\left[\frac{g(x+h)-g(h)}{h}\right]$$

$\therefore$ (fg)'(x)= f '(x) g(x) + f(x) g'(x)

Note :

Above formula can be taken as follows which is known as Product rule or uv rule :

If u and v are two differentiable function of x then their product uv is a differentiable function of x and $\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{du}{dx}$

Quotient Rule

THEOREM

If f and g are differentiable functions at x and g(x)≠0 then the quotient function $\frac{f}{g}$ is differentiable at x and $\left(\frac{f}{g}\right)'(x) = \frac{g(x).f'(x) - f(x)g'(x)}{[g(x)]^2}$

Proof:

Since f and g are differentiable at x,therefore

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

and $\lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = g'(x)$

$$\left(\frac{f}{g}\right)^1 = \lim_{h\to 0} \frac{(f/g)(x+h)-(f/g)(x)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{f(x+h)g(x) - g(x+h)f(x)}{g(x)\,g(x+h)}\right]$$

$$= \cdot \lim_{h\to 0} \frac{1}{g(x).g(x+h)}\left[\frac{\begin{array}{r} f(x+h)g(x) - f(x)g(x) \\ + f(x)g(x) - f(x)g(x+h) \end{array}}{h}\right]$$

$$= \frac{1}{[g(x)]^2}\cdot\left\{g(x)\lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right] - f(x)\lim_{h\to 0}\left[\frac{g(x+h)-g(x)}{h}\right]\right\}$$

$$= \frac{1}{[g(x)]^2}\left\{g(x)\lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right] - f(x)\lim_{h\to 0}\left[\frac{g(x+h)-g(x)}{h}\right]\right\}$$

$$= \frac{1}{[g(x)]^2}\{g(x)f'(x) - f(x)g'(x)\}$$

$$\therefore \left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Above formula can be taken as follows.

If u and v are two differentiable functions of x then $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

which is known as $\frac{u}{v}$ rule .

Derivative of the reciprocal of a function :

If f(x) is a differentiable function at x and

f(x)≠ 0 then $\frac{1}{f}$ is differentiable at x and $\left(\frac{1}{f}\right)'(x)=\frac{-f(x)}{[f(x)]^2}$

DERIVATIVE OF A COMPOSITE FUNCTION

If f is a differentiable function at x and g is a differentiable function at f(x) then gof is differentiable at x and $(gof)'(x)=g'(f(x))\,f'(x)$.

This is also known as chain rule. This rule can be further extended.

For example, if y is a function of u and u is a function x then y will be a function of x and therefore, $\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}$

DERIVATIVE OF THE INVERSE OF A FUNCTION

Let f : [a, b]→ [c, d] be a bijection and g be the inverse function of f. If f is differentiable at x ∈ (a, b),$f^1(x)\neq 0$ and g is continuous at f(x) then g is differentiable at f(x) and

$$g'(f(x))=\frac{1}{f'(x)}.$$

Note : If y =f(x) ⇔ x = f^{-1}(y) then

$$\frac{dx}{dy}=(f^{-1})'(y)=(f^{-1})'(f(x))=\frac{1}{f'(x)}=\frac{1}{\left(\frac{dy}{dx}\right)}$$

$$\therefore \frac{dx}{dy}=\frac{1}{\left(\frac{dy}{dx}\right)} and \frac{dy}{dx}=\frac{1}{\left(\frac{dx}{dy}\right)}$$

DIFFERENTIATION OF DETERMINANTS

THEOREM:

If $y = \begin{vmatrix} f(x) & g(x) \\ \phi(x) & \psi(x) \end{vmatrix}$ then $\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) \\ \phi(x) & \psi(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) \\ \phi'(x) & \psi'(x) \end{vmatrix}$

Proof: Given that $y = \begin{vmatrix} f(x) & g(x) \\ \phi(x) & \psi(x) \end{vmatrix}$

$\Rightarrow y = f(x)\psi(x) - g(x)\phi(x)$

$\therefore \frac{dy}{dx} = \frac{d}{dx}[f(x)\psi(x) - g(x)\phi(x)]$

$= [f'(x)\psi(x) + f(x)\psi'(x)] - [g'(x).\psi(x) + g(x)\psi'(x)]$

$= [f'(x)\psi(x) - g'(x)\phi(x)] + [f(x)\psi'(x) - g(x)\phi'(x)]$

$= \begin{vmatrix} f'(x) & g'(x) \\ \phi(x) & \psi(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) \\ \phi'(x) & \psi'(x) \end{vmatrix}$

EXERCISE 9(A)

1. Find the derivatives of the following functions.

i) $y = \left(4 + x^2\right).e^{2x}$

Sol : $y = \left(4 + x^2\right).e^{2x}$

Differentiating w.r. to x

$$\frac{dy}{dx} = \left(4 + x^2\right)\frac{d}{dx}\left(e^{2x}\right) + e^{2x}\frac{d}{dx}\left(4 + x^2\right)$$

$$= \left(4 + x^2\right).2e^{2x} + e^{2x}(0 + 2x)$$

$$= 2e^{2x}\left[4 + x^2 + x\right] = 2e^{2x}\left(x^2 + x + 4\right)$$

ii) $y = \left(\sqrt{x}+1\right)\left(x^2 - 4x + 2\right)(x > 0)$

Sol: $y = \left(\sqrt{x}+1\right)\left(x^2 - 4x + 2\right)(x > 0)$

Differentiating w.r. to x

$$\frac{dy}{dx} = \left(\sqrt{x}+1\right)\frac{d}{dx}\left(x^2 - 4x + 2\right) +$$

$$\left(x^2 - 4x + 2\right)\frac{d}{dx}\left(\sqrt{x}+1\right) = \left(\sqrt{x}+1\right)(2x-4) + \frac{x^2 - 4x + 2}{2\sqrt{x}}$$

iii) $y = a^x.e^{x^2}$

sol : $y = a^x.e^{x^2}$ Differentiating w.r. to x

$$\frac{dy}{dx} = \left(a^x\right)\frac{d}{dx}\left(e^{x^2}\right) + \left(e^{x^2}\right)\frac{d}{dx}\left(a^x\right) = a^x.e^{x^2}.2x + e^{x^2}.a^x.\log a$$

$$= a^x.e^{x^2}(2x + \log a) = y(2x + \log a)$$

iv) $y = \frac{a\,x + b}{c\,x + d}[|c| + |d| \neq 0]$

sol : $y = \frac{a\,x + b}{c\,x + d}[|c| + |d| \neq 0]$

Differentiating w .r. to x

$$\frac{dy}{dx} = \frac{(c\,x + d)\frac{d}{dx}(a\,x + b) - (ax + b)\frac{d}{dx}(c\,x + d)}{(c\,x + d)^2} = \frac{(c\,x + d)a - (a\,x + b).c}{(c\,x + d)^2}$$

$$= \frac{a\,cx + ad - a\,cx - bc}{(c\,x + d)^2} = \frac{ad - bc}{(cx + d)^2}$$

v) $y = e^{2x}.\log(3x+4)\left(x > \frac{-4}{3}\right)$

sol : $y = e^{2x}.\log(3x+4)\left(x > \frac{-4}{3}\right)$ Differentiating w.r. to x

$$\frac{dy}{dx} = e^{2x}\frac{d}{dx}\left[\log(3x+4) + \log(3x+4)\left(e^{2x}\right)\right] = e^{2x}.\frac{1}{3x+4}.3 + \log(3x+4).e^{2x}.2$$

$$= e^{2x}\left(\frac{3}{3x+4} + 2\log(3x+4)\right)$$

vi) $y = \sqrt{x} + 2x^{\frac{3}{4}} + 3x^{\frac{5}{6}}\,(x > 0)$

sol : $y = \sqrt{x} + 2x^{\frac{3}{4}} + 3x^{\frac{5}{6}}\,(x > 0)$

$$\frac{dy}{dx} = \frac{1}{2}.x^{-1/2} + 2.\ \frac{3}{4}.\ x^{-1/4} + 3.\ \frac{5}{6}\ .\ x^{-1/6}$$

$$= \frac{1}{2}\left[x^{-1/2} + 3.x^{-1/4} + 5.\ x^{-1/6}\right]$$

vii) $y = \sqrt{2x-3} + \sqrt{7-3x}$

sol : $y = \sqrt{2x-3} + \sqrt{7-3x}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{2x-3}}.2 + \frac{1}{2\sqrt{7-3x}}(-3)$$

$$= \frac{1}{\sqrt{2x-3}} - \frac{3}{2\sqrt{7-3x}}$$

viii) $y = e^{x} + \sin x.\cos x$

sol : $y = e^{x} + \sin x.\cos x$

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^x\right) + \frac{d}{dx}\left(\frac{1}{2}\sin 2x\right)$$

$$= e^x + \frac{1}{2}\cos 2x \quad \text{X } 2$$

$$= e^x + \cos\ 2x$$

ix) $y = \left(x^2 - 3\right)\left(4x^3 + 1\right)$

sol : $y = \left(x^2 - 3\right)\left(4x^3 + 1\right)$

$$\frac{dy}{dx} = \left(x^2 - 3\right)\frac{d}{dx}\left(4x^3 + 1\right) + \left(4x^3 + 1\right)\frac{d}{dx}\left(x^2 - 3\right)$$

$$= \left(x^2 - 3\right)\left(12x^2\right) + \left(4x^3 + 1\right)(2x)$$

$$= 12\,x^4 - 36\,x^2 + 8\,x^4 + 2x\ = 20\,x^4 - 36\ x^2 + 2\,x$$

x) $y = \left(\sqrt{x} - 3x\right)\left(x + \frac{1}{x}\right)$

sol : $y = \left(\sqrt{x} - 3x\right)\left(x + \frac{1}{x}\right)$

$$\frac{dy}{dx} = \left(\sqrt{x} - 3x\right)\left(1 - \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right)\left(\frac{1}{2\sqrt{x}} - 3\right)$$

$$= \sqrt{x} - \frac{1}{x\sqrt{x}} - 3x + \frac{3}{x} + \frac{\sqrt{x}}{2} + \frac{1}{2x\sqrt{x}} - 3x - \frac{3}{x}$$

$$= \frac{3}{2}\sqrt{x} + \frac{1}{x\sqrt{x}} - 6\,x$$

xi) $y = 5\sin x + e^x.\log\ x$

sol : $y = 5\sin x + e^x.\log\ x$

$$\frac{dy}{dx} = 5\cos x + e^x.\frac{d}{dx}(\log x) + \log\ x\frac{d}{dx}\left(e^x\right)$$

$$= 5\cos x + e^x.\frac{1}{x} + (\log x)\left(e^x\right)$$

xii) $y = 5^x + \log x + x^3e^x$

sol : $y = 5^x + \log x + x^3e^x$

$$\frac{dy}{dx} = 5^x + \log 5 + \frac{1}{x} + x^3.e^x + e^x.3x^2$$

$$= 5^x.\log 5 + \frac{1}{x} + x^3e^x + 3x^2e^x$$

xiii) $y = \frac{px^2 + q\,x + r}{a\,x + b}(|a| + |b| \neq 0)$

sol : $y = \frac{px^2 + q\,x + r}{a\,x + b}(|a| + |b| \neq 0)$

$$(ax + b)\frac{d}{dx}\left(p\,x^2 + q\,x + r\right)\frac{dy}{dx} = \frac{-\left(px^2 + qx + r\right)\frac{d}{dx}(a\,x + b)}{(a\,x + b)^2}$$

$$= \frac{(ax + b)(2px + q) - \left(px^2 + qx + r\right).a}{(ax + b)^2}$$

$$= \frac{2apx^2 + 2bpx + aqx + bq - apx^2 - aqx - ar}{(ax + b)^2}$$

$$= \frac{apx^2 + 2bpx + (bq - ar)}{(ax + b)^2}$$

xiv) $y = (ax+b)^n.(cx+d)^m$.

sol : $y = (ax+b)^n.(cx+d)^m$.

$$\frac{dy}{dx} = (ax+b)^n \frac{d}{dx}(cx+d)^m + (cx+d)^m \frac{d}{dx}(ax+b)^n$$

$$= (ax+b)^n\left[m(cx+d)^{m-1}.c\right] + (cx+d)^m\left[n(3x+b)^{n-1}\right]$$

$$= (ax+b)^{n-1}(cx+d)^{m-1}[cm(ax+b) + an(cx+d)] = (ax+b)^n(cx+d)^m\left[\frac{an}{ax+b} + \frac{cm}{cx+d}\right]$$

xv) $y = \frac{1}{ax^2+bx+c}(|a|+|b|+|c| \neq 0)$

sol : $y = \frac{1}{ax^2+bx+c}(|a|+|b|+|c| \neq 0)$

$$\frac{dy}{dx} = \frac{(-1)}{\left(ax^2+bx+c\right)^2}\frac{d}{dx}\left(ax^2+bx+c\right)$$

$$= \frac{-(2ax+b)}{\left(ax^2+bx+c\right)^2}$$

xvi) $y = \log_7(\log x)(x > 0)$

sol : $y = \log_7(\log x) = \frac{\log(\log x)}{\log 7}$

$$\frac{dy}{dx} = \frac{1}{\log_7}.\frac{1}{\log x}.\frac{1}{x}$$

2. **If** $f(x) = 1 + x + x^2 + ... + x^{100}$, **then find** $f'(1)$.

Sol : $f(x) = 1 + x + x^2 + ... + x^{100}$,

$\Rightarrow f'(x) = 1 + 2x + 3x^2 ..... + 100x^{99}$

$\Rightarrow f'(1) = 1 + 2 + 3..... + 100$

$= \frac{100 \times 101}{2} = 5050 \left(\sum x = \frac{x(x+1)}{2} \right)$

3. **if** $f(x) = 2x^2 + 3x - 5,$ **then prove that** $f'(0) + 3f'(-1) = 0$**.**

Sol : $f(x) = 2x^2 + 3x - 5,$

$\Rightarrow f'(x) = 4x + 3$

$\Rightarrow f'(0) = 0 + 3 = 3$

$\Rightarrow f'(-1) = -4 + 3 = -1$

$\Rightarrow f'(0) + 3f'(-1) 3 + 3(-1) = 3 - 3 = 0$

2. **Find the derivatives of the following functions** $f(x)$ **from the first principles.**

i) $f(x) = x^3$

Sol : $f^1(x) = \underset{h \to 0}{Lt} \frac{f(x+h) - f(x)}{h} = \underset{h \to 0}{Lt} \frac{(x+h)^3 - x^3}{h}$

$= \underset{h \to 0}{Lt} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$

$= \underset{h \to 0}{Lt} \frac{h\left(3x^2 + 3xh + h^2\right)}{h}$

$= \underset{h \to 0}{Lt} \left(3x^2 + 3xh + h^2\right) = 3x^2 + 0 + 0 = 3x^2$

ii) $f(x) = x^4 + 4$

sol : $f^1(x) = \underset{h \to 0}{Lt} \frac{f(x+h) - f(x)}{h}$

$$= \underset{h\to 0}{Lt} \frac{\left((x+h)^4+4\right)-\left(x^4+4\right)}{h}$$

$$= \underset{h\to 0}{Lt} \frac{(x+h)^4+4-x^4-4}{h}$$

$$= \underset{h\to 0}{Lt} \frac{h\left[4x^3+6x^2h+4xh^2+h^3\right]}{h} = \underset{h\to 0}{Lt}\, 4x^3+6x^2h+4xh^2+h^3$$

$$= 4x^3+0+0+4x^3$$

iii) $f(x) = ax^2 + bx + c$

Ans: $2ax + b$

iv) $f(x) = \sqrt{x+1}$

sol : $f^1(x) = \underset{h\to 0}{Lt} \frac{f(x+h)-f(x)}{h} = \underset{h\to 0}{Lt} \frac{\sqrt{x+h+1}-\sqrt{x+1}}{h}$

$$= \underset{h\to 0}{Lt} \frac{\left(\sqrt{x+h+1}-\sqrt{x+1}\right)\left(\sqrt{x+h+1}+\sqrt{x+1}\right)}{\sqrt{x+h+1}+\sqrt{x+1}} = \underset{h\to 0}{Lt} \frac{x+h+1-x-1}{\sqrt{x+h+1}+\sqrt{x+1}}$$

$$= \underset{h\to 0}{Lt} \frac{h}{h\left(\sqrt{x+h+1}+\sqrt{x+1}\right)}$$

$$= \underset{h\to 0}{Lt} \frac{1}{\sqrt{x+h+1}+\sqrt{x+1}}$$

$$= \frac{1}{\sqrt{x+1}+\sqrt{x+1}} = \frac{1}{2\sqrt{x+1}}$$

v) $f(x) = \sin 2x$

Ans: $= 2\cos 2x$

vi) $f(x) = \cos ax$

sol : $f^1(x) = \underset{h\to 0}{Lt} \dfrac{f(x+h) - f(x)}{h} = \underset{h\to 0}{Lt} \dfrac{\cos a(x+h) - \cos ax}{h}$

$$= \underset{h\to 0}{Lt} \frac{-2\sin\dfrac{ax+ah+ax}{2}.\sin\dfrac{ax+ah-ax}{2}}{h}$$

$$= \underset{h\to 0}{Lt} \frac{-2\sin\left(ax+\dfrac{ah}{2}\right).\sin\left(\dfrac{ah}{2}\right)}{h}$$

$$\underset{h\to 0}{Lt} -2\sin\left(ax+\frac{ah}{2}\right) \underset{h\to 0}{Lt} \frac{\sin\left(\dfrac{ah}{2}\right)}{h}$$

$$= -2\sin ax.\frac{a}{2} = -a.\sin ax$$

vii) $f(x) = \tan 2x$

sol : $f'(x) = \underset{h\to 0}{Lt} \dfrac{f(x+h) - f(x)}{h} = \underset{h\to 0}{Lt} \dfrac{\tan 2(x+h) - \tan 2x}{h}$

$$= \underset{h\to 0}{Lt} \frac{\dfrac{\sin(2x+2h)}{\cos(2x+2h)} - \dfrac{\sin 2x}{\cos 2x}}{h}$$

$$= \underset{h\to 0}{Lt} \frac{\dfrac{\sin(2x+2h).\cos 2x - \cos(2x+2h).\sin 2x}{\cos(2x+2h).\cos 2x}}{h}$$

$$= \underset{h\to 0}{Lt} \frac{1}{h}\frac{\sin(2x+2h-2x)}{\cos(2x+2h)\cos 2x}$$

$$= \underset{h\to 0}{Lt} \frac{\sin 2h}{h}.\underset{h\to 0}{Lt} \frac{1}{\cos(2x+2h)\cos 2x}$$

$$= 2.\frac{1}{\cos 2x.\cos 2x} = 2\sec^2 2x.$$

viii) $f(x) = \cot x$

sol : $f^1(x) = \underset{h\to 0}{Lt}\frac{f(x+h)-f(x)}{h} = \underset{h\to 0}{Lt}\frac{\cot(x+h)-\cot x}{h}$

$$= \underset{h\to 0}{Lt}\frac{\frac{\cos(x+h)}{\sin(x+h)}-\frac{\cos x}{\sin x}}{h}$$

$$= \underset{h\to 0}{Lt}\frac{\frac{\cos(x+h).\sin x-\sin(x+h).\cos x}{\sin(x+h).\sin x}}{h}$$

$$= \underset{h\to 0}{Lt}\frac{-\sin(x+h-x)}{\sin(x+h).\sin x}\frac{1}{h}$$

$$\underset{h\to 0}{Lt}\frac{-\sin h}{h}\ \underset{h\to 0}{Lt}\frac{1}{\sin(x+h).\sin x}$$

$$= -\frac{1}{\sin x.\sin x} = -\cos ec^2\ x$$

ix) $f(x) = \sec 3x$

sol : $f'(x) = \underset{h\to 0}{Lt}\frac{f(x+h)-f(x)}{h}$

$$= \underset{h\to 0}{Lt}\frac{1}{h}\left(\frac{1}{\cos(3x+3h)}-\frac{1}{\cos 3x}\right)$$

$$= \underset{h\to 0}{Lt}\frac{1}{h}\frac{\cos 3x-\cos(3x+3h)}{\cos(3x+3h).\cos 3x}$$

$$= \underset{h\to 0}{Lt}\frac{1}{h}\frac{2.\sin\frac{3x+3h+3x}{2}.\sin\frac{3x+3h-3x}{2}}{\cos(3x+3h).\cos 3x}$$

$$f'(x) = \frac{\underset{h\to 0}{Lt}\, 2\sin\left(3x+\frac{3h}{2}\right).\underset{h\to 0}{Lt}\,\frac{\sin\frac{3h}{2}}{h}}{\underset{h\to 0}{Lt}\cos(3x+3h).\cos 3x}$$

$$= \frac{2\sin 3x.\frac{3}{2}}{(\cos 3x)(\cos 3x)} = 3.\frac{\sin 3x}{\cos 3x}.\frac{1}{\cos 3x}$$

$$= 3.\tan 3x.\sec 3x$$

x) $f(x) = x\sin x$

sol : $$f^1(x) = \underset{h\to 0}{Lt}\,\frac{f(x+h)-f(x)}{h} = \underset{h\to 0}{Lt}\,\frac{(x+h)\sin(x+h)-x\sin x}{h}$$

$$= \underset{h\to 0}{Lt}\,\frac{x(\sin(x+h)-\sin x)+h.\sin(x+h)}{h}$$

$$= \underset{h\to 0}{Lt}\,\frac{x\left(2\cos\frac{x+h+x}{2}.\sin\frac{x+h-x}{2}\right)+h.\sin(x+h)}{h}$$

$$= 2x.\underset{h\to 0}{Lt}\cos\left(x+\frac{h}{2}\right).\underset{h\to 0}{Lt}\,\frac{\sin\frac{h}{2}}{h}+\underset{h\to 0}{Lt}\,h.\frac{\sin(x+h)}{h}$$

$$= 2x.\cos x.\frac{1}{2}+\sin x$$

$$= x.\cos x.+\sin x$$

xi) $f(x)=\cos^2 x$

sol : $$f^1(x) = \underset{h\to 0}{Lt}\,\frac{f(x+h)-f(x)}{h}$$

$$f^1(x) = \underset{h\to 0}{Lt}\,\frac{\cos^2(x+h)-\cos^2 x}{h}$$

$$= \underset{h\to 0}{Lt} \frac{-\left(\cos^2 x - \cos^2 (x+h)\right)}{h}$$

$$= \underset{h\to 0}{Lt} \frac{-\sin(x+h+x)\sin(x+h-x)}{h}$$

$$f'(x) = \underset{h\to 0}{Lt} -\sin(2x+h) \underset{h\to 0}{Lt} .\frac{\sin h}{h}$$

$$= -\sin 2x.1 = -\sin 2x$$

3. Show that the function $f(x) = |x| + |x+1|$ is differentiable for real numbers except for 0 and 1.

Sol : $f(x) = |x| + |x+1| \forall\, x \in R$

Case (i) : at $x = 0$

$$Rf'(0) = \underset{x\to 0+}{Lt} \frac{f(x) - f(0)}{x}$$

$$= \underset{x\to 0+}{Lt} \frac{2x+1-1}{x} = 2$$

$$Lf'(0) = \underset{x\to -}{Lt} \frac{f(x) - f(0)}{x}$$

$$= \underset{x\to 0-}{Lt} \frac{1-1}{x} = 0$$

$R f'(0) \neq L f'(0)$

$\therefore\ f'(0)$ does not exist.

$f(x)$ is not differentiable at $x = 0$.

Case (ii) : at $x = 1$

$$R\,f'(1) = \underset{x \to 1+}{Lt} \frac{f(x) - f(1)}{x - 1}$$

$$= \underset{x \to 1+}{Lt} \frac{2x + 1 - 3}{x - 1} = \underset{x \to 1+}{Lt} \frac{2x - 2}{x - 1} = 2$$

$$L\,f'(1) = \underset{x \to 1-}{Lt} \frac{f(x) - f(1)}{x - 1} = \underset{x \to 1-}{Lt} \frac{1 - 1}{x - 1} = 0$$

$$R\,f'(1) \neq Lf'(1)$$

$f'(1)$ Does not exist.

$f(x)$ is not differentiable at $x = 1$

$\therefore$ $f(x)$ is differentiable on $R - \{0,1\}$.

4. Verify whether the following function is differentiable at 1 and 3

$$f(x) = \begin{cases} x & \text{if} \quad x < 1 \\ 3 - x & \text{if} \quad 1 \leq x \leq 3 \\ x^2 - 4x + 3 & \text{if} \quad x > 3 \end{cases}$$

Sol : Case (i) : **at** $x = 1$

$$f'\left(1^-\right) = \underset{x \to 1-}{Lt} \frac{f(x) - f(1)}{x - 1}$$

$$= \underset{x \to 1-}{Lt} \frac{x - (3 - 1)}{x - 1} = \underset{x \to 1}{Lt} \frac{x - 2}{x - 1} = \infty$$

$$f'\left(1^+\right) = \underset{x \to 1+}{Lt} \frac{f(x) - f(1)}{x - 1}$$

$$\underset{x \to 1+}{Lt} \frac{(3 - x) - 2}{x - 1} = \underset{x \to 1+}{Lt} \frac{1 - x}{x - 1} = -1$$

$R\,f'(1) \neq L\,f'(1)$

$f(x)$ is not differentiable at $x = 1$

Case (ii) : at $x = 3$

$$f'(3^-) = \underset{x \to 3-}{Lt} \frac{f(x) - f(3)}{x - 3}$$

$$\underset{x \to 3-}{Lt} \frac{(3 - x) - 0}{x - 3} = \underset{x \to 3-}{Lt} \frac{3 - x}{x - 3} = -1$$

$$f'(3^+) = \underset{x \to 3+}{Lt} \frac{f(x) - f(3)}{x - 3}$$

$$= \underset{x \to 3+}{Lt} \frac{(x^2 - 4x + 3) - 0}{x - 3} = \underset{x \to 3+}{Lt} \frac{(x - 3)(x - 1)}{x - 3}$$

$$= \underset{x \to 3+}{Lt}\, x - 1 = 3 - 1 = 2$$

5. Is the following function f derivable at 2? Justify $f(x) = \begin{cases} x, & if \quad 0 \le x \le 2 \\ 2, & if \quad x \ge 2 \end{cases}$

Sol : $f'(2^-) = \underset{x \to 2^-}{Lt} \frac{f(x) - f(2)}{x - 2} = \underset{x \to 2-}{Lt} \frac{x - 2}{x - 2} = 1$

$$f'(2^+) = \underset{x \to 2^+}{Lt} \frac{f(x) - f(2)}{x - 2} = \underset{x \to 2+}{Lt} \frac{2 - 2}{x - 2} = 0$$

$f'(2-) \neq f'(2^+)$; $f(x)$ is not derivable at $x = 2$.