

JAM 2006

MATHEMATICS TEST PAPER

NOTATIONS USED

- $\mathbb{R}$: The set of all real numbers
 $\mathbb{Z}$: The set of all integers

IMPORTANT NOTE FOR CANDIDATES

Objective Part:

Attempt ALL the objective questions (Questions 1-15). Each of these questions carries six marks. Each incorrect answer carries minus two. Write the answers to the objective questions in the Answer Table for Objective Questions provided on page 7 only.

Subjective Part:

Attempt ALL subjective questions (Questions 16-29). Each of these questions carries fifteen marks.

1. $\lim_{n \rightarrow \infty} \frac{2^{n+1} + 3^{n+1}}{2^n + 3^n}$ equals
 - (A) 3
 - (B) 2
 - (C) 1
 - (D) 0
2. Let $f(x) = (x-2)^{17}(x+5)^{24}$. Then
 - (A) f does not have a critical point at 2
 - (B) f has a minimum at 2
 - (C) f has a maximum at 2
 - (D) f has neither a minimum nor a maximum at 2
3. Let $f(x, y) = x^5 y^2 \tan^{-1}\left(\frac{y}{x}\right)$. Then $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y}$ equals
 - (A) $2f$
 - (B) $3f$
 - (C) $5f$
 - (D) $7f$
4. Let G be the set of all irrational numbers. The interior and the closure of G are denoted by G^0 and $\overline{G}$, respectively. Then
 - (A) $G^0 = \phi$, $\overline{G} = G$
 - (B) $G^0 = \mathbb{R}$, $\overline{G} = \mathbb{R}$
 - (C) $G^0 = \phi$, $\overline{G} = \mathbb{R}$
 - (D) $G^0 = G$, $\overline{G} = \mathbb{R}$

5. Let $f(x) = \int_{\sin x}^{\cos x} e^{-t^2} dt$. Then $f'(\pi/4)$ equals
- (A) $\sqrt{1/e}$
 (B) $-\sqrt{2/e}$
 (C) $\sqrt{2/e}$
 (D) $-\sqrt{1/e}$
6. Let C be the circle $x^2 + y^2 = 1$ taken in the anti-clockwise sense. Then the value of the integral
- $$\int_C \left[(2xy^3 + y)dx + (3x^2y^2 + 2x)dy \right]$$
- equals
- (A) 1
 (B) $\pi/2$
 (C) π
 (D) 0
7. Let r be the distance of a point $P(x, y, z)$ from the origin O . Then ∇r is a vector
- (A) orthogonal to $\overrightarrow{OP}$
 (B) normal to the level surface of r at P
 (C) normal to the surface of revolution generated by OP about x -axis
 (D) normal to the surface of revolution generated by OP about y -axis
8. Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be defined by
- $$T(x_1, x_2, x_3) = (x_1 - x_2, x_1 - x_2, 0).$$
- If $N(T)$ and $R(T)$ denote the null space and the range space of T respectively, then
- (A) $\dim N(T) = 2$
 (B) $\dim R(T) = 2$
 (C) $R(T) = N(T)$
 (D) $N(T) \subset R(T)$
9. Let S be a closed surface for which $\iint_S \vec{r} \cdot \hat{n} d\sigma = 1$. Then the volume enclosed by the surface is
- (A) 1
 (B) $1/3$
 (C) $2/3$
 (D) 3

10. If $(c_1 + c_2 \ln x)/x$ is the general solution of the differential equation

$$x^2 \frac{d^2 y}{dx^2} + kx \frac{dy}{dx} + y = 0, \quad x > 0,$$

then k equals

- (A) 3
 - (B) -3
 - (C) 2
 - (D) -1
11. If A and B are 3×3 real matrices such that $\text{rank}(AB)=1$, then $\text{rank}(BA)$ **cannot** be
- (A) 0
 - (B) 1
 - (C) 2
 - (D) 3
12. The differential equation representing the family of circles touching y -axis at the origin is
- (A) linear and of first order
 - (B) linear and of second order
 - (C) nonlinear and of first order
 - (D) nonlinear and of second order
13. Let G be a group of order 7 and $\phi(x) = x^4, x \in G$. Then ϕ is
- (A) not one – one
 - (B) not onto
 - (C) not a homomorphism
 - (D) one – one, onto and a homomorphism
14. Let R be the ring of all 2×2 matrices with integer entries. Which of the following subsets of R is an integral domain?
- (A) $\left\{ \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} : x, y \in \mathbf{Z} \right\}$
 - (B) $\left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x, y \in \mathbf{Z} \right\}$
 - (C) $\left\{ \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} : x \in \mathbf{Z} \right\}$
 - (D) $\left\{ \begin{pmatrix} x & y \\ y & z \end{pmatrix} : x, y, z \in \mathbf{Z} \right\}$

15. Let $f_n(x) = n \sin^{2n+1} x \cos x$. Then the value of

$$\lim_{n \rightarrow \infty} \int_0^{\pi/2} f_n(x) dx - \int_0^{\pi/2} \left(\lim_{n \rightarrow \infty} f_n(x) \right) dx$$

is

- (A) $1/2$
- (B) 0
- (C) $-1/2$
- (D) $-\infty$

16. (a) Test the convergence of the series

$$\sum_{n=1}^{\infty} \frac{n^n}{n! 3^n} \quad (6)$$

- (b) Show that

$$\ln(1 + \cos x) \leq \ln 2 - \frac{x^2}{4} \quad \text{for } 0 \leq x \leq \pi/2. \quad (9)$$

17. Find the critical points of the function

$$f(x, y) = x^3 + y^2 - 12x - 6y + 40.$$

Test each of these for maximum and minimum. (15)

18. (a) Evaluate $\iint_R x e^{y^2} dx dy$, where R is the region bounded by the lines $x = 0$, $y = 1$ and the parabola $y = x^2$. (6)

- (b) Find the volume of the solid bounded above by the surface $z = 1 - x^2 - y^2$ and below by the plane $z = 0$. (9)

19. Evaluate the surface integral

$$\iint_S x(12y - y^4 + z^2) d\sigma,$$

where the surface S is represented in the form $z = y^2$, $0 \leq x \leq 1$, $0 \leq y \leq 1$. (15)

20. Using the change of variables, evaluate $\iint_R xy dx dy$, where the region R is bounded by the curves $xy = 1$, $xy = 3$, $y = 3x$ and $y = 5x$ in the first quadrant. (15)

21. (a) Let u and v be the eigenvectors of A corresponding to the eigenvalues 1 and 3 respectively. Prove that $u + v$ is not an eigenvector of A . (6)

- (b) Let A and B be real matrices such that the sum of each row of A is 1 and the sum of each row of B is 2. Then show that 2 is an eigenvalue of AB . (9)

22. Suppose W_1 and W_2 are subspaces of $\mathbb{R}^4$ spanned by $\{(1, 2, 3, 4), (2, 1, 1, 2)\}$ and $\{(1, 0, 1, 0), (3, 0, 1, 0)\}$ respectively. Find a basis of $W_1 \cap W_2$. Also find a basis of $W_1 + W_2$ containing $\{(1, 0, 1, 0), (3, 0, 1, 0)\}$. (15)

23. Determine y_0 such that the solution of the differential equation

$$y' - y = 1 - e^{-x}, \quad y(0) = y_0$$

has a finite limit as $x \rightarrow \infty$. (15)

24. Let $\phi(x, y, z) = e^x \sin y$. Evaluate the surface integral $\iint_S \frac{\partial \phi}{\partial n} d\sigma$, where S is the surface of the cube $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$ and $\frac{\partial \phi}{\partial n}$ is the directional derivative of ϕ in the direction of the unit outward normal to S . Verify the divergence theorem. (15)

25. Let $y = f(x)$ be a twice continuously differentiable function on $(0, \infty)$ satisfying

$$f(1) = 1 \text{ and } f'(x) = \frac{1}{2} f\left(\frac{1}{x}\right), \quad x > 0.$$

Form the second order differential equation satisfied by $y = f(x)$, and obtain its solution satisfying the given conditions. (15)

26. Let $G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}$ be the group under matrix addition and H be the subgroup of G consisting of matrices with even entries. Find the order of the quotient group G/H . (15)

27. Let

$$f(x) = \begin{cases} x^2 & 0 \leq x \leq 1 \\ \sqrt{x} & x > 1. \end{cases}$$

Show that f is uniformly continuous on $[0, \infty)$. (15)

28. Find $M_n = \max_{x \geq 0} \left\{ \frac{x}{n(1 + nx^3)} \right\}$, and hence prove that the series

$$\sum_{n=1}^{\infty} \frac{x}{n(1 + nx^3)}$$

is uniformly convergent on $[0, \infty)$. (15)

29. Let R be the ring of polynomials with real coefficients under polynomial addition and polynomial multiplication. Suppose

$$I = \{ p \in R : \text{sum of the coefficients of } p \text{ is zero} \}.$$

Prove that I is a maximal ideal of R . (15)