A

2009 – MS

Test Paper Code : MS

Time : 3 Hours Maximum Marks : 300

INSTRUCTIONS

1. The question-cum-answer booklet has **32** pages and has **25** questions. Please ensure that the copy of the question-cum-answer booklet you have received contains all the questions.
2. Write your **Roll Number, Name and the name of the Test Centre** in the appropriate space provided on the right side.
3. Write the answers to the objective questions against each Question No. in the **Answer Table for Objective Questions**, provided on Page No. **7**. Do not write anything else on this page.
4. Each objective question has **4 choices** for its answer: (A), (B), (C) and (D). Only **ONE** of them is the correct answer. There will be **negative marking** for wrong answers to objective questions. The following marking scheme for objective questions shall be used :
   (a) For each correct answer, you will be awarded **6 (Six)** marks.
   (b) For each wrong answer, you will be awarded **–2 (Negative two)** marks.
   (c) Multiple answers to a question will be treated as a wrong answer.
   (d) For each un-attempted question, you will be awarded **0 (Zero)** mark.
   (e) Negative marks for objective part will be carried over to total marks.
5. Answer the subjective question only in the space provided after each question.
6. Do not write more than one answer for the same question. In case you attempt a subjective question more than once, please cancel the answer(s) you consider wrong. Otherwise, the answer appearing last only will be evaluated.
7. All answers must be written in blue/black/blue-black ink only. Sketch pen, pencil or ink of any other colour should not be used.
8. All rough work should be done in the space provided and scored out finally.
9. No supplementary sheets will be provided to the candidates.
10. **Clip board, log tables, slide rule, calculator, cellular phone, pager and electronic gadgets in any form are NOT allowed.**
11. The question-cum-answer booklet must be returned in its entirety to the Invigilator before leaving the examination hall. Do not remove any page from this booklet.
12. Refer to special instructions/useful data on the reverse.

Somesh

Somesh

A

2009 – MS

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

ROLL NUMBER							

Name :

Test Centre :

Do not write your Roll Number or Name anywhere else in this question-cum-answer booklet.

I have read all the instructions and shall abide by them.

....

Signature of the Candidate

I have verified the information filled by the Candidate above.

....

Signature of the Invigilator

A

IMPORTANT NOTE FOR CANDIDATES

- **Questions 1-15 (objective questions) carry _six_ marks each and questions 16-25 (subjective questions) carry _twenty one_ marks each.**
- **Write the answers to the objective questions in the _Answer Table for Objective Questions_ provided on page 7 only.**

Q.1 For detecting a disease, a test gives correct diagnosis with probability 0.99. It is known that 1% of a population suffers from this disease. If a randomly selected individual from this population tests positive, then the probability that the selected individual actually has the disease is

(A) 0.01 (B) 0.05 (C) 0.5 (D) 0.99

Q.2 Let X be any random variable with mean μ and variance 9. Then the smallest value of m such that $P(|X-\mu|<m)\geq 0.99$, is

(A) 90 (B) $\sqrt{90}$ (C) $\sqrt{100/11}$ (D) 30

Q.3 If a random variable X has the cumulative distribution function

$$F(x)=\begin{cases} 0, & \text{if } x<0, \\ \frac{1}{3}, & \text{if } x=0, \\ \frac{1+x}{3}, & \text{if } 0<x<1, \\ 1, & \text{if } x\geq 1, \end{cases}$$

then $E(X)$ equals

(A) 1/3 (B) 1 (C) 1/6 (D) 1/2

Q.4 If $Y=\dfrac{\ln U_1}{\ln U_1+\ln(1-U_2)}$, where U_1 and U_2 are independent $U(0,1)$ random variables, then variance of Y equals

(A) 1/12 (B) 1/3 (C) 1/4 (D) 1/6

Q.5 If X is a Binomial (30, 0.5) random variable, then

(A) $P(X>15)=0.5$ (B) $P(X<15)=0.5$
(C) $P(X>15)>0.5$ (D) $P(X<15)<0.5$

Q.6 If the joint probability density function of (X, Y) is given by

$$f(x, y) = \frac{1}{y} e^{-\frac{x}{y}}, \quad x > 0, \quad 0 < y < 1,$$

then

(A) $E(X) = 0.5$ and $E(Y) = 0.5$
(B) $E(X) = 1.0$ and $E(Y) = 0.5$
(C) $E(X) = 0.5$ and $E(Y) = 1.0$
(D) $E(X) = 1.0$ and $E(Y) = 1.0$

Q.7 If X is an $F(m, n)$ random variable, where $m > 2,\ n > 2$, then $E(X)E\left(\frac{1}{X}\right)$ equals

(A) $\frac{n(n-2)}{m(m-2)}$
(B) $\frac{m(m-2)}{n(n-2)}$
(C) $\frac{mn}{(m-2)(n-2)}$
(D) $\frac{m(n-2)}{n(m-2)}$

Q.8 Let X be a random variable having probability mass function

$$f(x) = \begin{cases} \frac{2+4\alpha_1+\alpha_2}{6}, & \text{if } x = 1, \\ \frac{2-2\alpha_1+\alpha_2}{6}, & \text{if } x = 2, \\ \frac{1-\alpha_1-\alpha_2}{3}, & \text{if } x = 3, \end{cases}$$

where $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$ are unknown parameters such that $\alpha_1 + \alpha_2 \leq 1$. For testing the null hypothesis $H_0 : \alpha_1 + \alpha_2 = 1$ against the alternative hypothesis $H_1 : \alpha_1 = \alpha_2 = 0$, suppose that the critical region is $C = \{2, 3\}$. Then, this critical region has

(A) size = 1/2 and power = 2/3
(B) size = 1/4 and power = 2/3
(C) size = 1/2 and power = 1/4
(D) size = 2/3 and power = 1/3

Q.9 The observed value of mean of a random sample from $N(\theta, 1)$ distribution is 2.3. If the parameter space is $\Theta = \{0, 1, 2, 3\}$, then the maximum likelihood estimate of θ is

(A) 1
(B) 2
(C) 2.3
(D) 3

Q.10 The series

$$\sum_{n=1}^{\infty}\frac{\sqrt{n}}{x^n\sqrt{n^2+1}},\ x>0,$$

(A) converges for $x>1$ and diverges for $x\le 1$
(B) converges for $x\le 1$ and diverges for $x>1$
(C) converges for all $x>0$
(D) diverges for all $x>0$

Q.11 Let f be a differentiable function defined on $[0,1]$. If $\xi\in(0,1)$ is such that $f(x)<f(\xi)=f(0)$ for all $x\in(0,1]$, $x\ne\xi$, then

(A) $f'(\xi)=0$ and $f'(0)=0$
(B) $f'(\xi)=0$ and $f'(0)\le 0$
(C) $f'(\xi)>0$ and $f'(0)\le 0$
(D) $f'(\xi)=0$ and $f'(0)>0$

Q.12 The area of the region bounded by $y=x^3$, $x+y-2=0$ and $y=0$ is

(A) 0.25 (B) 0.5 (C) 0.75 (D) 1.0

Q.13 The system of equations

$$x+3y+2z=k$$
$$2x+y-4z=4$$
$$5x-14z=10$$

(A) has unique solution for $k=2$
(B) has infinitely many solutions for $k=2$
(C) has no solution for $k=2$
(D) has unique solution for any $k\ne 2$

Q.14 Let $A=((a_{ij}))$ be an orthogonal matrix of order n such that $a_{1j}=\frac{1}{\sqrt{n}}, j=1,\ldots,n$. If $\overline{a}=\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}$, then $\sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij}-\overline{a})^2$ equals

(A) $\frac{n+1}{n}$ (B) $\frac{n-1}{n}$ (C) $\frac{n^2+1}{n}$ (D) $\frac{n^2-1}{n}$

Q.15 The solution of the differential equation

$$\frac{dy}{dx}=\frac{y^2\cos x+\cos y}{x\sin y-2y\sin x};\ y\left(\frac{\pi}{2}\right)=0$$

is

(A) $y^2\cos x+x\sin y=0$

(B) $y^2\sin x+x\cos y=\frac{\pi}{2}$

(C) $y^2\sin x+x\sin y=0$

(D) $y^2\cos x+x\cos y=\frac{\pi}{2}$

Space for rough work

Q.16 Let $U_1, U_2, \ldots, U_n$ be n urns such that urn U_k contains k white and k^2 black balls, $k = 1, \ldots, n$. Consider the random experiment of selecting an urn and drawing a ball out of it at random. If the probability of selecting urn U_k is proportional to $(k+1)$, then

(a) find the probability that the ball drawn is black. (12)

(b) find the probability that urn U_n was selected, given that the ball drawn is white. (9)

Q.17 Let the joint probability density function of (X, Y) be given by

$$f(x, y) = \frac{1}{4}(1 + x^3 y^3), \quad -1 \le x \le 1, -1 \le y \le 1.$$

(a) Find the joint probability density function of (X^2, Y^2). (12)

(b) Calculate the correlation coefficient between X and Y. (9)

Q.18 (a) Let X be a random variable having probability mass function

$$p(x)=\begin{cases}2\theta, & \text{if } x=-1,\\ \theta^2, & \text{if } x=0,\\ 1-2\theta-\theta^2, & \text{if } x=1,\end{cases}$$

where $\theta\in[0, \sqrt{2}-1]$. Show that there is one, and only one, unbiased estimator of $(\theta+1)^2$ based on a single observation. (12)

(b) Let $X_1,\ldots,X_n$ be a random sample from a population having probability density function

$$f(x)=\frac{1}{\theta}e^{-\frac{(x-\theta)}{\theta}}, \quad x\geq\theta,$$

where $\theta\in(0, \infty)$. Find the maximum likelihood estimator of θ. (9)

Q.19 (a) A box contains M white and $3-M$ black balls. To test the null hypothesis $H_0: M=2$ against the alternative hypothesis $H_1: M=1$, five balls are drawn at random from the box with replacement. If X is the number of white balls drawn, then find the most powerful test of size $\alpha = \frac{11}{243}$. Also find its power. (12)

(b) Let $X_1, \ldots, X_5$ be a random sample from $Exp(\lambda)$ distribution, where $\lambda > 0$. Find a pair of constants (c_1, c_2) such that $P(c_1 \overline{X} \le \lambda \le c_2 \overline{X}) = 0.9$, where $\overline{X}$ denotes the sample mean.

(Useful Data : If X has $\chi^2(10)$ distribution, then $P(X \le 3.94) = 0.05$ and $P(X \le 18.3) = 0.95$) (9)

Q.20 (a) Using the Central Limit Theorem, evaluate

$$\lim_{n \to \infty} \sum_{j=0}^{n} \binom{j+n-1}{j} \frac{1}{2^{j+n}}.$$ (12)

(b) Suppose that n balls are randomly placed in $2n$ cells. Let X denote the number of balls occupying the first cell. Find $\lim_{n \to \infty} P(X=5)$. (9)

Q.21 (a) Suppose that the joint probability density function of (X, Y) is given by

$$f(x, y)=\begin{cases}\frac{1}{2}, & \text{if } (x, y)\in R,\\ 0, & \text{otherwise,}\end{cases}$$

where R is the region in xy-plane bounded by the lines $y=x$, $y=-x$, $y=x-2$, $y=-x+2$. Define $U=\frac{X+Y}{2}$ and $V=\frac{X-Y}{2}$. Show that U and V are independent $U(0, 1)$ random variables. (12)

(b) The probability mass function $p(x)$ of a discrete random variable X satisfies $p(x+1)=\lambda\, p(x)$, $x=1, 2, 3,\ldots$, where $0<\lambda<1$. For positive integers m and n, find $P(X \geq m+n \mid X \geq m)$. (9)

Q.22 (a) Let f be a differentiable function on $(0, 1)$ such that $|f'(x)|<1$, for all $x \in (0, 1)$. Show that the sequence $\{a_n\}$ defined by $a_n = f\left(\frac{1}{n}\right)$, $n = 1, 2, \ldots$ is convergent. (12)

(b) Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers such that $\lim_{n \to \infty} a_n = l$ and $b_n \leq l$, for $n = 1, 2, \ldots$. Define $c_n = \max(a_n, b_n)$, $n = 1, 2, \ldots$. Show that $\lim_{n \to \infty} c_n = l$. (9)

Q.23 Let A be a 3×3 real non-diagonal matrix with $A^{-1} = A$. Show that $tr(A) = -\det(A) = \pm 1$. (21)

Q.24 Let $\alpha(t)$ and $\beta(t)$ be differentiable functions on R such that $\alpha(0)=2$ and $\beta(0)=1$. If $\alpha(t)+\beta'(t)=1$ and $\alpha'(t)+\beta(t)=1$ for all $t\in[0,\infty)$, find the value of $\alpha(\ln 2)$. (21)

Q.25 (a) Find the global minimum value of the function

$$g(x, y) = 2\,(x-2)^2 + 3\,(y-1)^2$$

in the region $R = \{(x, y) : 0 \le x \le y \le 3\}$. Also find the point(s) at which it is attained.

(12)

(b) Let f be a continuous function on $[a, b]$. If $a \le x_1 < x_2 < \cdots < x_n \le b\ (n \ge 2)$, then show that there exists a $\xi \in (a, b)$ such that $\sum_{k=1}^{n} \alpha_k f(x_k) = f(\xi)$, where $\alpha_i > 0,\ i = 1, 2, \ldots, n$ and $\alpha_1 + \cdots + \alpha_n = 1$.

(9)