Mathematics

Q.1 A line passes through the point $P(h, k)$ is parallel to the x- axis. It forms a triangle with the lines $y = x$ & $x + y = 2$ of area $4h^2$ then find the locus of P. **[2]**
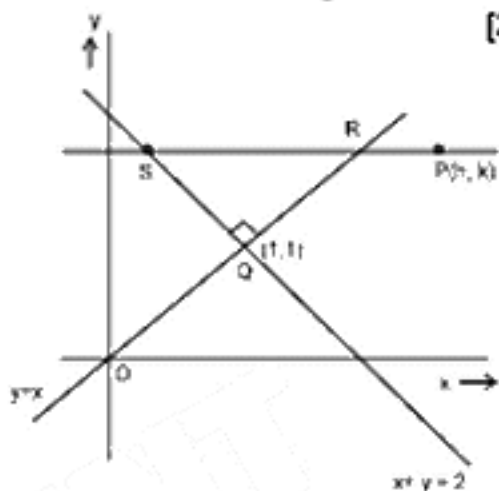
Sol. On solving the lines we get S & R points

S : $(2 - k, k)$, R : (k, k)

Two lines intersect at Q $(1, 1)$

Area of the $(\Delta QRS) = \frac{1}{2} \times 2 (k - 1)^2$ $[\because QS = QR = \sqrt{2}\, SR]$

$4h^2 = [k - 1]^2$

$\Rightarrow$ locus is the pair of straight lines $\boxed{4x^2 = (y - 1)^2}$ **Ans.**

Q.2 A cricket player in his career plays n match and scores total no. of $\dfrac{(n + 1)(2^{n+1} - n - 2)}{4}$ runs. **[2]**

Sol. If he scores $k \cdot 2^{n-k+1}$ runs in k^{th} match, where $1 \le k \le n$. Find n.

Let S_n be the total scores in his career plays n matches.

$S_n = \sum_{k=1}^{n} k.2^{n+1-k} = 2^{n+1} \sum_{k=1}^{n} k.2^{-k}$

$= 2^{n+1} \left[\dfrac{\frac{1}{2}\left(1 - \frac{1}{2^n}\right)}{1 - \frac{1}{2}} - \dfrac{n}{2^{n+1}} \right] \dfrac{1}{1 - \frac{1}{2}}$

$= 2^{n+2} \left[1 - \dfrac{1}{2^n} - \dfrac{n}{2^{n+1}} \right] = 2^{n+2} - 4 - 2n = 2^{n+2} - 2n - 4$

But $S_n = \dfrac{n-1}{4} (2^{n+1} - 2 - n)$ (as given)

so $\dfrac{n+1}{4} (2^{n+1} - 2 - n) = 2(2^{n+1} - n - 2)$

$n + 1 = 8$ so $\boxed{n = 7}$ **Ans.**

Q.3 Ramesh goes to office either by car, scooter, bus or train probability of which being $\frac{1}{7}, \frac{3}{7}, \frac{2}{7}$ and $\frac{1}{7}$ respectively and probability that he is reaching office late if he takes car, scooter, bus or train is $\frac{2}{9}, \frac{1}{9}, \frac{4}{9}$ and $\frac{1}{9}$ respectively. Find the probability that he has travelled by car, if he reaches office in time. **[2]**

Sol. Let A, B, C, D be the events when Ramesh is going by car, scooter , bus or train respectively.

$P(A) = \frac{1}{7}$, $P(B) = \frac{3}{7}$, $P(C) = \frac{2}{7}$, $P(D) = \frac{1}{7}$

Let $\bar{x}$ be the event when Ramesh reaching the office in time.

$P = \frac{7}{9}$, $P\left(\frac{\bar{x}}{B}\right) = \frac{8}{9}$, $P\left(\frac{\bar{x}}{C}\right) = \frac{5}{9}$, $P\left(\frac{\bar{x}}{D}\right) = \frac{8}{9}$

$P\left(\frac{A}{\bar{x}}\right) = \frac{P\left(\frac{\bar{x}}{A}\right).P(A)}{P(\bar{x})} = \frac{\frac{7}{9} \times \frac{1}{7}}{\frac{1}{7} \times \frac{7}{9} + \frac{3}{7} \times \frac{8}{9} + \frac{2}{7} \times \frac{5}{9} + \frac{8}{9} \times \frac{1}{7}} = \frac{1}{7}$ **Ans.**

Q.4 A light ray incident along the unit vector $\hat{v}$ and the unit vector along the normal to reflecting surface at the point P is $\hat{n}$ outwards. If reflecting ray is along the unit vector $\hat{w}$, find $\hat{w}$ in terms of $\hat{v}$ and $\hat{n}$. **[2]**

Sol. $\hat{n}$ is bisector of the angle between the incident and reflected ray and can be represencted as

$\hat{n} = \frac{\hat{w} - \hat{v}}{|\hat{w} - \hat{v}|}$
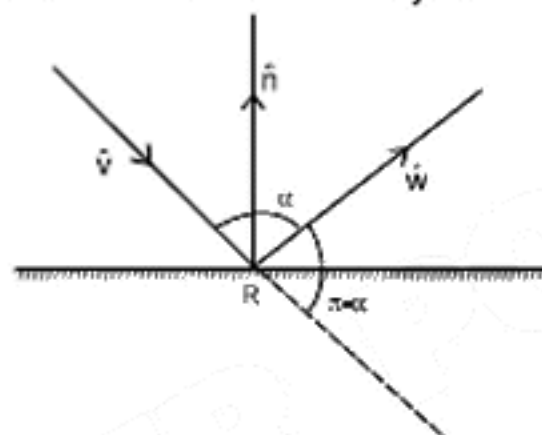
where $|\hat{w} - \hat{v}| = \sqrt{1 + 1 - 2(\hat{w}.\hat{v})}$ [where $\pi - \alpha$ is angle between $\hat{w}$ and $\hat{v}$]

$= |2 \cos \alpha/2|$ where $\alpha \in (0, \pi)$

$= -2(\hat{n}.\hat{v})$

$\Rightarrow \boxed{\hat{w} = \hat{v} - 2(\hat{n}.\hat{v})\,\hat{n}}$ **Ans.**

Q.5 Find the equation of the plane at distance of $\frac{1}{\sqrt{6}}$ from the point $(2, 1, -1)$ and containing the line $2x - y + z - 3 = 0 = 3x + y + z - 5$. **[2]**

Sol. $(2x - y + z - 3) + \lambda(3x + y + z - 5) = 0$

$(3\lambda + 2)x + (\lambda - 1)y + (\lambda + 1)z - 5\lambda - 3 = 0$

$\left|\frac{6\lambda + 4 + \lambda - 1 - \lambda - 1 - 5\lambda - 3}{\sqrt{(3\lambda - 2)^2 + (\lambda - 1)^2 + (\lambda + 1)^2}}\right| = \frac{1}{\sqrt{6}}$

$6(\lambda - 1)^2 = 11\lambda^2 + 12\lambda + 6$

$\lambda = 0, -\frac{24}{5}$

So the planes are $2x - y + z - 3 = 0$ and $62x + 29y + 19z - 105 = 0$ **Ans.**

Q.6 For any two distinct real numbers x_1 & x_2, $y = f(x)$ is satisfying the condition. $|f(x_1) - f(x_2)| < (x_1 - x_2)^2$. Find the equation of the tangent at the point $(1, 2)$ to the curve $y = f(x)$. [2]

Sol. Given : $|f(x_1) - f(x_2)| < |x_1 - x_2|^2$

Let $x_2 = 1$ & $x_1 = 1 + h$

$|f(1+h) - f(1)| < |h|^2$

so $\lim_{h \to 0} \left| \frac{f(1+h) - f(1)}{h} \right| < \lim_{h \to 0} |h|$

$|f'(1)| < \varepsilon$; where ε is a very small positive quantity.

so $f'(1) = 0$

Hence the tangent to $y = f(x)$ at $(1, 2)$ is $y = 2$. **Ans.**

Q.7 3, 4, 5 are radii of three circles touch each other externally if P is the point of intersection of tangents of these circles at their points of contact, find the distances of P from the points of contact. [2]

Sol. Let x, y, z be the centre of the three circles clearly the point P is the in- centre of the $\triangle xyz$ and

$r = \dfrac{\sqrt{s(s-a)(s-b)(s-c)}}{s}$; where $\triangle$ & S have their usual meaning

$2s = 7 + 8 + 9 \Rightarrow s = 12$

$r = \sqrt{\dfrac{5 \cdot 4 \cdot 3}{12}} = \sqrt{5}$ unit. **Ans.**
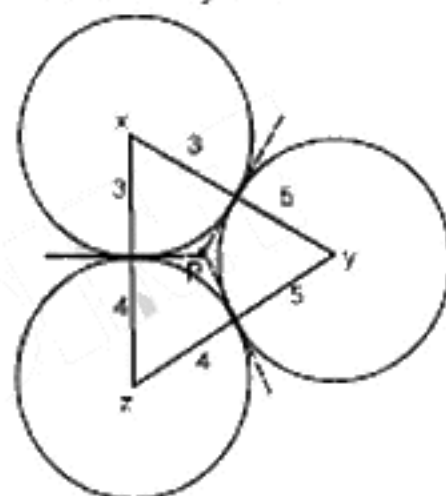
Q.8 Evaluate : $\int_0^\pi e^{|\cos x|}[2\sin(1/2\cos x) - 3\cos(1/2\cos x)]\sin x \, dx$ [2]

Sol. Let $\dfrac{1}{2}\cos x = t$

$\cos x = 2t$ so $-\sin x \, dx = 2dt$

so $I = \int_{-1/2}^{1/2} e^{|2t|} [2\sin(t) + 3\cos(t)] 2dt$

$= 2 \int_{-1/2}^{1/2} 2 \cdot e^{|2t|} \sin t \, dt + 6 \int_{-1/2}^{1/2} e^{|2t|} \cos t \, dt$
(Using $\int_{-a}^{a} f(x)dx = 0$; if $f(x)$ is odd function

$= 2 \int_0^a f(x) \, dx$; if $f(x)$ is even function)

$= 0 + 12 \int_0^{1/2} e^{2t} \cos t \, dt$

$= \dfrac{12}{\sqrt{5}} [e^{2t} \cos(t - \tan^{-1}\dfrac{1}{2})]_0^1$

$= \dfrac{12}{\sqrt{5}} [e \cos(\dfrac{1}{2} - \tan^{-1}\dfrac{1}{2}) - \cos(\tan^{-1}\dfrac{1}{2})]$

$= \dfrac{12}{\sqrt{5}} [e \cos(\dfrac{1}{2} - \tan^{-1}\dfrac{1}{2}) - \dfrac{2}{\sqrt{5}}]$ **Ans.**

Q.9 Find the area bounded by the curves $x^2 = y$, $x^2 = -y$ & $y^2 = 4x - 3$. [4]
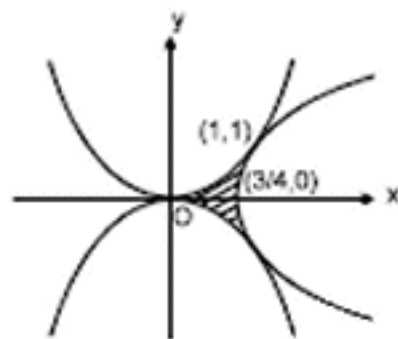
Sol. On solving the curves $x^2 = y$ and $y^2 = 4x - 3$

We get their point of contact $(1,1)$ and the area bounded is symmetric about x- axis.

so that the area bounded is

$$= 2\left[\int_0^{3/4} x^2 + \int_{3/4}^1 (x^2 - \sqrt{4x-3})\, dx\right]$$

$$= \frac{1}{3} \text{ sq. units. } \textbf{Ans.}$$

Q.10 Find the equation of the common tangent in 1st quadrnt to the circle $x^2 + y^2 = 16$ and the ellipse $\frac{x^2}{25} + \frac{y^2}{4} = 1$. Also find

the length of the intercept of the tangent between the coordinates axes. [4]

Sol. Let the equations of tangents to the given circle and the ellipse respectively.

$$y = mx + 4\sqrt{1+m^2} \qquad ....(1)$$

and $y = mx + \sqrt{25m^2 + 4} \qquad ....(2)$

since (1) & (2) are coincident lines, so

$$4\sqrt{1+m^2} = \sqrt{25m^2 + 4}$$

$$16(1 + m^2) = 25 m^2 + 4$$

$$m = \pm\frac{2}{\sqrt{3}}$$

$m < 0$ because common tangent in 1st quadrant

so $m = -\frac{2}{\sqrt{3}}$

so the equation of the common tangent is : (from (1))

$$y = -\frac{2}{\sqrt{3}} x + 4\sqrt{\frac{7}{3}}$$

It meets the coordinates axes at A $(2\sqrt{7}, 0)$ and $(0, 4\sqrt{\frac{7}{3}})$ so length of the intercept of the tangent between the

coordinate axes is $\frac{14}{\sqrt{3}}$. **Ans.**

Q.11 A tangent is drawn from a point on the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ to circle $x^2 + y^2 = 9$ find the locus of mid-point of chord

of contact. [4]

Sol. Let any point $P(3\sec\theta, 2\tan\theta)$ on the hyperbola

chord of contact of the circle $x^2 + y^2 = 9$ with respect to the point $(3\sec\theta, 2\tan\theta)$is

$(3\sec\theta) . x + (2\tan\theta) y = 9 \qquad .....(1)$

Let (h, k) be the mid point of the chord of contact.

equation of chord in mid-point form is $xh + yk = h^2 + k^2 \qquad ....(2)$

By (1) and (2)

$$\frac{3\sec\theta}{h} = \frac{2\tan\theta}{k} = \frac{9}{h^2+k^2}$$

$$\sec\theta = \qquad , \tan\theta = \frac{9k}{2(h^2+k^2)}$$

as $\sec^2\theta - \tan^2\theta = 1$, so

$$\frac{81h^2}{9(h^2+k^2)^2} - \frac{81k^2}{4(h^2+k^2)^2} = 1$$

so the required locus is $\boxed{(x^2+y^2)^2 = 9x^2 - \dfrac{81}{4}y^2}$ **Ans.**

Q.12 A square having one vertex as $2+\sqrt{3}\,i$ is circumscribed on the circle $|z-1|=\sqrt{2}$. Then find the other vertices of square. **[4]**

Sol. Centre of circle is $(1, 0)$ and is also the mid point of diagonals of square

$$z_3 - z_0 = (z_1 - z_0)e^{i\pi/2}$$
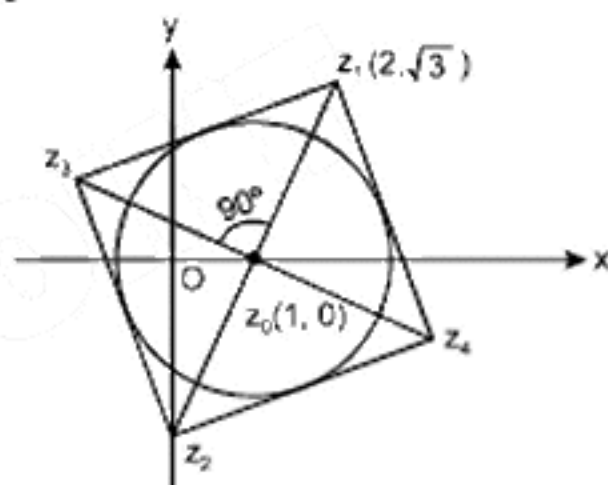$$= (2+\sqrt{3}\,i - 1)\,i$$

$$z_3 = i - \sqrt{3} + 1$$
$$= (1 - \sqrt{3} + i)$$

Since z_0 is mid point of z_1 & z_2 as well as z_3 & z_4 also

so $z_4 = 2z_0 - z_3 = (1+\sqrt{3}) - i$

and $z_2 = 2z_0 - z_1 = 2 - (2+\sqrt{3}\,i) = -\sqrt{3}\,i$

$\boxed{\text{so other vertices are } (0, -\sqrt{3}),\ (1-\sqrt{3},\ 1),\ (1+\sqrt{3} \cdot \frac{9h}{3(h^2-k^2)})} \text{ Ans.}$

Q.13 Find all the curves $y = f(x)$, tangents at any point on which are drawn such that the segment of tangent intercepted between the contact point and x-axis is of unit length. **[4]**

Sol. Given : $|AP| = 1$

Equation of tangent at $P(x, y)$ point is

$$Y - y = \frac{dy}{dx}(X - x)$$

Putting $X = 0$ & $Y = 0$ respectively we get pts A & B and length of the segment AB is

$|AP|$ is length of tangent, so

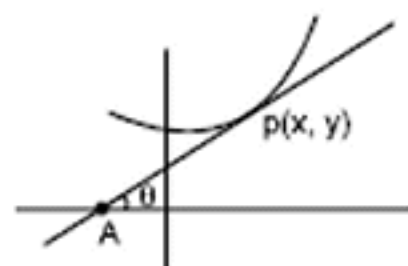$$|AP| = \left|\frac{y}{y'}\sqrt{1+(y')^2}\right| = 1 \qquad \text{where } y' = \left(\frac{dy}{dx}\right)_P$$

$$y^2\left[1+\left(\frac{dy}{dx}\right)^2\right] = \left(\frac{dy}{dx}\right)^2$$

$$\int\frac{\sqrt{1-y^2}}{y}\,dy = \pm\int dx$$

on integrating we get,

$$\boxed{\sqrt{1-y^2} + \ln\left|\frac{1-\sqrt{1-y^2}}{y}\right| = \pm x + c} \qquad \text{Ans.}$$

Q.14 If two functions 'f' and 'g' satisfying given conditions for $\forall\, x, y \in R$. $f(x-y) = f(x)\,g(y) - f(y)\cdot g(x)$ and $g(x-y) = g(x)\cdot g(y) + f(x)\,f(y)$. If right hand deviative at $x = 0$ exists for $f(x)$ then find the deviative of $g(x)$ at $x = 0$. **[4]**

Sol. Given $\quad f(x-y) = f(x)\cdot g(y) - f(y)\cdot g(x)$

Put $\quad y = x \quad$ and you get $f(0) = 0$

put $\quad y = 0 \quad$ and you get $g(0) = 1$

R.H.D. of $f(x)$: $\quad f'(0^+) = \lim_{h\to 0} \dfrac{f(0+h) - f(0)}{h} \qquad$ ($h \in R^+$ and tending to zero)

$$= \lim_{h\to 0} \frac{f(0)g(-h) - g(0)f(-h) - f(0)}{h} = \lim_{h\to 0} \frac{-f(-h)}{h} = f'(0^-) = \text{L.H.D.} \qquad .....(1)$$

& L.H.D. of $f(x)$: $f'(0^-) = \lim_{h\to 0} \dfrac{f(0-h) - f(0)}{-h} = -f'(0^+) = -\text{R.H.D.} \qquad ....(2)$

Hence from (1) and (2) $\qquad f'(0) = 0$

Put $y = x$ in given condition $g(x-y) = g(x)\cdot g(y) + f(x)\cdot f(y)$

$\Rightarrow g(0) = g^2(x) + f^2(x)$

on diff. w.r.t.x we get $\quad \Rightarrow g^2(x) + f^2(x) = 1 \qquad \Rightarrow g^2(x) = 1 - f^2(x)$

$\Rightarrow 2g(x)\cdot g'(x) + 2f(x)\,f'(x) = 0$

$\qquad\qquad g'(0) = 0$

[Note : 'g' is differentiable at zero because 'f' is diff. at 0 & $g^2(x) = 1 - f^2(x)$]

Q.15 Find the range of values of t for which $2\sin t = \dfrac{1 - 2x - 5x^2}{3x^2 - 2x - 1}$; $t \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ **[4]**

Sol. Given : $2\sin t = \dfrac{1 - 2x + 5x^2}{3x^2 - 2x - 1}$

First we have to check the range of R.H.S. $\dfrac{1 - 2x + 5x^2}{3x^2 - 2x - 1} = y$ (say)

$(3x^2 - 2x - 1)\,y - (1 - 2x + 5x^2) = 0$

$(3y - 5)\,x^2 - 2x\,(y - 1) - (y + 1) = 0$

So $D \ge 0$

since $x \in R - \{1, -\dfrac{1}{3}\}$

$y^2 - y - 1 \ge 0$

or $\quad y \ge \dfrac{1 + \sqrt{5}}{2} \qquad$ or $\qquad y \le \dfrac{1 - \sqrt{5}}{2} \qquad\qquad (\because \sin\dfrac{\pi}{10} = \dfrac{\sqrt{5} - 1}{4}$ & $\sin\dfrac{3\pi}{10} = \dfrac{\sqrt{5} + 1}{4})$

or $\quad \sin t \ge \dfrac{1 + \sqrt{5}}{4} \qquad$ or $\qquad \sin t \le \dfrac{1 - \sqrt{5}}{4}$

$\Rightarrow \sin t \ge \sin\dfrac{3\pi}{10} \qquad$ or $\qquad \sin t \le \sin\left(-\dfrac{\pi}{10}\right)$

$\Rightarrow \boxed{\dfrac{\pi}{2} \ge t \ge \dfrac{3\pi}{10} \qquad \text{or} \qquad -\dfrac{\pi}{2} \le t \le -\dfrac{\pi}{10}} \qquad$ **Ans.**
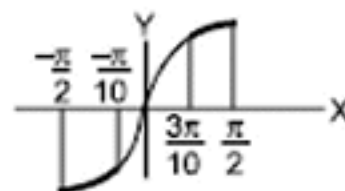
Q.16 If $P(x)$ be the cubic polynomial satisfying $p(-1) = 10$, $p(1) = -6$ and $p(x)$ has maximum at $x = -1$ and $p'(x)$ has minima at $x = 1$. Find the points of local maxima and minima, also find the distance between these two points. **[4]**

Sol. Let $P(x) = ax^2 + bx^2 + cx + d$

$P(-1) = -a + b - c + d = 10$ ....(1)

& $P(1) = a + b + c + d = -6$ ....(2)

$P'(x) = 3ax^2 + 2bx + c$

since $P'(-1) = 3a - 2b + c = 0$ ....(3)

and $P''(x) = 6ax + 2b$

$P''(1) = 6a + 2b = 0$

solving (1), (2), (3) & (4) for a, b, c, d

$a = 1, b = -3, c = -9, d = 5$

so $P(x) = x^3 - 3x^2 - 9x + 5$

Now $P'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x-3)(x+1)$

$$\frac{+ \quad - \quad +}{-1 \quad\quad 3}$$
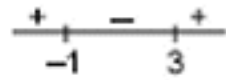
So P(x) has minima at $x = 3$ & maxima at $x = -1$

so the point of local max. $(3, -22)$

so the point minima $(-1, 10)$

distance $= \sqrt{16 + (32)^2} = \sqrt{16 + 1024} = \sqrt{1040} = 4\sqrt{65}$ unit **Ans.**

Q.17 f(x) be a quadratic polynomial , a, b,c are three distinct real numbers, such that :

$$\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix} \begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 3a^2 + 3a \\ 3b^2 - 3b \\ 3c^2 - 3c \end{bmatrix}$$

V is the point where f (x) attains maximum. A & B are the points on f(x) such that f (x) cuts x –axis at A in the first quadrant and chord AB subtends right angle at V. Find the area bounded by curve y = f (x) and chord AB. **[6]**

Sol. As given $4a^2 f(-1) + 4a f(1) + f(2) = 3a^2 + 3a$.

$4b^2 f(-1) + 4b f(1) + f(2) = 3b^2 + 3b$ and

$4c^2 f(-1) + 4c f(1) + f(2) = 3c^2 + 3c$

here it can be considered that the equation :

$4x^2 f(-1) + 4x f(1) + f(2) = 3x^2 + 3x$ has three distinct roots i.e. a, b, c

which is possible if and only if

$4 f(-1) = 3 \Rightarrow f(-1) = 3/4$

and $4 f(1) = 3 \quad f(1) = 3/4$

$f(2) = 0$
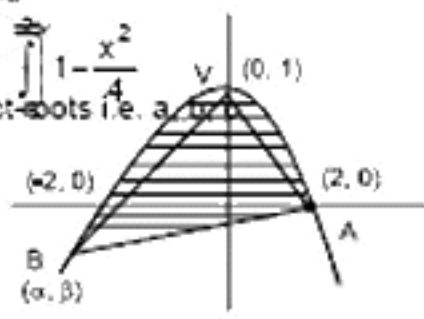
so the function $f(x) = \dfrac{-x^2}{4} + 1$

Let A be at $(2, 0)$. let B be (α, β)

so $\beta = -\dfrac{\alpha^2}{4} + 1$ & ....(1)

Since $BV \perp AV$ so

$-\dfrac{1}{2}\left(\dfrac{\beta - 1}{\alpha}\right) = -1 \Rightarrow \beta = 2\alpha + 1$ ....(2)

from (1) & (2)

$(\alpha, \beta) = (-8, -15) =$ point B

so the required area $= \left. -\dfrac{(-6 + 3x)}{2}\right) dx$

$$\boxed{\text{area} = \frac{125}{3} \text{ sq. units}}$$ **Ans.**

Q.18 If 'f' is differentiable and 'g' is a double differentiable function such that $f(x) \in [-1, 1]$ and $f'(x) = g(x)$.

If $f^2(0) + g^2(0) = 9$ then prove that there exist some $c \in (-3, 3)$ such that $g(c) g''(c) < 0$ **[6]**

Sol. We have to prove that $g(x) \cdot g''(x) < 0$ for some $c \in (-3, 3)$ means both can not be +ve or −ve for all $x \in (-3, 3)$ simultaneously. First we assume both are +ve for all $x \in (-3, 3)$.

$f(x) \in [-1, 1]$ and $f^2(0) = 9$

$$\Rightarrow g(0) \in [\quad, 3]$$

given $f'(x) = g(x)$

$$\int_{-3}^{x} f'(x)dx = \int_{-3}^{x} g(x)dx$$

$$f(x) = \int_{-3}^{x} g(x)dx + f(-3) \quad [f(-3) \text{ can take min}^m \text{ value } -1]$$

since $g''(x) > 0 \Rightarrow$ curve is opening upwards

for any $g(x)$ satisfying the conditions, why ?

you can understand by following cases –

Case I : If $g(x)$ is decreasing.

$$\int_{-3}^{x} g(x)dx > 3 \times 2\sqrt{2} \quad \text{(area of rectangle)}$$

Case II : If $g(x)$ is increasing

$$\int_{-3}^{x} g(x)dx > 6\sqrt{2} \quad \text{(area of rectangle)}$$

$$\int_{-3}^{2\sqrt{2}} g(x)\, dx > 6\sqrt{2}$$

Case II : If $g(x)$ takes min^m at $x = 0$

$$\int_{-3}^{g(x)} g(x)\, dx > 6 \times 2\sqrt{2} > 6\sqrt{2} \quad \text{(area of rectangle)}$$

$f(x) > 6\sqrt{2} - 1$

but $f(x)$ can not be greater than one so that their shows contradiction means assumed condition can not be true any how.

$\Rightarrow g(x)$ and $g''(x)$ can not be both +ve for simultaneously all $x \in (-3 \; 3)$

for some $c \in (-3, 3)$ $g(c) \cdot g''(c) < 0$

similarly you can prove that both can not be −ve simultaneously.