Theory of Equations

SOME DEFINITIONS

REAL POLYNOMIAL Let $a_o, a_1, \ldots, a_n$ be real numbers and x is a real variable. Then, $f(x)=a_0 + a_1x+a_2x^2 +\ldots+a_nx^n$ is called a real polynomial of real variable x with real coefficients.
For example, $2x^3 - 6x^2 + 11x-6$, x^2-4x+3 etc. are real polynomials.

COMPLEX POLYNOMIAL If $a_0, a_1, a_2\ldots a_n$ be complex numbers and x is a varying complex number, then $f(x) = a_0 + a_1x+a_2x^2 +\ldots+a_{n-1}x^{n-1}+a_nx^n$ is called a complex polynomial or apolynomial of complex variable with complex coefficients.

POLYNOMIAL EQUATION If f(x) is a polynomial, real or complex, then f(x) = 0 is called a polynomial equation.
If f(x) is a polynomial of second degree, then f(x) =0 is called a quadratic equation. The general form of a quadratic equation is $ax^2+bx+c=0$, where $a, b, c \in C$, set of all complex numbers, and $a \neq 0$.

ROOTS OF AN EQUATION The values of the variable satisfying the given equation are called its roots.
In other words, $x = \alpha$ is a root of the equation f(x)=0, if $f(\alpha)=0$.
The real roots of an equation f(x)=0 are the x-coordinates of the points where the curve y=f(x) crosses x-axis.

SOME RESULTS ON ROOTS OF AN EQUATION

The following are some results on the roots of a polynomial equation with rational coefficients:

I An equation of degree n has n roots, real or imaginary
II Surd and imaginary roots always occur in pairs i.e. if 2-3i is a root of an equation, then 2+3i is also its root. Similarly, if $2+\sqrt{3}$ is a root of a given equation, then $2-\sqrt{3}$ is also its roots.
III An odd degree equation has at least one real root, whose sign is opposite to that of its last term provided that the coefficient of highest degree term is positive.
IV Every equation of an even degree whose constant term is negative and the coefficient of highest degree term is positive, has at least two reals, one positive and one negative.

POSITION OF ROOTS OF A POLYNOMIAL EQUATION

If f(x) = 0 is an equation and a, b are two real numbers such that f(a) f(b) < 0, then the equation f(x) = 0 has at least one real root or an odd number of real roots between a and b. In case f(a) and f(b) are of the same sign, then either no real root or an even number of real roots f(x)=0 lie between a and b.

DEDUCTIONS

1. Every equation of an odd degree has at least one real root, whose sign is opposite to that of its last term, provided that the coefficient of first term is positive.
2. Every equation of an even degree whose last term is negative and the coefficient of first term positive, has at least two real roots, one positive and one negative.
3. If an equation has only one change of sign, it has one positive root and no more.
4. If all the terms of an equation are positive and the equation involves no odd powers of x, then all its roots are complex.

ILLUSTRATION 1 If $a, b, c, d \in R$ such that $a < b < c < d$, then show that the roots of the equation $(x-a)(x-c)+2(x-b)(x-d)=0$ are real and distinct.

ROOTS OF A QUADRATIC EQUTION WITH REALCOEFFICIENTS

An equation of the form $ax^2 + bx + c= 0$ where $a \neq 0$, $a, b, c \in R$ is a called a quadratic equation with real coefficients.

The quantity $D=b^2 - 4ac$ is known as the discriminant of the quadratic equation in (i) whose roots are given by $\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\beta = \frac{-b+\sqrt{b^2-4ac}}{2a}$

The nature of the roots is as given below:

1. The roots are real and distinct if D > 0.
2. The roots are real and equal if D = 0.
3. The roots are complex with non-zero imaginary part if D < 0.
4. The roots are rational if a, b, c are rational and D is a perfect square.
5. The roots are of the form P + $\sqrt{q}$ $(p, q \in Q)$ if a, b, c are rational and D is not a perfect square.
6. If a = 1, b, c ∈ 1 and the roots are rational numbers, then these roots must be integers.
7. If a quadratic equation in x has more than two roots, then it is an identity in x that is a = b = c = 0.

COMMON ROOTS

Let $a_1x^2 + b_1x+c_1=0$ and $a_2x^2 + b_2x+c_2=0$ be two quadratic equation such that $a_1, a_2 \neq 0$ and $a_1b_2 \neq a_2b_1$. Let α be the common root of these two equations. Then,

$$a_1\alpha^2 + b_1\alpha + c_1 = 0$$

$$a_2\alpha^2 + b_2\alpha + c_2 = 0$$

Eliminating α, we get

$$\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} = \left(\frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)^2$$

SIGN OF A QUADRTIC EXPRESSION

Let $f(x) = ax^2 + bx+c$ be a quadratic expression, where a, b, c ∈ R and $a \neq 0$. In this section, we shall determine the sign of $f(x) = ax^2 + bx + c$ for real values of x. As the discriminate of $f(x) = ax^2 + bx + c$ can be positive, zero or negative. So, we shall discuss the following three cases.

CASE I : When $D = b^2 - 4ac < 0$

If D < 0, then it is evident from Figs 20.12 and 20.13 that f(x) > 0 iff a > 0 and f(x) < 0 iff a < 0.

CASE II When $D = b^2 - 4ac = 0$

From Figs, 20.10 and 20.11, we abserve that:

When D = 0, we have

f(x) ≥ 0 iff a > 0 and f(x) ≤ 0 iff a < 0.

CASE III When $D = b^2 - 4ac > 0$

From Fig. 20.8 and 20.9, we observe the following if $D = b^2 - 4ac > 0$ and a > 0, then

$$f(x)\begin{cases} > 0 \ \ for \ \ x < \alpha \ or \ x > \beta \\ < 0 \ \ for \ \ \alpha < x < \beta \\ =0 \ \ for \ \ x = \alpha, \ \beta \end{cases}$$

If $D = b^2 - 4ac > 0$ and a < 0, then $f(x)\begin{cases} < 0 \ \ for \ \ x < \alpha \ or \ x > \beta \\ > 0 \ \ for \ \ \alpha < x < \beta \\ \ \ 0 \ \ for \ \ x = \alpha, \ \beta \end{cases}$

CONDITION FOR RESOLUTION INTO LINEAR FACTORS

THEOREM: The quadratic function $ax^2 + 2hxy + by^2 + 2gx + 2fy+c$ is resolvable into linear rational factors if $abc+2fgh-af^2-bg^2-ch^2=0$ i.e. $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$