Maxima and Minima

MAXIMUM Let f(x) be a function with domain D $\subset$ R. Then, f(x) is said to attaining the maximum value at a point a $\in$ D if f(x) $\le$ f (a0 for all x $\in$ D.

MINIMUM Let f(x) be a function with domain D $\subset$ R. Then, f(x) is said to attain the minimum value at a point a $\in$ D if f(x) $\ge$f(a0 for all x $\in$ D.

LOCAL MAXIMUM A function f(x) is said to attain a local maximum at x = 0 a if there exists a neighbourhood (a-δ m a+δ) of a such that
f(x) < f(a) for all x $\in$ (a - δ, a + δ), x $\ne$ a (or)
f(x) – f(a) < 0 for all x$\in$ (a - δ, a+δ), x $\ne$ a. In such a case f(a) is called the local maximum value of f(x) at x = a.

LOCAL MINIMUM A function f(x) is said to attain a local minimum at x = a if there exists a neighbourhood (a - δ, a+δ) of a such that
f(x) > f(a) for all x $\in$ (a - δ, a + δ), x $\ne$ a
(or) f(x) – f(a0 > 0 for all x$\in$ (a - δ, a + δ), x $\ne$ a
The value of the function at x=a i.e., f(a0 is called the local minimum value of f(x) at x = a.

NECESSARY CONDITION FOR EXTREME VALUES:
We have the following theorem which we state without proof.

THEOREM A necessary condition for f (a) to be an extreme value of a function f(x) is that f'(a) = 0, in case it exists.

ILLUSTRATION Let f(x) = $\begin{cases} x^3 + x^2 + 10x, x < 0 \\ -3\sin x \quad , x \ge 0 \end{cases}$
Investigate x = 0 for local maximum/minimum.

PROPERTIES OF MAXIMA AND MINIMA

(I) If f(x) is continuous function in its domain, then at least one maxima and one minima must lie between two equal values of x.

(II) Maxima and Minima occur alternately, that is, between two maxima there is one minimum and vice-versa.

(III) If f(x)$\to\infty$ as x $\to$a or b and f'(x) = 0 only for one value of x (say c) between a and b, then f© is necessarily the minimum and the least value.
If f(x) $\to -\infty$ as x $\to$a or b, f(c) is necessarily the maximum the greatest value.

(1) The circle x^2+y^2 =1 cuts the x-axis at P and Q. Another circle with center at Q and variable radius intersects the first circle at R above the x-axis and the line segment PQ at S. Find the maximum area of ΔQSR.

(2) P is a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whose center is O and N is the foot of the perpendicular from O upon the tangent at P. Find the maximum area of ΔOPN and the coordinates of P.

(3) Let A(p^2, -p), B(q^2, q) and C(r^2, -r) be the vertices of the triangle ABC. A parallelogram AFDE is drawn with D, E and F on the lines segments BC, CA and AB respectively. Using calculus show that the maximum area of such a parallelogram is $\frac{1}{4}$ (p+q) (q+r) (p-r)

(4) $(at_i^2, 2at_i)$; i=1, 2, 3 are the vertices of a triangle inscribed in the parabola $y^2 = 4ax$. A parallelogram AFDE is drawn with D, E, F on the line segments BC, CA and AB respectively. Show that the maximum area of such a parallelogram is $\frac{a^2}{2}(t_1 - t_2)(t_2 - t_3)(t_1 - t_3)$.

(8) From a fixed point P on the circumference of a circle of radius a, the perpendicular PM is let fall on the tangent at point Q. Prove that the maximum area of ΔPQM is $\frac{3\sqrt{3}a^2}{8}$.

(9) Find the values of p for which $f(x) = x^3 + 6(p-3)x^2+3(p^2-4)x+10$ has positive point of maximum.

(10) Find the condition that $f(x) = x^3 + ax^2 + bx + c$ has

(i) a local minimum at a certain $x \in R^+$

(ii) a local maximum at a certain $x \in R^-$

(iii) a local maximum at certain $x \in R^-$ and minimum at certain $x \in R^+$.

(11) If $f(x) = \cos^3 x + \lambda \cos^2 x$, $x \in (0, \pi)$. Find the range of λ so that f(x) has exactly one maximum and exactly one minimum.