Complex Numbers

COMPLEX NUMBER:

If a, b are two real numbers, then a number of the form a+ib is called a complex number.

A complex number z is purely real if its imaginary part is zero i.e. Im(z)=0 and purely imaginary if its real part is zero i.e. Re (z) = 0.

SET OF COMPLEX NUMBERS: The set of all complex numbers is denoted by C

I.e. C = {a + ib | a, b ∈ R}.

Since a real number 'a' can be written as a+0 i, therefore every real number is a complex number. Hence, $\mathbb{R} \subset \mathbb{C}$, where $\mathbb{R}$ is the set of all real numbers.

DEFINITION: Two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are equal if $a_1 = a_2$ and $b_1 = b_2$ i.e. Re (z_1) = Re (z_2) and Im(z_1) = Im(z_2).

POLAR FORM OF z = x + iy FOR DIFFERENT SIGNS OF x and y.

Let | z | = r and be the acute angle given by $\tan^{-1}\left(\frac{y}{x}\right)$. Let θ be the argument of z.

<u>CASE I</u> Polar form of z = x + iy when x > 0 and y > 0 : In this case, we have $\theta = \alpha$. So, the polar form of z = x + iy is r(cosα + isinα).

<u>CASE II</u> Polar form of z = x + iy when x < 0 and y > 0 : In this case, we have $\theta = \pi - \alpha$. So the polar form of z=x+iy is $r[\cos(\pi-\alpha)+i\sin(\pi-\alpha)$ or, $r(-\cos\alpha + i\sin\alpha)$

<u>CASE III</u> Polar form of z = x +iy when x < 0 and y < 0 : In this case, we have $\theta = -(\pi-\alpha)$. So, the polar form of z is $r[\cos(\pi-\alpha)+i\sin(-(\pi-\alpha))]$ or $r[-\cos\alpha - i\sin\alpha)$

<u>CASE IV</u> Polar form of z=x+iy when x > 0 and y<0 : In this case, we have $\theta = -\alpha$ So, the polar form of z is $r[\cos(-\alpha)+i\sin(-\alpha)]$ or, $r[\cos\alpha - i\sin\alpha]$

ILLUSTRATION1 Write the following complex numbers in the polar form:

(i) $-3\sqrt{2}+3\sqrt{2}i$ (ii) 1 + i (iii) -1 - i (iv) 1 - i

EXAMPLE 1: If z_1 z_1 and z_2 are complex numbers, prove that:

(i) arg $\overline{(z)}$ = - arg(z). In general, arg $\overline{(z)}$ $= 2n\pi - \arg(z)$

(ii) arg $(z_1\ z_2)$ = arg(z_1)+arg(z_2) (iii) arg$(z_1\overline{z_2})$=arg(z_1)-arg(z_2)

(iv) $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

EXAMPLE 3:

(i) If z_1 and z_2 are two computers such both of them satisfy the relation $\Leftrightarrow \frac{z_1}{z_2}$ and arg$(z_1 - z_2)$ = , then $\frac{\pi}{4}$ find the imaginary part of $(z_1 + z_2)$.

(ii) If z is a complex number satisfying |z-1|=1, then prove that $\frac{z-2}{z} = i\tan\{\arg(z)\}$

(iii) If z is a complex number such that $\arg\left(\frac{z-2}{z+2}\right) = \frac{\pi}{4}, a$ then prove that |z-2i|= $2\sqrt{2}$

(iv) Number such that $\left(\frac{z-z_1}{z+z_2}\right)=\frac{\pi}{4}$ arg, then prove that |z-7-9i|= $3\sqrt{2}$

THEOREM 1: If z_1, z_2, z_3 are the affixes of the points A, B, and C in the Arg and plane, then

(i) $\angle BAC=\arg\left(\frac{z_3-z_1}{z_2-z_1}\right)$ (ii) $\frac{z_3-z_1}{z_2-z_1}=\frac{|z_3-z_1|}{|z_2-z_1|}$ where $\alpha=\angle BAC$

THEOREM 2: If z_1, z_2, z_3 and z_4 are the affixes of the points A, B, C and D respectively in the Arg and plane. Then, AB is inclined to CD at the angle.

$$\arg\left(\frac{z_2-z_1}{z_4-z_3}\right)$$

(i) Prove that the triangle whose vertices are the points z_1, z_2, z_3 on the Argand plane is an equilateral triangle if and only if $\frac{1}{z_2-z_3}+\frac{1}{z_3-z_1}+\frac{1}{z_1-z_2}=0$

(ii) Let z_1, z_2, z_3 be three distinct complex numbers satisfying $|z_1-1|=|z_2-1|=|z_3-1|$. Let A, B and C be the points in the Arg and plane representing z_1, z_2 and z_3 respectively. Prove that $z_1+z_2+z_3=3$ if and only if ΔABC is an equilateral triangle.

THEOREM : The equation of a circle described on a line segment joining A(z_1) and B(z_2) as diameter is $(z-z_1)(\overline{z}-\overline{z}_2)+(z-z_2)(\overline{z}-\overline{z}_1)=0$

EXAMPLE 3

(i) A point z moves in the Arg and plane such that arg(z-2-i)= $\frac{\pi}{4}$. Find the path traced by $\frac{1}{z}$.

(ii) Let $z_1=10+6i$ and $z_2=4+2i$ be two complex numbers and z be a complex numbers such that arg $\left(\frac{z-z_1}{z-z_2}\right)=\frac{\pi}{4}$. Find the centre and radius of the locus of complex number z.

De' MOIVERE' S THEOREM

STATEMENT:

(i) If $n\in Z$ (the set of integers), then $(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$

(ii) If $n\in Q$ (the set of rational numbers), then $\cos n\theta+i\sin n\theta$ is one of the values of $(\cos\theta+i\sin\theta)^n$.

(1) If z=x+iy is a complex number with rational x and y and |z|=1, show that $|z^{2n}-1|$ is a rational number for every $n\in N$.

(2) Form an equation whose roots are $\sin^2\frac{\pi}{2n+1},\sin^2\frac{2\pi}{2n+1},\sin\frac{3\pi}{2n+1},...,\sin^2\frac{2n\pi}{2n+1}$

ROOTS OF A COMPLEX NUMBER

let z = a + ib be a complex number, and let r (cos+isin) be the polar form of z. Then by De moivere's theorem $r^{1/n}\left(\cos\left(\frac{\theta}{n}\right)+i\sin\left(\frac{\theta}{n}\right)\right)$ is one of the values of $z^{1/n}$. Here, we shall show that $z^{1/n}$ has n distinct values. We know that $\cos\theta+i\sin\theta=\cos(2m\pi+\theta)+i\sin(2m\pi+\theta)$,

m=0, 1, 2, ... So, $z^{1/n}=r^{1/n}[\cos(2m\pi+\theta)+i\sin(2m\pi+\theta]^{1/n}=$

$$r^{1/n}\left[\cos\frac{2m\pi+\theta}{n}+i\sin\frac{2m\pi+\theta}{n}\right]$$

Now by giving m the values 0, 1, 2, ..., (n - 1); we shall obtain distinct values of $z^{1/n}$.

For the values m = n, n+1, ..., the valuesof $z^{1/n}$ will repeat for example, if m=n,then

$$z^{1/n} = r^{1/n}\left[\cos\frac{2n\pi+\theta}{n} + i\sin\frac{2n\pi+\theta}{n}\right]$$

$$= r^{1/n}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$$

which is same as the value obtained by taking m=0. On taking m=n+1, the value comes out to be identical with corresponding to m=1, Hence, $z^{1/n}$ has n distinct values.

(i) $\cos\frac{\pi}{7}\cos\frac{3\pi}{7}\cos\frac{5\pi}{7} = -\frac{1}{8}$ (ii) $\cos\frac{\pi}{4}\cos\frac{3\pi}{14}\cos\frac{5\pi}{14} = \frac{\sqrt{7}}{8}$

ROOTS OF UNITY

In this section, we shall obtain various order roots of unity by using De' Moivere's theorem. We shall also learn about some poperties of these roots.

nth ROOTS OF UNITY : Let $z = 1^{1/n}$. Then, z = (cos0^o + isin0^{o}0$^{1/n}$

$$\Rightarrow z = (\cos 2r\pi + i\sin 2\pi)^{1/n}, r \in Z$$

$$z = \cos\frac{2r\pi}{n} + i\sin\frac{2r\pi}{n},\ r = 0,1,2,...,(n-1)$$

[Using De' Moivere' s

$1,\alpha,\alpha^2,....,\alpha^{n-1}$, where $\alpha = e^{i2\pi/n} = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$

PROPERTIES OF nTH ROOTS OF UNITY

(i) nth roots of unity form a G.P. With common ratio $e^{i2\pi/n}$

(ii) Sum of nth roots of unity is always zero

(iii) Sum of pth power of nth roots of unity is zero, if P is not a multiple of n.

(iv) Sum of pth powers of nth roots of unity is n, if p is a multiple of n.

(v) Product of nth roots of unity is $(-1)^{n-1}$

(vi) nth roots of unity lie on the unit circle | z | = 1 and divide its circumference into n equal parts.

EXAMPLE 1

(i) If $1,\alpha_1,\alpha_2,....,\alpha^{n-1}$ are the nth roots of unity then prove that

(i) $(1+\alpha_1)(1+\alpha_2)....(1+\alpha_{n-1}) = 1$, if n is an odd natural number

(ii) $(1+\alpha_1)(1+\alpha_2)....(1+\alpha_{n-1}) = 0$, if n is an even natural number.

(ii) If $1,\alpha_1,\alpha_2,\alpha_3,\alpha_4$ are the roots of the equation x^5-1=0, then prove that

$$\frac{\omega-\alpha_1}{\omega^2-\alpha_1},\frac{\omega-\alpha_2}{\omega^2-\alpha_2}.\frac{\omega-\alpha_3}{\omega^2-\alpha_3},\frac{\omega-\alpha_4}{\omega^2-\alpha_4} = \omega$$