Chapter 4

ELLIPSE

TOPICS:

1. STANDARD FORM

2.PARAMETRIC FORM

3. TANGENTS AND NORMALS

4.CHORDS,CHORD OF CONTACT.

5.POLE –POLAR

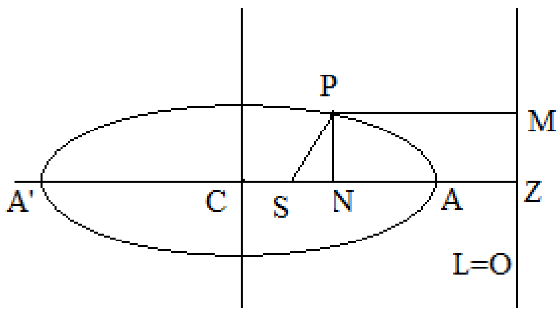
ELLIPSE

A conic section is said to be an ellipse if it's eccentricity e is less than 1.

EQUATION OF AN ELLIPSE

The equation of an ellipse in the standard form is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$**.(a<b)**

Proof :



Let S be the focus, e be the eccentricity and L = 0 be the directrix of the ellipse.

Let P be a point on the ellipse.

Let M, Z be the projections (foot of the perpendiculars) of P, S on the directrix L = 0 respectively.

Let N be the projection of P on SZ. Let A, A′ be the points of division of SZ in the ratio e : 1 internally and externally respectively.

Let AA′ = 2a. Let C be the midpoint of AA′.

The points A, A′ lie on the ellipse and $\frac{SA}{AZ}=e, \frac{SA'}{A'Z}=e$.

$\therefore$ SA = eAZ, SA′ = eA′Z

Now SA + SA′ = eAZ + eA′Z

$\Rightarrow AA' = e(AZ + A'Z)$

$\Rightarrow 2a = e(CZ - CA + A'C + CZ)$

$\Rightarrow 2a = e \cdot 2CZ \ (\because CA = A'C)$

$\Rightarrow CZ = a/e$

Also $SA' - SA = eA'Z - eAZ$

$\Rightarrow A'C + CS - (CA - CS) = e(A'Z - AZ)$

$\Rightarrow 2CS = eAA' \ (\because CA = A'C)$

$\Rightarrow 2CS = e2a \Rightarrow CS = ae$

Take CS, the principal axis of the ellipse as x-axis and Cy perpendicular to CS as y-axis. Then S(ae,0) and the ellipse is in the standard form. Let $P(x_1,y_1)$.

Now PM = NZ = CZ – CN = $\frac{a}{e} - x_1$

P lies on the ellipse :

$$\Rightarrow \frac{PS}{PM} = e \Rightarrow PS = ePM \Rightarrow PS^2 = e^2 PM^2$$

$$\Rightarrow (x_1 - ae)^2 + (y_1 - 0)^2 = e^2\left(\frac{a}{e} - x_1\right)^2$$

$$\Rightarrow (x_1 - ae)^2 + y_1^2 = (a - x_1 e)^2$$

$$\Rightarrow x_1^2 + a^2e^2 - 2x_1ae + y_1^2 = a^2 + x_1^2e^2 - 2x_1ae$$

$$\Rightarrow (1 - e^2)x_1^2 + y_1^2 = (1 - e^2)a^2$$

$$\Rightarrow \frac{x_1^2}{a^2} + \frac{y_1^2}{a^2(1 - e^2)} = 1 \Rightarrow \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$$

where $b^2 = a^2(1 - e^2) > 0$

The locus of P is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$\therefore$ The equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

NATURE OF THE CURVE $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

i) The curve is symmetric about the coordinate axes.

ii) The curve is symmetric about the origin O and hence O is the midpoint of every chord of the ellipse through O. Therefore the origin is the centre of the ellipse.

iii) put y = 0 in the equation of the ellipse $\Rightarrow x^2 = a^2 \Rightarrow x = \pm a$.

Thus the curve meets x-axis (Principal axis) at two points A(a, 0), A′(–a, 0). Hence the ellipse has two vertices. The axis AA′ is called major axis. The length of the major axis is AA′ = 2a

iv) put x = 0 $\Rightarrow y^2 = b^2 \Rightarrow y = \pm b$.

Thus the curve meets y-axis (another axis) at two points B(0, b), B′(0, –b). The axis BB′ is called minor axis and the length of the minor axis is BB′ = 2b.

V) The focus of the ellipse is S(ae, 0). The image of S with respect to the minor axis is S′(–ae,0).

The point S′ is called second focus of the ellipse.

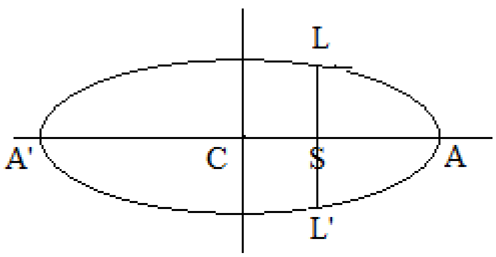
Vi) The directrix of the ellipse is x = a/e. The image of x = a/e with respect to the minor axis is x = –a/e. The line x = –a/e is called second directrix of the ellipse.

Vii) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow y^2=b^2\left(1-\frac{x^2}{a^2}\right) \Rightarrow y=\frac{b}{a}\sqrt{a^2-x^2}$

Thus y has real values only when $-a \le x \le a$. Similarly x has real values only when $-b \le y \le b$. Thus the curve lies completely with in the rectangle $x = \pm a$, $y = \pm b$. Therefore the ellipse is a closed curve.

THEOREM

The length of the latus rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ **(a > b > 0) is** $\frac{2b^2}{a}$**. The length of the latus rectum of the ellipse** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ **(0 < a < b) is** $\frac{2a^2}{b}$**.**



Proof :

Let LL' be the length of the latus rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Focus S =(ae, 0)

If SL = l, then L = (ae, l)

L lies on the ellipse $\Rightarrow \frac{(ae)^2}{a^2}+\frac{l^2}{b^2}=1$

$\Rightarrow e^2+\frac{l^2}{b^2}=1 \Rightarrow \frac{l^2}{b^2}=1-e^2=\frac{b^2}{a^2} \Rightarrow l^2=\frac{b^4}{a^2}$

$\Rightarrow l=\frac{b^2}{a} \Rightarrow SL=\frac{b^2}{a} \quad \therefore LL'=2SL=\frac{2b^2}{a}$

Note : The coordinates of the four ends of the latusrecta of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (a > b > 0) are $L=\left(ae,\frac{b^2}{a}\right), L'=\left(ae,-\frac{b^2}{a}\right), L_1=\left(-ae,\frac{b^2}{a}\right),\ L_1'=\left(-ae,-\frac{b^2}{a}\right)$.

Note : The coordinates of the four ends of the latusrecta of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (0 < a < b) are $L=\left(\frac{a^2}{b},be\right), L'=\left(-\frac{a^2}{b},be\right), L_1=\left(\frac{a^2}{b},-be\right),\ L_1'=\left(-\frac{a^2}{b},-be\right)$.

THEOREM

If P is a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with foci S and S′ then PS + PS′ = 2a.

Proof :

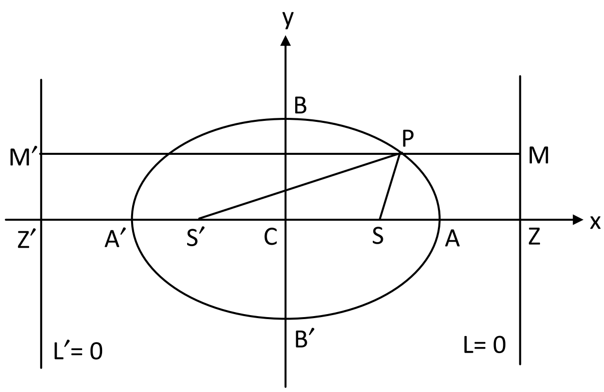
Let e be the eccentricity and L = 0, L′ = 0 be the directrices of the ellipse.

Let C be the centre and A, A′ be the vertices of the ellipse.

$\therefore$ AA′ = 2a.

Foci of the ellipse are S(ae, 0), S′(–ae, 0).

Let P(x_1, y_1) be a point on the ellipse.



Let M, M′ be the projections of P on the directrices L = 0, L′ = 0 respectively.

$\therefore\ \frac{SP}{PM}=e,\ \frac{S'P}{PM'}=e$.

Let Z, Z′ be the points of intersection of major axis with directrices.

$\therefore$ MM′ = ZZ′ = CZ + CZ′ = 2a/e.

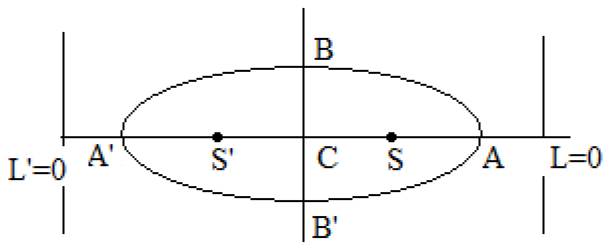
PS + PS′ = ePM + ePM′

= e(PM + PM′) = e(MM′) = e(2a/e) = 2a.

DIFFERENT FORMS OF ELLIPSE

Case I :

In the equation of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ = 1 (a > b)



i) Centre C = (0,0)

ii) Vertices A = (a,0), A^1 = (-a,0)

iii) Length of Major axis AA^1 = 2a and length of Minor axis BB^1 = 2b

iv) Length of latus rectum is $2b^2/a$

v) Foci = ($\pm$ ae,0)

vi) Equation of directrices x = $\pm\frac{a}{e}$

vii) Equation of latus recta x = $\pm$ ae and eccentricity e = $\sqrt{\frac{a^2-b^2}{a^2}}$

viii) ends of latus rectum = $\left(ae,\pm\frac{b^2}{a}\right)$

Case II :

In the equation of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ = 1 (b > a)

i) Centre C = (0,0)

ii) Vertices B = (0,b), B^1 = (0, -b)

iii) Length of Major axis BB^1 = 2b and length of Minor axis AA^1 = 2a

iv) Length of Latus rectum = $\frac{2a^2}{b}$

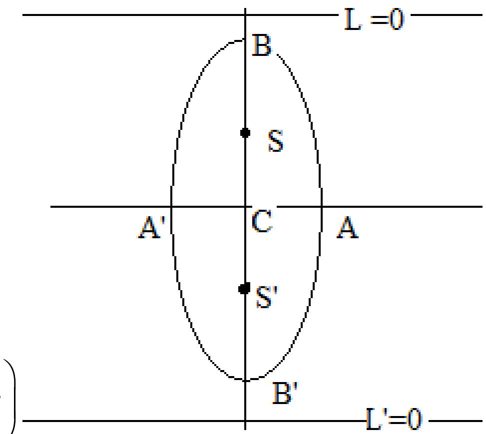
v) Foci = (0, $\pm$be)

vi) Equation of directrices y = $\frac{b}{e}$

vii) Equation of latus recta y = $\pm$be

viii) Eccentricity e = $\sqrt{\frac{b^2 - a^2}{b^2}}$

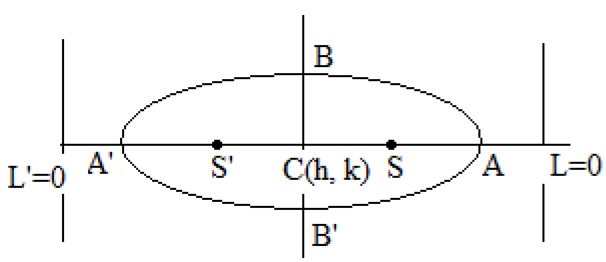
ix) ends of latus rectum = $\left(\pm\frac{a^2}{b}, be\right)$



Case III :

Equation of ellipse with centre (h, k) and axes are parallel to coordinate axes is

$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2}$ = 1; (a > b)



i) Centre C = (h,k)

ii) Vertices (h $\pm$a,k)

iii) foci = $(h \pm ae, k)$

iv) Eccentricity e = $\sqrt{\frac{a^2 - b^2}{a^2}}$

v) Length of latus rectum = $\frac{2b^2}{a}$

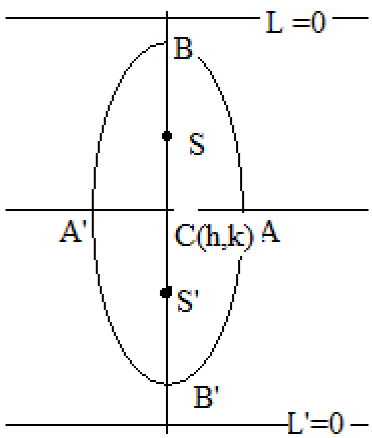
vi) Equation of directrices x = $h \pm \frac{a}{e}$

vii) Equation of latus rectum x = $h \pm ae$

viii) Length of Major axis = 2a and length of minor axes is 2b.

Case IV :

In the equation of ellipse $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$; (b > a)



i) Centre C = (h,k)

ii) Vertices $(h, k \pm b)$

iii) foci = $(h, k \pm be)$

iv) Eccentricity e = $\sqrt{\frac{b^2 - a^2}{b^2}}$

v) Length of latus rectum = $\frac{2a^2}{b}$

vi) Equation of directrices y = k $\pm \frac{b}{e}$

vii) Equation of latus rectum y = k $\pm$ be

viii) Length of Major axis = 2b and length of minor axes is 2a.

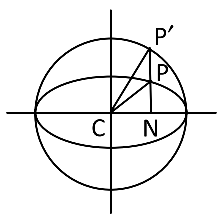
THEOREM

Two tangents can be drawn to an ellipse from an external point.

ECCENTRIC ANGLE

DEFINITION

Let P(x, y) be a point on the ellipse with centre C. Let N be the foot of the perpendicular of P on the major axis. Let NP meets the auxiliary circle at P′. Then ∠NCP′ is called eccentric angle of P. The point P′ is called the corresponding point of P.



PARAMETRIC EQUATIONS

If P(x, y) is a point on the ellipse then x = a cos θ, y = b sin θ where θ is the eccentric angle of P. These equations x = a cos θ, y = b sin θ are called parametric equations of the ellipse. The point P(acos θ, b sin θ) is simply denoted by θ.

THEOREM

The equation of the chord joining the points with eccentric angles α and β on the ellipse S = 0 is $\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2}$.

Proof :

Given points on the ellipse are P(acos α, b sin α), Q(acos β, b sin β).

Slope of $\overleftrightarrow{PQ}$ is $\frac{b\sin\alpha-b\sin\beta}{a\cos\alpha-a\cos\beta}=\frac{b(\sin\alpha-\sin\beta)}{a(\cos\alpha-\cos\beta)}$

Equation of $\overleftrightarrow{PQ}$ is : $y-\sin\alpha=\frac{b(\sin\alpha-\sin\beta)}{a(\cos\alpha-\cos\beta)}(x-a\cos\alpha)$

$$\Rightarrow \frac{(x - a\cos\alpha)}{a}(\sin\alpha - \sin\beta) = \frac{y - b\sin\alpha}{b}(\cos\alpha - \cos\beta)$$

$$\Rightarrow \left(\frac{x}{a} - \cos\alpha\right) 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$= \left(\frac{y}{b} - \sin\alpha\right)(-2)\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$\Rightarrow \left(\frac{x}{a} - \cos\alpha\right)\cos\frac{\alpha+\beta}{2} = -\left(\frac{y}{b} - \sin\alpha\right)\sin\frac{\alpha+\beta}{2}$$

$$\Rightarrow \frac{x}{a}\cos\frac{\alpha+\beta}{2} + \frac{y}{b}\sin\frac{\alpha+\beta}{2} \quad = \cos\alpha\cos\frac{\alpha+\beta}{2} + \sin\alpha\sin\frac{\alpha+\beta}{2} = \cos\left(\alpha - \frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$$

Let $P(x_1, y_1)$ be a point and $S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ be an ellipse. Then

(i) P lies on the ellipse $\Leftrightarrow S_{11} = 0$,

(ii) P lies inside the ellipse $\Leftrightarrow S_{11} < 0$,

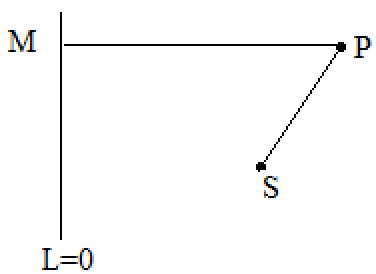
III) P lies outside the ellipse $\Leftrightarrow S_{11} > 0$.

EXERCISE –4(A)

1. Find the equation of the ellipse with focus at (1, –1)e = 2/3 and directrix is x + y + 2 = 0.

Sol. Let $P(x_1, y_1)$ be any point on the ellipse. Equation of the directrix is

$L = x + y + 2 = 0$



By definition of ellipse SP = e PM

$$SP^2 = e^2 \cdot PM^2$$

$$\Rightarrow (x_1 - 1)^2 + (y_1 + 1)^2 = \left(\frac{2}{3}\right)^2 \left[\frac{x_1 + y_1 + 2}{\sqrt{1+1}}\right]^2$$

$$\Rightarrow (x_1 - 1)^2 + (y_1 + 1)^2 = \frac{4}{9}\frac{(x_1 + y_1 + 2)^2}{2}$$

$$\Rightarrow 9\left[(x_1 - 1)^2 + (y_1 + 1)^2\right] = 2(x_1 + y_1 + 2)^2$$

$$\Rightarrow 9\left[x_1^2 - 2x_1 + 1 + y_1^2 + 2y_1 + 1\right] = 2\left[x_1^2 + y_1^2 + 4 + 2x_1y_1 + 4x_1 + 4y_1\right]$$

$$\Rightarrow 9x_1^2 + 9y_1^2 - 18x_1 + 18y_1 + 18 = \quad 2x_1^2 + 2y_1^2 + 4x_1y_1 + 8x_1 + 8y_1 + 8$$

$$\Rightarrow 7x_1^2 - 4x_1y_1 + 7y_1^2 - 26x_1 + 10y_1 + 10 = 0$$

Locus of $P(x_1, y_1)$ is $7x^2 - 4xy + 7y^2 - 26x + 10y + 10 = 0$

2. Find the equation of the ellipse in the standard form whose distance between foci is 2 and length of latus rectum is 15/2.

Sol. Latus rectum = 15/2

$$\Rightarrow \frac{2b^2}{a} = \frac{15}{2}$$

Distance between foci is $2ae = 2$

$\Rightarrow ae = 1$

But $b^2 = a^2 - a^2e^2$

$\Rightarrow b^2 = a^2 - 1$

$$\Rightarrow \frac{15}{4}a = a^2 - 1 \Rightarrow 4a^2 - 15a - 4 = 0$$

$$a = 4 \text{ or } a = -\frac{1}{4}$$

Equation of the ellipse is $\frac{x^2}{16} + \frac{y^2}{15} = 1.$

3. Find the equation of the ellipse in the standard form such that the distance between the foci is 8 and the distance between directrices is 32.

Sol. Distance between foci is $2ae = 8 \Rightarrow ae = 4$

Distance between directrices = 32

$$\Rightarrow \frac{2a}{e} = 32 \Rightarrow \frac{a}{e} = 16$$

$$\Rightarrow (ae)\left(\frac{a}{e}\right) = 64$$

$$\Rightarrow a^2 = 64$$

$$\Rightarrow b^2 = a^2 - a^2e^2 = 64 - 16 = 48$$

Equation of the ellipse is $\frac{x^2}{64} + \frac{y^2}{48} = 1.$

4. Find the eccentricity of the ellipse, in standard form, if its length of the latus rectum is equal to half of its major axis.

Sol.

Given, latus rectum is equal to half of its major axis $\Rightarrow \frac{2b^2}{a} = a$

$\Rightarrow 2b^2 = a^2$

But $b^2 = a^2 (1 - e^2)$

$\Rightarrow 2a^2 (1 - e^2) = a^2$

$\Rightarrow 1 - e^2 = \frac{1}{2} \Rightarrow e^2 = \frac{1}{2} \Rightarrow e = \frac{1}{\sqrt{2}}$

5. The distance of a point on the ellipse $x^2 + 3y^2 = 6$ from its centre is equal to 2. Find the eccentric angles.

Sol. Equation of the ellipse is $x^2 + 3y^2 = 6$

$\Rightarrow \frac{x^2}{6} + \frac{y^2}{2} = 1$

$\Rightarrow a = \sqrt{6}, b = \sqrt{2}$

Any point on the ellipse is $P(\sqrt{6}\cos\theta, \sqrt{2}\sin\theta)$

Given CP = 2 $\Rightarrow CP^2 = 4$

$\Rightarrow 6\cos^2\theta + 2\sin^2\theta = 4$

$\Rightarrow 6(1 - \sin^2\theta) + 2\sin^2\theta = 4$

$\Rightarrow 6 - 6\sin^2\theta + 2\sin^2\theta = 4$

$\Rightarrow 4\sin^2\theta = 2 \Rightarrow \sin^2\theta = \frac{2}{4} = \frac{1}{2}$

$\sin\theta = \pm\frac{1}{\sqrt{2}}$

$\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}$

$\sin\theta = -\frac{1}{2} \Rightarrow \theta = \frac{5\pi}{4}, \frac{7\pi}{4}$

Eccentric angles are : $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

6. Find the equation of the ellipse in the standard form, if it passes through the points (–2, 2) and (3, –1).

Sol. Equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

It is passing through (–2, 2), (3, –1)

(–2, 2) $\Rightarrow \frac{4}{a^2}+\frac{4}{b^2}=1$ …(i)

(3, –1) $\Rightarrow \frac{9}{a^2}+\frac{1}{b^2}=1$ …(ii)

Solving (i) and (ii), we get

$\frac{1}{a^2}=\frac{3}{32},\ \frac{1}{b^2}=\frac{5}{32}$

$\Rightarrow \frac{3x^2}{32}+\frac{5y^2}{32}=1$

$\Rightarrow 3x^2+5y^2=32$

7. If the ends of major axis of an ellipse are (5, 0) and (–5, 0). Find the equation of the ellipse in the standard form if its focus lies on the line 3x – 5y – 9 = 0.

Sol. Vertices (±a, 0)= (±5, 0) $\Rightarrow$ a = 5,

focus S = (ae,o)

Focus lies on the line 3x – 5y – 9 = 0

$\Rightarrow 3(ae) - 5(0) - 9 = 0$

$\Rightarrow 5e=\frac{9}{3}\Rightarrow e=\frac{3}{5}$

$b^2 = a^2(1 - e^2) \Rightarrow b^2=25\left(1-\frac{9}{25}\right)=25\left(\frac{16}{25}\right)=16$

Equation of the ellipse is $\frac{x^2}{25}+\frac{y^2}{16}=1\Rightarrow 16x^2+25y^2=400$

II.

1. Find the length of major axis, minor axis, latus rectum, eccentricity, coordinates of the centre, foci and equations of directrices of the following ellipse.

i) $9x^2 + 16y^2 = 144$

ii) $4x^2 + y^2 - 8x + 2y + 1 = 0$

iii) $x^2 + 2y^2 - 4x + 12y + 14 = 0$

Sol. Given equation is $9x^2 + 16y^2 = 144 \Rightarrow \frac{x^2}{16}+\frac{y^2}{9}=1$

$\therefore$ a = 4, b = 3 where a>b

Length of major axis = 2a = 2×4 = 8

Length of minor axis = 2b = 2×3 = 6

Length of latus rectum = $\frac{2b^2}{a}=\frac{2\cdot 9}{4}=\frac{9}{2}$

Eccentricity = $\sqrt{\frac{a^2-b^2}{a^2}}=\sqrt{\frac{16-9}{16}}=\frac{\sqrt{7}}{4}$

Centre is C(0, 0)

Foci are $(\pm ae, 0) = (\pm\sqrt{7}, 0)$

Equations of the directrices are

$$x=\pm\frac{a}{e}\Rightarrow x=\pm 4\cdot\frac{4}{\sqrt{7}}=\pm\frac{16}{\sqrt{7}}$$

$$\Rightarrow \sqrt{7}x=\pm 16$$

ii) Given equation is $4x^2 + y^2 - 8x + 2y + 1=0$

$$\Rightarrow 4(x^2-2x)+(y^2+2y)=-1$$

$$\Rightarrow 4\left((x-1)^2-1\right)+\left((y+1)^2-1\right)=-1$$

$$\Rightarrow 4(x-1)^2+(y+1)^2=4+1-1=4$$

$$\Rightarrow \frac{(x-1)^2}{1}+\frac{(y+1)^2}{4}=1$$

a = 1, b = 2 where a < b $\Rightarrow$ y-axis is major axis

Length of major axis = 2b = 4

Length of minor axis = 2a = 2

Length of latus rectum = $\frac{2a^2}{b}=\frac{2}{2}=1$

Eccentricity = $\sqrt{\frac{b^2 - a^2}{b^2}} = \sqrt{\frac{4-1}{4}} = \frac{\sqrt{3}}{2}$

Centre is c(–1, 1)

be = $2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$

Foci are $(-1, 1 \pm \sqrt{3})$

Equations of the directrices are

$$y + 1 = \pm\frac{b}{e} = \pm\frac{4}{\sqrt{3}}$$

$$\sqrt{3}y + \sqrt{3} = \pm 4$$

$$\sqrt{3}y + \sqrt{3} \pm 4 = 0$$

iii) **TRY YOURSELF**

2. Find the equation of the ellipse in the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ **given the following data. i) Centre (2, –1), one end of major axis (2,–5), e = 1/3.**

Sol. Centre C =(h, k) = (2, –1) $\Rightarrow$ h = 2, k = –1

End of major axis A =(2, –5).

The x coordinates of centre and end of the major axis are same, therefore major axis is parallel to y axis.

$$b = CA = \sqrt{(2-2)^2 + (-5+1)^2} = \sqrt{(-4)^2} = 4$$

$$a^2 = b^2(1 - e^2) = 16\left(1 - \frac{1}{9}\right) = \frac{128}{9}$$

Equation of the ellipse is

$$\Rightarrow \frac{(x-2)^2}{\frac{128}{9}} + \frac{(y+1)^2}{16} = 1$$

$$\Rightarrow \frac{9(x-2)^2}{128} + \frac{(y+1)^2}{16} = 1$$

$$\Rightarrow 9(x-2)^2 + 8(y+1)^2 = 128$$

$\Rightarrow$ i.e. $8(x-2)^2 + 9(y+1)^2 = 128$.

ii) Centre (4, –1), one end of major axis is (–1,–1) and passing through (8, 0).

Sol. Centre C (4, –1)

ONE end of major axis is A =(–1,–1).

Y coordinates of above points are same, major axis is parallel to x axis

$$a = CA = \sqrt{(4+1)^2 + (-1+1)^2} = 5$$

Ellipse is passing through (8, 0)

$$\Rightarrow \frac{(8-4)^2}{25} + \frac{(0+1)^2}{b^2} = 1 \Rightarrow \frac{1}{b^2} = 1 - \frac{16}{25} = \frac{9}{25}$$

Equation of ellipse is

$$\frac{(x-4)^2}{25} + \frac{9}{25}(y+1)^2 = 1$$

$$\Rightarrow (x-4)^2 + 9(y+1)^2 = 25$$

iii) Centre (0, –3), e = 2/3, semi-minor axis = 5.

Sol.

Centre C (0, –3) , e = 2/3

Semi minor axis b = 5

$\Rightarrow b^2 = a^2 - a^2 e^2$

$$\Rightarrow 25 = a^2 - a^2 \frac{4}{9} = a^2 \left(\frac{5}{9}\right)$$

$\Rightarrow 45 = a^2$

Equation of ellipse is

$$\frac{(x-0)^2}{45} + \frac{(y+3)^2}{25} = 1$$

$$\Rightarrow \frac{x^2}{45} + \frac{(y+3)^2}{25} = 1$$

iv) Centre (2, –1), e = 1/2, latus rectum = 4.

Sol.

Centre (2, –1), e = 1/2

latus rectum = 4 $\Rightarrow \frac{2b^2}{a} = 4 \Rightarrow b^2 = 2a$

$b^2 = a^2 - a^2 e^2$

$\Rightarrow b^2 = a^2 - a^2 \frac{1}{4}$

$\Rightarrow b^2 = \frac{3}{4}a^2$

$\Rightarrow 2a = \frac{3}{4}a^2$

$\Rightarrow \frac{8}{3} = a$ or $a^2 = \frac{64}{9}$

$\Rightarrow b^2 = \frac{16}{3}$

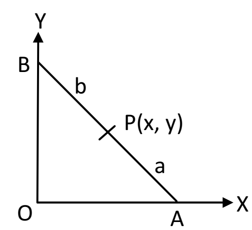
Equation of the ellipse is $\frac{9(x-2)^2}{64} + \frac{3(y+1)^2}{16} = 1$

$\Rightarrow 9(x - 2)^2 + 12(y + 1)^2 = 64$

III.

1. A line of fixed length (a + b) moves so that its ends are always on two perpendicular straight lines prove that a marked point on the line, which divides this line into portions of lengths 'a' and 'b' describes an ellipse and also find the eccentricity of the ellipse when a = 8, b = 12.

Sol.



Let the perpendicular lines as coordinate axes.

Let OA = α and OB = β then A (α , 0) and B(0, β)

And the equation of AB is $\frac{x}{\alpha} + \frac{y}{\beta} = 1$.

Given length of the line AB =(a + b)

$\Rightarrow \alpha^2 + \beta^2 = (a+b)^2 \quad \ldots \text{(i)}$

Let P(x, y)be the point which divides AB in the ratio a : b

$\Rightarrow P = \left(\frac{b\alpha}{a+b}, \frac{a\beta}{a+b}\right) = (x, y)$

$\frac{b\alpha}{a+b} = x \Rightarrow \alpha = \frac{a+b}{b} \cdot x, \frac{a\beta}{a+b} = y \Rightarrow \beta = \frac{a+b}{a} \cdot y$

Substituting the values of α, β in (i), we get,

$$\frac{(a+b)^2}{b^2}\cdot x^2+\frac{(a+b)^2}{a^2}\cdot y^2=(a+b)^2$$

or $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$

P describes an ellipse.

Given a = 8, b = 12, so that b > a.

Eccentricity =

$$\sqrt{\frac{b^2-a^2}{b^2}}=\sqrt{\frac{144-64}{144}}=\sqrt{\frac{80}{144}}=\frac{\sqrt{5}}{3}$$

2. Prove that the equation of the chord joining the points α and β on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\left(\frac{\alpha-\beta}{2}\right)$.

Sol. The given points on the ellipse are P(a cos α, b sin α) and Q(a cos β, b sin β)

Slope of PQ = $\frac{y_1-y_2}{x_1-x_2}=\frac{b(\sin\alpha-\sin\beta)}{a(\cos\alpha-\cos\beta)}=\frac{b\left(2\cos\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2}\right)}{a\left(-2\sin\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2}\right)}=-\frac{b\cdot\cos\frac{\alpha+\beta}{2}}{a\cdot\sin\frac{\alpha+\beta}{2}}$

Equation of the chord PQ is $y-b\sin\alpha=-\frac{b\cos\frac{\alpha+\beta}{2}}{a\sin\frac{\alpha+\beta}{2}}(x-a\cos\alpha)$

$$\frac{y}{b}\sin\frac{\alpha+\beta}{2}-\sin\alpha\cdot\sin\frac{\alpha+\beta}{2}=-\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\cos\alpha\cdot\cos\frac{\alpha+\beta}{2}$$

$$\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\quad\cos\alpha\cdot\cos\frac{\alpha+\beta}{2}+\sin\alpha\cdot\sin\frac{\alpha+\beta}{2}$$

$$=\cos\left(\alpha-\frac{\alpha+\beta}{2}\right)=\cos\frac{\alpha-\beta}{2}$$

THEOREM

The equation of the tangent to the ellipse S = 0 at $P(x_1, y_1)$ is $S_1 = 0$.

THEOREM

The equation of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $P(x_1, y_1)$ is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$.

Proof : The equation of the tangent to S = 0 at P is $S_1 = 0$

$$\Rightarrow\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=0$$

The equation of the normal to S = 0 at P is

$$\frac{y_1}{b^2}(x-x_1)-\frac{x_1}{a^2}(y-y_1)=0$$

$$\Rightarrow \quad \frac{xy_1}{b^2}-\frac{yx_1}{a^2}=\frac{x_1y_1}{b^2}-\frac{x_1y_1}{a^2}$$

$$\Rightarrow \quad \frac{a^2b^2}{x_1y_1}\left(\frac{xy}{b^2}-\frac{yx_1}{a^2}\right)=\frac{a^2b^2}{x_1y_1}\left(\frac{x_1y_1}{b^2}-\frac{x_1y_1}{a^2}\right)$$

$$\Rightarrow \quad \frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2.$$

THEOREM

The condition that the line y = mx + c may be a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ **is** $\mathbf{c^2 = a^2m^2 + b^2}$**.**

Proof : Suppose y = mx + c …(1) is a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Let $P(x_1, y_1)$ be the point of contact.

The equation of the tangent at P is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=0 \quad \ldots(2)$$

Now (1) and (2) represent the same line.

$$\therefore \frac{x_1}{a^2m}=\frac{y_1}{b^2(-1)}=\frac{-1}{c}\Rightarrow x_1=\frac{-a^2m}{c}, y_1=\frac{b^2}{c}.$$

P lies on the line y = mx + c $\Rightarrow y_1 = mx_1 + c$

$$\Rightarrow \frac{b^2}{c}=m\left(\frac{-a^2m}{c}\right)+c\Rightarrow b^2=-a^2m^2+c^2$$

$$\Rightarrow c^2=a^2m^2+b^2.$$

Note : The equation of a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ may be taken as $y=mx\pm\sqrt{a^2m^2+b^2}$. The point of contact is $\left(\frac{-a^2m}{c},\frac{b^2}{c}\right)$ where $c^2 = a^2m^2 + b^2$.

DIRECTOR CIRCLE THEOREM

The points of intersection of perpendicular tangents to an ellipse S = 0 lies on a circle, concentric with the ellipse.(WHICH IS CALLED DIRECTOR CIRCLE)

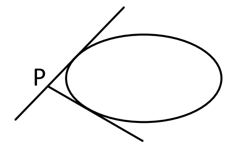
Proof :

Equation of the ellipse $S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$

Let $P(x_1, y_1)$ be the point of intersection of perpendicular tangents drawn to the ellipse.

Let $y = mx \pm \sqrt{a^2m^2 + b^2}$ be a tangent to the ellipse S = 0 passing through P.

Then $y_1 = mx_1 \pm \sqrt{a^2m^2 + b^2}$



$\Rightarrow y_1 - mx_1 = \pm\sqrt{a^2m^2 + b^2}$

$\Rightarrow (y_1 - mx_1)^2 = a^2m^2 + b^2$

$\Rightarrow y_1^2 + m^2x_1^2 - 2x_1y_1m = a^2m^2 + b^2$

$\Rightarrow (x_1^2 - a^2)m^2 - 2x_1y_1m + (y_1^2 - b^2) = 0$ …(1)

If m_1, m_2 are the slopes of the tangents through P then m_1, m_2 are the roots of (1).

The tangents through P are perpendicular.

$\Rightarrow m_1m_2 = -1 \Rightarrow \frac{y_1^2 - b^2}{x_1^2 - a^2} = -1$

$\Rightarrow y_1^2 - b^2 = -x_1^2 + a^2 \Rightarrow x_1^2 + y_1^2 = a^2 + b^2$

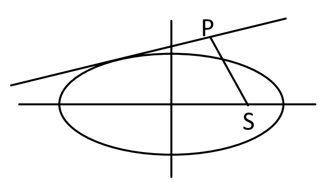
$\therefore$ Locus of P is $x^2 + y^2 = a^2 + b^2$, which is a circle with centre as origin, the centre of the ellipse.

AUXILIARY CIRCLE THEOREM

The feet of the perpendiculars drawn from either of the foci to any tangent to the ellipse S = 0 lies on a circle, concentric with the ellipse.(called auxiliary circle)

Proof :

Equation of the ellipse $S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$



Let $P(x_1, y_1)$ be the foot of the perpendicular drawn from either of the foci to a tangent.

The equation of the tangent to the ellipse S = 0 is $y = mx \pm \sqrt{a^2m^2 + b^2}$ …(1)

The equation to the perpendicular from either foci $(\pm ae, 0)$ on this tangent is

$y = -\frac{1}{m}(x \pm ae)$ …(2)

Now P is the point of intersection of (1) and (2).

$$\therefore\ y = mx \pm \sqrt{a^2m^2 + b^2}, y_1 = -\frac{1}{m}(x_1 \pm ae)$$

$$\Rightarrow y_1 - mx_1 = \pm\sqrt{a^2m^2 + b^2}, my_1 + x_1 = \pm ae$$

$$\Rightarrow (y_1 - mx_1)^2 + (my_1 + x_1)^2 = a^2m^2 + b^2 + a^2e^2$$

$$\Rightarrow y_1^2 + m^2x_1^2 - 2x_1y_1m + m^2y_1^2 + x_1^2 + 2x_1y_1m$$

$$= a^2m^2 + a^2(1 - e^2) + a^2e^2$$

$$\Rightarrow x_1^2(m^2 + 1) + y_1^2(1 + m^2) = a^2m^2 + a^2$$

$$\Rightarrow (x_1^2 + y_1^2)(m^2 + 1) = a^2(m^2 + 1) \Rightarrow x_1^2 + y_1^2 = a^2$$

$\therefore$ Locus of P is $x^2 + y^2 = a^2$, which is a circle with centre as origin, the centre of the ellipse.

THEOREM

The equation to the chord of contact of $P(x_1, y_1)$ with respect to the ellipse $S = 0$ is $S_1 = 0$.